G54FOP: Lecture 3

Programming Language Semantics: Introduction

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Why Does PL Semantics Matter? (1)

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- Documentation
  - Programmers ("What does X mean? Did the compiler get it right?")
  - Implementers ("How to implement X?")

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  - Implementers (“How to implement X?”)
- Formal Reasoning
  - Proofs about programs
  - Proofs about programming languages (E.g. “Well-typed programs do not go wrong”)
  - Proofs about tools (E.g. compiler correctness)

Why Does PL Semantics Matter? (2)

- Language Design
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- Research

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  - Scope rules
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  - What are the effects of a computation?

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  Example: the meaning of 1+2 is the integer 3.

Static vs. Dynamic Semantics (2)

Distinction between static and dynamic semantics not always clear cut. E.g.

- Multi-staged languages (“more than one run-time”)
- Dependently typed languages (computation at the type level)

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Main examples:

- ***Operational Semantics***: Meaning given by ***Abstract Machine***, often a ***Transition Function*** mapping a state to a “more evaluated” state.

  Kinds:

  - ***small-step*** semantics: each step is atomic; more machine like
  - ***structural operational semantics (SOS)***: compound, but still simple, steps
  - ***big-step*** or ***natural*** semantics: Single, compound step evaluates term to final value.

Styles of Semantics (2)

- ***small-step*** semantics:

$$t \longrightarrow t^{(1)} \longrightarrow t^{(2)} \longrightarrow \ldots \longrightarrow v$$

- ***structural operational semantics***:

$$t \xrightarrow{\nabla} t^{(1)} \xrightarrow{\nabla} t^{(2)} \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} v$$

- ***big-step*** or ***natural*** semantics:

$$t \xrightarrow{\nabla} v$$

where ∇ suggests a proof (tree) justifying the step.

Example: Small Step Semantics (1)

Consider a simple machine for evaluating arithmetic expressions. Its state is denoted

$$(\bar{t}, \bar{v})$$

where

- $\bar{t}$ is a sequence of expressions (including values) and operators
- $\bar{v}$ is a stack (sequence) of values

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- If head of $\bar{t}$ is an expression, prepend the individual subexpressions and the operator to the tail of $\bar{t}$.
- If head of $\bar{t}$ is an operator, apply it to appropriate number of arguments on top of the value stack and replace them with the result.

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Exercise: Evaluate $(\langle \texttt{(5-3)*(1+2)} \rangle, \langle\rangle)$

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- Each step describes a small, essentially mechanical, ***syntactic*** transformation of the machine state; i.e., a very operational view of computation.
- Our description was informal. We will discuss at length how to formalise operational semantics in a mathematically precise way, typically using ***inference rules***.

Styles of Semantics (3)

- ***Denotational Semantics***: More abstract view:
  - meaning of a term is a mathematical object (like a ***number*** (e.g. $\mathbb{N}$ or $\mathbb{Z}$) or ***function*** (e.g. $\mathbb{Z} \to \mathbb{Z}$);
  - an ***interpretation function*** maps terms ((abstract) syntax) to their meaning (semantics).

Example: Denotational Semantics (1)

Given the abstract syntax for expressions:

e	$\longrightarrow$		*expressions:*
	$\vert$	$\mathbb{Z}$	*integer literals*
	$\vert$	e **+** e	*addition*
	$\vert$	e **-** e	*subtraction*
	$\vert$	e ***** e	*multiplication*

Example: Denotational Semantics (2)

the denotational semantics (the interpretation function mapping the (abstract) syntax of an expression to its meaning) might be specified as:

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\begin{aligned}
[\![\cdot]\!] &: e \longrightarrow \mathbb{Z} \\
[\![n]\!] &= n \\
[\![e_1 \texttt{+} e_2]\!] &= [\![e_1]\!] + [\![e_2]\!] \\
[\![e_1 \texttt{-} e_2]\!] &= [\![e_1]\!] - [\![e_2]\!] \\
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Not vacuous: e.g., note the difference between **+**, ***syntax***, and $+$, the ordinary function plus.

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Calculate the denotation (meaning) of:

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1 + (2 - 3)
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Styles of Semantics (4)

- ***Axiomatic Semantics***: More abstract still:
  - An operational or denotational semantics ***implies*** certain properties or ***laws***.
  - An axiomatic semantics takes such laws as the starting point: the laws defines the semantics and the meaning is just what can be proved.
  - Closely related to Hoare logic.

Example: Axiomatic Semantics (1)

The meaning of a command c is given by specifying a ***precondition*** and a ***postcondition***:

$$\{PRE\}\ c\ \{POST\}$$

This says: If PRE holds for a program state, then executing c in that state will terminate and $POST$ will hold in the resulting state.

Example: Axiomatic Semantics (2)

For example, the meaning of assignment is given by:

$$\{P[v \mapsto e]\}\ v\ \texttt{:=}\ e\ \{P\}$$

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This says: For ***any*** predicate P whatsoever that holds in a program state when e is substituted for free occurrences of v, it is the case that P holds in the state resulting after the assignment.

Note: Nothing is said about ***how*** execution is carried out, but the focus is on ***what*** its effect is.

Example: Axiomatic Semantics (3)

We can now prove e.g.

$$\{\texttt{x} = 7\}\ \texttt{x := x+1}\ \{\texttt{x} = 8\}$$

(often easier to work backwards from postcondition):

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$\{x = 8 - 1\}$	arithmetic
$\{x = 7\}$	simplification yields precondition

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Works for ***any*** predicate. Consider proving:

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$\{\mathbf{x} \geq 1 - 1\}$	arithmetic
$\{\mathbf{x} \geq 0\}$	simplification yields precondition

Example: Axiomatic Semantics (5)

Exercise. Prove:

$$\{\mathtt{i} = 6 \land \mathtt{j} = 7\}\ \mathtt{i\ :=\ i\ *\ j}\ \{\mathtt{i} = 42 \land \mathtt{j} = 7\}$$