The University of Nottingham

SCHOOL OF COMPUTER SCIENCE and IT

A LEVEL 1 MODULE, AUTUMN 2006–2007

MATHEMATICS FOR COMPUTER SCIENCE

Time allowed 90 minutes

Candidates must NOT start writing their answers until told to do so.

Model Solutions

1 a)

i) true
ii) false
iii) false
iv) true
v) false

b)

i) false
ii) true
iii) true
iv) true
v) false

c)

i) true
ii) true
iii) true
iv) true
v) true

d)

i) false
ii) false
iii) true
iv) true
v) true

e)

i) true
ii) false
iii) true
iv) true
v) false

2 a) i)

p	q	(	p	$\Rightarrow$	q	)	$\wedge$	p	$\equiv$	p	$\wedge$	q
t	t			t			t		t		t	
t	f			f			f		t		f	
f	t			t			f		t		f	
f	f			t			f		t		f	

ii)

p	q	$\neg p$	$\equiv$	p	$\Leftarrow$	q
t	t	f	f		t	
t	f	f	f		t	
f	t	t	f		f	
f	f	t	t		t	

iii)

p	q	p	$\Rightarrow$	q	$\equiv$	$\neg p$	$\Leftarrow$	$\neg q$
t	t		t		t	f	t	f
t	f		f		t	f	f	t
f	t		t		t	t	t	f
f	f		t		t	t	t	t

b)

$(p{\Leftarrow}q)\vee(p{\Rightarrow}q)$

$=$ { Definitions }

$(p\equiv p\vee q)\vee(q\equiv p\vee q)$

$=$ { distributivity of disjunction over equivales }

$p\vee(q\equiv p\vee q) \;\equiv\; p\vee q\vee(q\equiv p\vee q)$

$=$ { distributivity of disjunction over equivales }

$p\vee q \;\equiv\; p\vee p\vee q \;\equiv\; p\vee q\vee q \;\equiv\; p\vee q\vee p\vee q$

$=$ { idempotence of disjunction }

$p\vee q \;\equiv\; p\vee q \;\equiv\; p\vee q \;\equiv\; p\vee q$

$=$ { reflexivity of equivales }

true .

(Note that associativity of equivales is used implicitly from step 3 onwards.)

3 a) i)

$m<n{+}1\leq m{+}1$

$=$ { meaning of continued inequality }

$m < n+1 \ \wedge \ n+1 \leq m+1$

$=$ { left operand: property of integers
right operand: monotonicity of addition and multiplication }

$m \leq n \ \wedge \ n \leq m$

$=$ { antisymmetry }

$m = n$

ii)

$2\times m \ < \ 2\times n \ \leq \ 2\times m+1$

$=$ { meaning of continued inequalities }

$2\times m < 2\times n \ \wedge \ 2\times n \ \leq \ 2\times m+1$

$=$ { left conjunct: multiplication by 2 preserves inequalities,
right conjunct: property of integers }

$m < n \ \wedge \ 2\times n \ < \ 2\times m+2$

$=$ { right conjunct: multiplication by 2 preserves inequalities }

$m < n \ \wedge \ n < m+1$

$=$ { there is no integer between m and $m+1$ }

false .

b)i) By instantiating k to $m\div n$, the left side is truthified; hence the right side is truthified. Formally,

true

$=$ { definition, $k := m\div n$ }

$m\div n \leq m\div n \quad \equiv \quad (m\div n)\times n \leq m$

$=$ { reflexivity of $\leq$ }

true $\quad \equiv \quad (m\div n)\times n \leq m$

$=$ { reflexivity of $\equiv$ }

$(m\div n)\times n \leq m$

ii) We have, for all k,

$k \leq m\div 1$

$=$ { definition, m }

$k\times 1 \leq m$

$=$ { arithmetic }

$k \leq m$.

Substituting m for k, we get $m \leq m\div 1$, and substituting $m\div 1$ for k, we get $m\div 1 \leq m$. Thus, by anti-symmetry of $\leq$, $m\div 1 = m$. Also,

$k \leq m\div m$

$$
\begin{array}{ll}
= & \{ \quad \text{definition, } n := m \quad \} \\
& k \times m \leq m \\
= & \{ \quad 1 \times m = m \text{, multiplication preserves inequalities} \\
& \quad (m \text{ is strictly positive}) \quad \} \\
& k \leq 1 \quad .
\end{array}
$$

Substituting 1 for k, we get $1 \leq m \div m$, and substituting $m \div m$ for k, we get $m \div m \leq 1$. Thus, by anti-symmetry of $\leq$, $m \div m = 1$.

iii) Choose k equal to 0, n equal to 2 and m equal to 1.

From the definition, we have $0 \leq (-1) \div 2 \equiv 0 \times 2 \leq -1$. That is, $0 \leq (-1) \div 2 \equiv$ false. Also,

$$
\begin{array}{ll}
& k \leq 1 \div 2 \\
= & \{ \quad (3.1) \quad \} \\
& 2 \times k \leq 1 \\
= & \{ \quad \text{integer inequalities} \quad \} \\
& 2 \times k < 2 \\
= & \{ \quad \text{multiplication preserves } < \quad \} \\
& k < 1 \\
= & \{ \quad \text{integer inequalities} \quad \} \\
& k \leq 0 \quad .
\end{array}
$$

It follows that $1 \div 2 = 0$. But then, $0 \leq -(1 \div 2) \equiv$ true. So $(-1) \div 2$ and $-(1 \div 2)$ cannot be equal.

4 a) The three inscriptions are formalised as:

$$(g \equiv G) \wedge (s \equiv \neg S) \wedge (l \equiv l \equiv \neg s) \quad .$$

In addition, we have that the portrait is placed inside at least one of the caskets. That is,

$$G \vee S \vee L \quad .$$

b) We calculate:

$$
\begin{array}{ll}
& (g \equiv G) \wedge (s \equiv \neg S) \wedge (l \equiv l \equiv \neg s) \wedge (G \vee S \vee L) \\
= & \{ \quad \text{reflexivity of } \equiv \quad \} \\
& (g \equiv G) \wedge (s \equiv \neg S) \wedge \neg s \wedge (G \vee S \vee L) \\
= & \{ \quad \text{substitution of equals for equals} \quad \} \\
& (G \equiv g) \wedge S \wedge \neg s \wedge (G \vee S \vee L)
\end{array}
$$

We conclude that the portrait could be in any of the caskets, but it is certainly in the silver casket. The inscription on the silver casket is false. Nothing can be concluded about the inscription on the lead casket. All that can be concluded about the gold casket is $G \equiv g$.

5 a) This is $0^2+1^2+2^2+3^2+4^2$, which equals 30 .

b) This is $2\times 25\leq 60 \wedge 2\times 27\leq 60 \wedge 2\times 29\leq 60$, which equals true.

c)

$$
\begin{array}{ll}
& \langle \exists j \;:\; 0\leq j\leq 250 \;:\; \langle \exists k \;:\; k=1 \vee k=3 \;:\; 2\times k=3\times j\rangle\rangle \\
= & \quad\{ \quad \text{rearrangement and one-point} \quad \} \\
& \langle \exists j \;:\; 0\leq j\leq 250 \;:\; 2\times 1=3\times j \;\vee\; 2\times 3=3\times j\rangle \\
= & \quad\{ \quad \text{rearrangement, arithmetic} \quad \} \\
& \langle \exists j : 0\leq j\leq 250 : 2=3\times j\rangle \;\vee\; \langle \exists j : 0\leq j\leq 250 : 2=j\rangle \\
= & \quad\{ \quad \text{arithmetic, one-point rule} \quad \} \\
& \text{true} \quad .
\end{array}
$$

d)

$$
\begin{array}{ll}
& even.\left\langle \Pi\, k : 1\leq k\leq 500 \;\wedge\; odd.k : k^2\right\rangle \\
= & \quad\{ \quad \text{distributivity and idempotence of disjunction} \quad \} \\
& \langle \exists k : 1\leq k\leq 500 \;\wedge\; odd.k : even.k\rangle \\
= & \quad\{ \quad \text{trading} \quad \} \\
& \langle \exists k : 1\leq k\leq 500 \wedge odd.k \wedge even.k : \text{true}\rangle \\
= & \quad\{ \quad odd.k \wedge even.k \equiv \text{false} \quad \} \\
& \langle \exists k : \text{false} : \text{true}\rangle \\
= & \quad\{ \quad \text{empty range} \quad \} \\
& \text{false}
\end{array}
$$

e) This is true (empty range rule and unit of conjunction is true).