Recursion in Coalgebras

Mauro Jaskelioff
mjj@cs.nott.ac.uk

School of Computer Science & IT
The University of Nottingham

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Outline

- Brief overview of coalgebras.
- The problem of divergence when considering unguarded recursion.
- Different approaches to solving the problem.

The coalgebraic Approach
A Quick Overview

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- Coalgebras are defined over a *behaviour functor* B
- B determines what is observable in the system.
- More concretely: A coalgebra is an arrow

$$X \to BX$$

The carrier X can be thought of as a set of states.

A Simple Coalgebra: LTS

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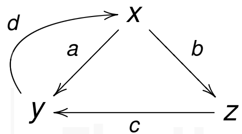

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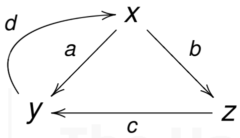

is given by the following coalgebra

$$\begin{aligned} \alpha &: X \to \mathcal{P}(A \times X) \\ \alpha(x) &= \{(a, y), (b, z)\} \\ \alpha(y) &= \{(d, x)\} \\ \alpha(z) &= \{(c, y)\} \end{aligned}$$

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- The unique map $!_\alpha$ into the final coalgebra is often called *unfold*

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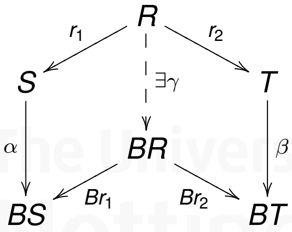

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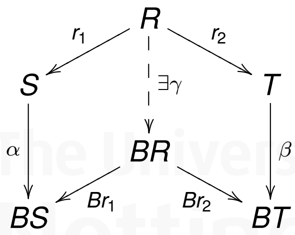

Theorem:

$$\langle s, \alpha \rangle \sim_B \langle t, \beta \rangle \qquad \Leftrightarrow \qquad !_\alpha(s) = !_\beta(t)$$

Example: Bisimulation for LTS

For the case of labelled transition systems, the previous diagram means $(s, t) \in R$ iff

$$\forall (a, s') \in \alpha(s). \quad \exists (a, t') \in \beta(t) \ \wedge \ (s', t') \in R$$

$$\forall (a, t') \in \beta(t). \quad \exists (a, s') \in \alpha(s) \ \wedge \ (s', t') \in R$$

$$\alpha(s) = \emptyset \quad \Leftrightarrow \quad \beta(t) = \emptyset$$

which corresponds which the ordinary notion of bisimulation.

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  - φ is syntactically but not behaviourally guarded

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- How to express divergence coalgebraically?

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- Programs can be infinite.
- This approach needs a category with more structure like CPO.
- Approach followed by Bartek Klin, JLAP 2004.
- It's a domain-theory-oriented solution.

2) Adding divergence to the behaviour

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- Consider the behaviour $B + 1$, where we denote the element of 1 by $\bot$.
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- Consider the behaviour $B + 1$, where we denote the element of 1 by $\bot$.
- We can then define $\varphi \mapsto \bot$.
- Drawback: A coalgebra may detect divergence.
- $\mathit{naughty}(t) \mapsto$ if $\alpha(t) = \bot$ then *stop* else $\bot$
- If we work in the category *Set*, this might be acceptable!

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- We need to consider a notion of observation that ignores equation expansion.

3) Transforming the coalgebra

- We define an endofunctor of $B_\perp$-coalgebras

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\Phi_n &: B_\perp\text{-Coalg} \rightarrow B_\perp\text{-Coalg} \\
\Phi_0(k) &= X \xrightarrow{k} X + BX \\
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- Given $\alpha, \beta : B_\perp$-Coalg. We define

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- Claim: if we have n equations, considering Φ_n is enough to eliminate all finite sequences of expansions.

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Future Work

- Remove dependence from n by some Φ_ω
- Correspondence between $\approx_{B_\perp}$ and what's expected in concrete cases.