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Universes for Generic Programming in Epigram

Constructing Strictly Positive Families

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The challenge

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To make a system for generic programming in Epigram

Strictly Positive Types

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- A large and very useful class of types
  - Built from $\mu, 0, +, 1, \times, K \rightarrow$
  - $e.g.\text{List}\ A = \mu X.1 + (A \times X)$

Strictly Positive Types

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- A large and very useful class of types
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  - $e.g.\text{List } A = \mu X.1 + (A \times X)$

- We build a universe of strictly positive types
  - Define a type of SPT codes
  - Reflect these codes as types
  - This gives us a generic programming

SPT codes

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$$\underline{\text{data}}\ \ \frac{n\ :\ \mathsf{Nat}}{\mathsf{SPT}\ n\ :\ \star}\ \ \underline{\text{where}}$$

$$\frac{}{\mathsf{vz}\ :\ \mathsf{SPT}\ (\mathsf{s}\ n)} \qquad \frac{T\ :\ \mathsf{SPT}\ n}{\mathsf{vs}\ T\ :\ \mathsf{SPT}\ (\mathsf{s}\ n)} \qquad \frac{}{\text{`0'}\ :\ \mathsf{SPT}\ n}$$

$$\frac{S, T\ :\ \mathsf{SPT}\ n}{S\ \text{`}+\text{'}\ T\ :\ \mathsf{SPT}\ n} \qquad \frac{}{\text{`1'}\ :\ \mathsf{SPT}\ n} \qquad \frac{S, T\ :\ \mathsf{SPT}\ n}{S\ \text{`}\times\text{'}\ T\ :\ \mathsf{SPT}\ n}$$

$$\frac{K\ :\ \star \quad T\ :\ \mathsf{SPT}\ n}{K\ \text{`}\rightarrow\text{'}\ T\ :\ \mathsf{SPT}\ n} \qquad \frac{F\ :\ \mathsf{SPT}\ (\mathsf{s}\ n)}{\text{`}\mu\text{'}\ F\ :\ \mathsf{SPT}\ n}$$

Building a universe

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$$\underline{\text{data}}\ \frac{T\ :\ \mathsf{SPT}\ n \quad \vec{T}\ :\ \mathsf{Tel}\ \mathsf{SPT}\ n}{\mathsf{El}\ T\ \vec{T}\ :\ \star}\ \underline{\text{where}}\ \ \ldots$$

$$\underline{\text{data}}\ \frac{X\ :\ \mathsf{Nat}\ \rightarrow\ \star \quad n\ :\ \mathsf{Nat}}{\mathsf{Tel}\ X\ n\ :\ \star}\ \underline{\text{where}}$$

$$\frac{}{\varepsilon\ :\ \mathsf{Tel}\ X\ n} \qquad \frac{\vec{T}\ :\ \mathsf{Tel}\ X\ n \quad x\ :\ X\ n}{\vec{T} :: x\ :\ \mathsf{Tel}\ X\ (\mathsf{s}\ n)}$$

Sums and Products

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$$\frac{s \;:\; \mathsf{El}\ S\ \vec{T}}{\mathsf{inl}\ s \;:\; \mathsf{El}\ (S\ \text{`}+\text{'}\ T)\ \vec{T}} \qquad \frac{t \;:\; \mathsf{El}\ T\ \vec{T}}{\mathsf{inr}\ t \;:\; \mathsf{El}\ (S\ \text{`}+\text{'}\ T)\ \vec{T}}$$

$$\frac{}{\mathsf{void} \;:\; \mathsf{El}\ \text{`}1\text{'}\ \vec{T}} \qquad \frac{s \;:\; \mathsf{El}\ S\ \vec{T} \quad t \;:\; \mathsf{El}\ T\ \vec{T}}{\mathsf{pair}\ s\ t \;:\; \mathsf{El}\ (S\ \text{`}\times\text{'}\ T)\ \vec{T}}$$

Fixed Point, variables

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$$\frac{f \ : \ \mathsf{El}\ F\ (\vec{T} :: \text{`}\mu\text{'}\ F)}{\mathsf{in}\ f \ : \ \mathsf{El}\ (\text{`}\mu\text{'}\ F)\ \vec{T}}$$

$$\frac{e \ : \ \mathsf{El}\ T\ \vec{T}}{\mathsf{top}\ e \ : \ \mathsf{El}\ \mathsf{vz}\ (\vec{T} :: T)} \qquad \frac{e \ : \ \mathsf{El}\ T\ \vec{T}}{\mathsf{pop}\ e \ : \ \mathsf{El}\ (\mathsf{vs}\ T)\ (\vec{T} :: S)}$$

List, nil, cons

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- $\underline{\text{let}}$ ‘List’ : SPT (s z) $\Rightarrow$ ‘μ’ (‘1’‘+’(vz‘$\times$’(vs vz))

List, nil, cons

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- $\underline{\text{let}}$ ‘List’ : SPT (s z) $\Rightarrow$ ‘μ’ (‘1’ ‘$+$’ (vz ‘$\times$’ (vs vz))

- $\underline{\text{let}}$ $\dfrac{}{\text{‘}\varepsilon\text{’} : \text{El ‘List’ } [A]}$

  ‘ε’ $\Rightarrow$ in (inl void)

List, nil, cons

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- $\underline{\text{let}}\ \text{`List'} : \mathsf{SPT}\ (\mathsf{s}\ \mathsf{z}) \Rightarrow \text{`}\mu\text{'}\ (\text{`}1\text{'}\ \text{`}+\text{'}\ (\mathsf{vz}\ \text{`}\times\text{'}\ (\mathsf{vs}\ \mathsf{vz}))$
- $\underline{\text{let}}\ \dfrac{}{\text{`}\varepsilon\text{'} : \mathsf{El}\ \text{`List'}\ [A]}$

  $\text{`}\varepsilon\text{'} \Rightarrow \mathsf{in}\ (\mathsf{inl}\ \mathsf{void})$
- $\underline{\text{let}}\ \dfrac{a : \mathsf{El}\ A\ []\quad as : \mathsf{El}\ \text{`List'}\ [A]}{a\text{`::'}as : \mathsf{El}\ \text{`List'}\ [A]}$

  $a\text{`::'}as \Rightarrow \mathsf{in}\ (\mathsf{inr}\ (\mathsf{pair}\ (\mathsf{pop}\ (\mathsf{top}\ a))\ (\mathsf{top}\ as)))$

Generic Programming

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- Generic programs are parametic in an SPT code

  $$\underline{\text{let}}\ \frac{T : \mathsf{SPT}\ n \quad t : \mathsf{EI}\ T\ \vec{T}}{\mathbf{gfoo}_T\, t : T}$$

  gfoo will work for any SPT

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  gfoo will work for any SPT
- Can define functorial map, remove '$\rightarrow$' and we can define decidable equality
- Different classes of types support different operations... so we need a number of universes like this

Inductive Families

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- There's an obvious imbalance, modelling SPTs using inductive families

Inductive Families

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- There's an obvious imbalance, modelling SPTs using inductive families
- Can we genelalise this technique to 'Strictly Positive Families'?

SPF

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$$\underline{\text{let}} \quad \frac{\vec{I} \ : \ \mathsf{Vec} \ \star \ n \quad O \ : \ \star}{\mathsf{SPF} \ \vec{I} \ O \ : \ \star} \quad \underline{\text{where}} \quad \ldots$$

SPF

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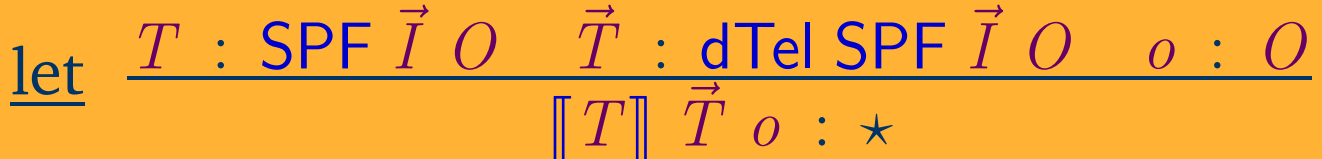
$$\underline{\text{let}}\ \frac{\vec{I} : \mathsf{Vec}\ \star\ n \quad O : \star}{\mathsf{SPF}\ \vec{I}\ O : \star}\ \underline{\text{where}}\ \ldots$$

$$\underline{\text{data}}\ \frac{X : \mathsf{Nat} \to \star \quad n : \mathsf{Nat}}{\mathsf{Tel}\ X\ n : \star}\ \underline{\text{where}}\ \ldots$$

$$\underline{\text{let}}\ \frac{}{\text{`Tel'} : \mathsf{SPF}\ [\mathsf{Nat}]\ \mathsf{Nat}}\ \ldots$$

Interpreting SPF



Interpreting SPF

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$$\underline{\text{let}}\ \frac{T : \mathsf{SPF}\ \vec{I}\ O \quad \vec{T} : \mathsf{dTel}\ \mathsf{SPF}\ \vec{I}\ O \quad o : O}{[\![T]\!]\ \vec{T}\ o : \star}\ \underline{\text{where}}\ \ldots$$

$$[\![\text{`Tel'}]\!]\ [\text{`SPT'}] : \mathsf{Nat} \to \star$$

Interpreting SPF

$$\underline{\text{let}}\ \frac{T : \mathsf{SPF}\ \vec{I}\ O \quad \vec{T} : \mathsf{dTel}\ \mathsf{SPF}\ \vec{I}\ O \quad o : O}{[\![T]\!]\ \vec{T}\ o : \star}\ \underline{\text{where}}\ \ldots$$

$$[\![\text{`Tel'}]\!]\ [\text{`SPT'}] : \mathsf{Nat} \to \star$$

But how do we build families?

Constant families

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$$\frac{}{\text{`0'}, \text{`1'} : \mathsf{SPF}\ \vec{I}\ O}$$

Constant families

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$$\frac{}{\text{`0', `1'} : \mathsf{SPF}\ \vec{I}\ O}$$

$$\frac{}{\mathsf{void} : [\![\text{`1'}]\!]\ \vec{T}\ o}$$

Constant families

$$\frac{}{\text{`0'}, \text{`1'} \;:\; \mathsf{SPF}\ \vec{I}\ O}$$

$$\frac{}{\mathsf{void} \;:\; [\![\text{`1'}]\!]\ \vec{T}\ o}$$

No rule for `0', obviously. Just like SPT these are the basic building blocks, we now build new codes from old.

Re-indexing

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The simplest this we can do is to ‘re-index’ a family

$$\frac{f \;:\; O \to O' \quad T \;:\; \mathsf{SPT}\ \vec{I}\ O'}{\text{`}\Delta\text{'}\ f\ T \;:\; \mathsf{SPF}\ \vec{I}\ O}$$

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$$\frac{v \;:\; [\![T]\!]\ \vec{T}\ (f\ o)}{\delta v \;:\; [\![\text{`}\Delta\text{'}\ f\ T]\!]\ \vec{T}\ o}$$

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$$\frac{f \ : \ O \to O' \quad T \ : \ \mathsf{SPT} \ \vec{I} \ O'}{\text{`}\Delta\text{'} \ f \ T \ : \ \mathsf{SPF} \ \vec{I} \ O}$$

$$\frac{v \ : \ [\![T]\!] \ \vec{T} \ (f \ o)}{\delta v \ : \ [\![\text{`}\Delta\text{'} \ f \ T]\!] \ \vec{T} \ o}$$

What happens if $f \ : \ O' \to O$?

Two things

Two things

$$\frac{f \;:\; O' \to O \qquad T \;:\; \mathsf{SPT}\ \vec{I}\ O'}{\text{`}\Sigma\text{'}\ f\ T \qquad\qquad :\ \mathsf{SPF}\ \vec{I}\ O}$$

We can pick an $o' \;:\; O'$ s.t. $f\ o' = o$

$$\frac{v \;:\; \exists o' \;:\; O' \Rightarrow f\ o' = o \to [\![T]\!]\ \vec{T}\ o'}{\sigma v \;:\; [\![\text{`}\Sigma\text{'}\ f\ T]\!]\ \vec{T}\ o}$$

Two things

$$\frac{f\ :\ O' \to O \quad T\ :\ \mathsf{SPT}\ \vec{I}\ O'}{\text{`}\Sigma\text{'}\ f\ T, \text{`}\Pi\text{'}\ f\ T\ :\ \mathsf{SPF}\ \vec{I}\ O}$$

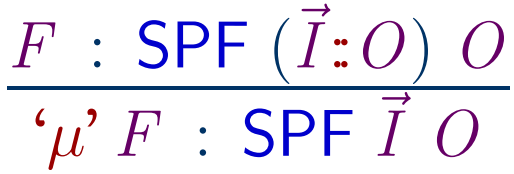
We can pick an $o'\ :\ O'$ s.t. $f\ o'{=}o$

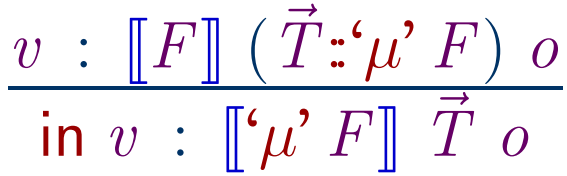
$$\frac{v\ :\ \exists o'\ :\ O' \Rightarrow f\ o'{=}o \to [\![T]\!]\ \vec{T}\ o'}{\sigma v\ :\ [\![\text{`}\Sigma\text{'}\ f\ T]\!]\ \vec{T}\ o}$$

Or we can consider *all* such values

$$\frac{v\ :\ \forall o'\ :\ O' \Rightarrow f\ o'{=}o \to [\![T]\!]\ \vec{T}\ o'}{\pi v\ :\ [\![\text{`}\Pi\text{'}\ f\ T]\!]\ \vec{T}\ o}$$

Fixed Points, variables





Fixed Points, variables

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$$\frac{F \;:\; \mathsf{SPF}\ (\vec{I} :: O)\ O}{\text{`}\mu\text{'}\ F \;:\; \mathsf{SPF}\ \vec{I}\ O}$$

$$\frac{v \;:\; [\![F]\!]\ (\vec{T} :: \text{`}\mu\text{'}\ F)\ o}{\mathsf{in}\ v \;:\; [\![\text{`}\mu\text{'}\ F]\!]\ \vec{T}\ o}$$

$$\frac{}{\mathsf{vz} \;:\; \mathsf{SPF}\ (\vec{I} :: I)\ I}$$

$$\frac{v \;:\; [\![T]\!]\ \vec{T}\ o}{\mathsf{top}\ v \;:\; [\![\mathsf{vz}]\!]\ (\vec{T} :: T)\ o}$$

$$\frac{T \;:\; \mathsf{SPF}\ \vec{I}\ O}{\mathsf{vs}\ T \;:\; \mathsf{SPF}\ (\vec{I} :: I)\ O}$$

$$\frac{v \;:\; [\![T]\!]\ \vec{T}\ o}{\mathsf{pop}\ v \;:\; [\![\mathsf{vs}\ T]\!]\ (\vec{T} :: S)\ o}$$

One last code

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So far there's no way to combine SPFs, how to do that?

One last code

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So far there's no way to combine SPFs, how to do that? Finite choice between constructors, based on part of the index:

$$\frac{f \ : \ \forall t \ : \ \mathsf{Fin}\ n \Rightarrow \mathsf{SPF}\ \vec{I}\ O}{\text{`Tag'}\ f \ : \ \mathsf{SPF}\ \vec{I}\ (O \times \mathsf{Fin}\ n)}$$

$$\frac{v \ : \ [\![f \ t]\!]\ \vec{T}\ o}{\mathsf{tag}\ v \ : \ [\![\text{`Tag'} f]\!]\ \vec{T}\ (o; t)}$$

defining Sum and Product

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$$\underline{\text{let}}\ \frac{A, B\ :\ \mathsf{SPF}\ \vec{I}\ O}{A\ \text{`}+\text{'}\ B\ :\ \mathsf{SPF}\ \vec{I}\ O}$$

$$A\ \text{`}+\text{'}\ B\ \Rightarrow\ \text{`}\Sigma\text{'}\mathbf{fst}\ \left(\text{`Tag'}\ \left(\lambda\ \begin{array}{lcl}\mathsf{fz} & \Rightarrow & A\\ \mathsf{fs}\ \mathsf{fz} & \Rightarrow & B\end{array}\right)\right)$$

$$\underline{\text{let}}\ \frac{A, B\ :\ \mathsf{SPF}\ \vec{I}\ O}{A\ \text{`}\times\text{'}\ B\ :\ \mathsf{SPF}\ \vec{I}\ O}$$

$$A\ \text{`}\times\text{'}\ B\ \Rightarrow\ \text{`}\Pi\text{'}\mathbf{fst}\ \left(\text{`Tag'}\ \left(\lambda\ \begin{array}{lcl}\mathsf{fz} & \Rightarrow & A\\ \mathsf{fs}\ \mathsf{fz} & \Rightarrow & B\end{array}\right)\right)$$

Generic Epigram

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- All Epigram data types have a code in this universe

$$\text{`Tel'} \Rightarrow \text{`}\mu\text{'} \left(\begin{array}{l} (\text{`}\Sigma\text{'}\ (\mathbf{const}\ \mathsf{z})\ \text{`}1\text{'}) \\ \quad \text{`}+\text{'}\ (\text{`}\Sigma\text{'}\mathsf{s}\ (\mathsf{vz}\ \text{`}\times\text{'}\ \mathsf{vs}\ \mathsf{vz})) \end{array} \right)$$

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- This SPF universe gives us generic programming for inductive families, we can define functorial map, equality (with restricted 'Π') etc...

The End

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- Just as strictly positive types are made from $\mu, 0, +, 1, \times$ and $K \to$, strictly positive families are constricted from $\mu, 0, 1, \Sigma, \Delta, \Pi$ and Tag

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- The question is how to create a usable generic programming system for Epigram using these codes, in the pipeline

The End

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- Just as strictly positive types are made from $\mu, 0, +, 1, \times$ and $K\ \rightarrow$, strictly positive families are constricted from $\mu, 0, 1, \Sigma, \Delta, \Pi$ and Tag
- The question is how to create a usable generic programming system for Epigram using these codes, in the pipeline
- More information: “Exploring the Regular Tree Types”, “Constructing Strictly Positive Families” and my thesis, soon.

SPF

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$$\underline{\text{data}}\ \frac{\vec{I} : \mathsf{Vec}\ \star\ n \quad O : \star}{\mathsf{SPF}\ \vec{I}\ O : \star}\ \underline{\text{where}}$$

$$\frac{}{\mathsf{vz} : \mathsf{SPF}\ (\vec{I} :: O)\ O} \qquad \frac{T : \mathsf{SPF}\ \vec{I}\ O}{\mathsf{vs}\ T : \mathsf{SPF}\ (\vec{I} :: I)\ O}$$

$$\frac{f : \forall t : \mathsf{Fin}\ n \Rightarrow \mathsf{SPF}\ \vec{I}\ O}{\text{`Tag'}\ f : \mathsf{SPF}\ \vec{I}\ (O \times \mathsf{Fin}\ n)} \qquad \frac{}{\text{`0'}, \text{`1'} : \mathsf{SPF}\ \vec{I}\ O}$$

$$\frac{f : O \to O' \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Sigma\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'} \qquad \frac{f : O' \to O \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Delta\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'}$$

$$\frac{f : O \to O' \quad T : \mathsf{SPF}\ \vec{I}\ O}{\text{`}\Pi\text{'}O\ f\ T : \mathsf{SPF}\ \vec{I}\ O'} \qquad \frac{T : \mathsf{SPF}\ (\vec{I} :: O)\ O}{\text{`}\mu\text{'}\ T : \mathsf{SPF}\ \vec{I}\ O}$$

El

$$\underline{\text{data}}\ \frac{T : \mathsf{F}\ \vec{I}\ O \quad \vec{T} : \mathsf{Tel}\ \vec{I} \quad o : O}{[\![T]\!]\,\vec{T}\ o : \star}\ \underline{\text{where}}$$

$$\frac{v : [\![T]\!]\,\vec{T}\ \vec{X}\ o}{\mathsf{top}\ v : [\![\mathsf{vz}]\!](\vec{T} :: T)\ o} \qquad \frac{v : [\![T]\!]\,\vec{T}\ \vec{X}\ o}{\mathsf{pop}\ v : [\![\mathsf{vs}\ T]\!](\vec{T} :: T)\ o}$$

$$\frac{v : [\![f\ t]\!]\,\vec{T}\ o}{\mathsf{tag}\ v : [\![\text{`Tag'}f]\!]\,\vec{T}\ (o; t)} \qquad \frac{}{\mathsf{void} : [\![\text{`1'}]\!]\,\vec{T}\ o}$$

$$\frac{v : [\![T]\!]\,\vec{T}\ o}{\sigma\, o\, v : [\![\text{`}\Sigma\text{'}f\ T]\!]\,\vec{T}\ (f\ o)} \qquad \frac{v : [\![T]\!]\,\vec{T}\ (f\ o)}{\delta v : [\![\text{`}\Delta\text{'}f\ T]\!]\,\vec{T}\ o}$$

$$\frac{\vec{v} : \forall o : O;\ p : (f\ o) = o'[\![T]\!]\,\vec{T}\ o}{\pi\vec{v} : [\![\text{`}\Pi\text{'}f\ T]\!]\,\vec{T}\ o'} \qquad \frac{v : [\![T]\!](\vec{T} :: (\text{`}\mu\text{'}\ T))\ o}{\mathsf{in}\ v : [\![\text{`}\mu\text{'}\ T]\!]\,\vec{T}\ o}$$