G52MAL
Machines and Their Languages
Lecture 14
The Language of a PDA

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Recap: Definition of PDA

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where

- Q is a finite set of states
- Σ is a finite set of ***input*** symbols
- Γ is a finite set of ***stack*** symbols
- $\delta \in Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to \mathcal{P}_{\text{fin}}(q \times \Gamma^*)$ is the transition function
- $q_0 \in Q$ is the initial state
- $Z_0 \in \Gamma$ is the initial stack symbol
- $F \subseteq Q$ is the accepting states

PDA recognising $\{a^n b^n \mid n \in \mathbb{N}\}$

$$P_1 = (\quad Q = \{q_0, q_1, q_2\},\ \Sigma = \{a, b\},\ \Gamma = \{a, \#\},\ \delta,\ q_0,\ Z_0 = \#,\ F = \{q_2\} \quad)$$

where

$$\begin{array}{rcl} \delta(q_0, a, \#) & = & \{(q_0, a\#)\} \\ \delta(q_0, \epsilon, \#) & = & \{(q_2, \#)\} \\ \delta(q_0, a, a) & = & \{(q_0, aa)\} \\ \delta(q_0, b, a) & = & \{(q_1, \epsilon)\} \\ \delta(q_1, b, a) & = & \{(q_1, \epsilon)\} \\ \delta(q_1, \epsilon, \#) & = & \{(q_2, \#)\} \\ \delta(q, w, x) & = & \emptyset \text{ everywhere else} \end{array}$$

Instantaneous Description (ID)

An Instantaneous Description (ID)

$$(q, w, \gamma) \in Q \times \Sigma^* \times \Gamma^*$$

describes the ***state*** of a PDA computation.

Relations on IDs

$\underset{P}{\vdash} \subseteq ID \times ID$: Read:

$$id_1 \underset{P}{\vdash} id_2$$

“PDA P can move in one step from id_1 to id_2.”

1. $(q, xw, z\gamma) \underset{P}{\vdash} (q', w, \alpha\gamma)$ if $(q', \alpha) \in \delta(q, x, z)$
2. $(q, w, z\gamma) \underset{P}{\vdash} (q', w, \alpha\gamma)$ if $(q', \alpha) \in \delta(q, \epsilon, z)$

where $q, q' \in Q$, $x \in \Sigma$, $w \in \Sigma^*$, $z \in \Gamma$, $\alpha, \gamma \in \Gamma^*$

Example

Consider PDA P_1 again on $aabb$:

$$\begin{array}{llll} (q_0, aabb, \#) & \underset{P_1}{\vdash} & (q_0, abb, a\#) & \text{as } (q_0, a\#) \in \delta(q_0, a, \#) \\ & \underset{P_1}{\vdash} & (q_0, bb, aa\#) & \text{as } (q_0, aa) \in \delta(q_0, a, a) \\ & \underset{P_1}{\vdash} & (q_1, b, a\#) & \text{as } (q_1, \epsilon) \in \delta(q_0, b, a) \\ & \underset{P_1}{\vdash} & (q_1, \epsilon, \#) & \text{as } (q_1, \epsilon) \in \delta(q_1, b, a) \\ & \underset{P_1}{\vdash} & (q_2, \epsilon, \#) & \text{as } (q_2, \#) \in \delta(q_1, \epsilon, \#) \end{array}$$

showing that P_1 accepts $aabb$ by final state as $q_2 \in F$ and all input consumed.

Relations on IDs (cont.)

$\overset{*}{\underset{P}{\vdash}} \subseteq ID \times ID$: The reflexive, transitive closure of $\underset{p}{\vdash}$.
Read:

$$id_1 \overset{*}{\underset{P}{\vdash}} id_2$$

“PDA P can move from id_1 to id_2 in 0 or more steps.”

Examples:

$(q_0, aabb, \#) \overset{*}{\underset{P_1}{\vdash}} (q_2, \epsilon, \#)$

For any PDA P and ID id: $id \overset{*}{\underset{P}{\vdash}} id$

The Language of a PDA (1)

Two “flavours” of PDAs. ***Acceptance by final state***:

$$L(P) = \{w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \epsilon, \gamma) \ \wedge\ q \in F\}$$

Acceptance by empty stack:

$$L(P) = \{w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \epsilon, \epsilon)\}$$

(F plays no role and can be left out from the definition of P.)

The Language of a PDA (2)

A PDA that accepts by final state can be converted to an equivalent PDA that accepts by empty stack and vice versa.

Both types of PDAs thus describe the same class of languages, the ***Context-Free Languages*** (CFLs).

PDAs and CFGs

Theorem: For a language $L \subseteq \Sigma^*$,

$L = L(G)$ for a CFG G iff $L = L(P)$ for a PDA P.

I.e., the CFGs and the PDAs describe the same class of languages.

Proof: By constructing a PDA P from a CFG G and vice versa such that $L(P) = L(G)$.

We will look at constructing a PDA from a CFG.

Translating a CFG into a PDA

Given CFG $G = (N, T, P, S)$,

$$P(G) = (\{q_0\}, \Sigma = T, \Gamma = N \cup T, \delta, q_0, Z_0 = S)$$

where

$$\begin{aligned} \delta(q_0, \epsilon, A) &= \{(q_0, \alpha) \mid A \to \alpha \in P\} \\ \delta(q_0, a, a) &= \{(q_0, \epsilon)\} \text{ for all } a \in T \\ \delta(q_0, w, \gamma) &= \emptyset \text{ everywhere else} \end{aligned}$$

Acceptance by empty stack.

Note: Highly non-deterministic!

Example: Translating a CFG into a PDA

Consider the grammar G_2:

$$A \;\to\; 0A0 \mid 1A1 \mid \epsilon$$

Contruct PDA $P_2 = P(G_2)$. On white board.

Deterministic PDAs (DPDAs) (1)

A DPDA is a PDA that has ***no*** choice:

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is ***deterministic*** iff
$|\delta(q, x, z)| + |\delta(q, \epsilon, z| \leq 1$ for all $q \in Q$, $x \in \Sigma$, $z \in \Gamma$.

Example: P_2 is not a DPDA.
E.g. $|\delta(q_0, 0, A)| + |\delta(q_0, \epsilon, A)| = 0 + 3 \not\leq 1$

Deterministic PDAs (DPDAs) (2)

DPDAs important because can be implemented efficiently. (See lectures on predictive recursive descent parsing.)

But unfortunately:

Theorem: The set of languages accepted by the DPDAs is a ***strict subset*** of the languages accepted by PDAs: $L(\mathrm{DPDA}) \subset L(\mathrm{PDA}) = \mathrm{CFL}$.

However, most context-free langauges of practical importance can be described by DPDAs.