COMP4075: Lecture 6

Type Classes

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Key reason why many practitioners like Haskell: lots of "programming" can happen automatically!

Haskell Overloading (1)

What is the type of $(==)$?

E.g. the following both work:

```
1 == 2
'a' == 'b'
```

I.e., $(==)$ can be used to compare both numbers and characters.

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No!!! Cannot work uniformly for arbitrary types!

Haskell Overloading (2)

A function like the identity function

$$id :: a \to a$$
$$id\ x = x$$

is ***polymorphic*** precisely because it works uniformly for all types: there is no need to "inspect" the argument.

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is ***polymorphic*** precisely because it works uniformly for all types: there is no need to "inspect" the argument.

In contrast, to compare two "things" for equality, they very much have to be inspected, and an ***appropriate method of comparison*** needs to be used.

Haskell Overloading (3)

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- We may want to be able to add numbers of any kind.
- But to add properly, we must understand what we are adding.

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Similar remarks apply to many other types. E.g.:

- We may want to be able to add numbers of any kind.
- But to add properly, we must understand what we are adding.
- Not every type admits addition.

Haskell Overloading (4)

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- Introduce the notion of a ***type class***: a set of types that support certain related operations.
- ***Constrain*** those operations to ***only*** work for types belonging to the corresponding class.
- Allow a type to be ***made an instance of*** (added to) a type class by providing ***type-specific implementations*** of the operations of the class.

The Type Class Eq

$$\textbf{class } Eq\ a \textbf{ where}$$
$$(==) :: a \to a \to Bool$$

$(==)$ is not a function, but a ***method*** of the ***type class*** Eq. It's type signature is:

$$(==) :: Eq\ a \Rightarrow a \to a \to Bool$$

$Eq\ a$ is a ***class constraint***. It says that that the equality method works for any type belonging to the type class Eq.

Instances of Eq (1)

Various types can be made instances of a type class like Eq by providing implementations of the class methods for the type in question:

$$\textbf{instance } Eq\ Int \textbf{ where}$$
$$\quad x == y = primEqInt\ x\ y$$
$$\textbf{instance } Eq\ Char \textbf{ where}$$
$$\quad x == y = primEqChar\ x\ y$$

Instances of Eq (2)

Suppose we have a data type:

$$\textbf{data}\ Answer = Yes \mid No \mid Unknown$$

We can make $Answer$ an instance of Eq as follows:

$$\begin{array}{l}\textbf{instance}\ Eq\ Answer\ \textbf{where} \\ \quad \begin{array}{lll} Yes & == Yes & = True \\ No & == No & = True \\ Unknown & == Unknown & = True \\ \_ & == \_ & = False \end{array}\end{array}$$

Instances of Eq (3)

Consider:

$$\begin{aligned}\mathbf{data}\ Tree\ a = &\ Leaf\ a \\ &\mid Node\ (Tree\ a)\ (Tree\ a)\end{aligned}$$

Can $Tree$ be made an instance of Eq?

Instances of Eq (4)

Yes, for any type a that is already an instance of Eq:

$$
\begin{array}{l}
\textbf{instance}\ (Eq\ a) \Rightarrow Eq\ (Tree\ a)\ \textbf{where} \\
\quad \begin{array}{lll}
Leaf\ a1 & == Leaf\ a2 & = a1 == a2 \\
Node\ t1l\ t1r & == Node\ t2l\ t2r & = t1l == t2l \\
 & & \quad \&\& \ t1r == t2r \\
\_ & == \_ & = False
\end{array}
\end{array}
$$

Note that $(==)$ is used at type a (whatever that is) when comparing $a1$ and $a2$, while the use of $(==)$ for comparing subtrees is a recursive call.

Derived Instances (1)

Instance declarations are often obvious and mechanical. Thus, for certain ***built-in*** classes (notably Eq, Ord, $Show$), Haskell provides a way to ***automatically derive*** instances, as long as

- the data type is sufficiently simple
- we are happy with the standard definitions

Thus, we can do:

$$\begin{aligned}&\mathbf{data}\ Tree\ a = Leaf\ a\\ &\qquad\qquad\qquad\qquad \mid Node\ (Tree\ a)\ (Tree\ a)\\ &\qquad\qquad\qquad\mathbf{deriving}\ Eq\end{aligned}$$

Derived Instances (2)

GHC provides ***many*** additional possibilities. With the extension `-XGeneralizedNewtypeDeriving`, a new type defined using **newtype** can "inherit" any of the instances of the representation type:

$$\textbf{newtype}\ \mathit{Time} = \mathit{Time}\ \mathit{Int}\ \textbf{deriving}\ \mathit{Num}$$

Derived Instances (2)

GHC provides ***many*** additional possibilities. With the extension `-XGeneralizedNewtypeDeriving`, a new type defined using **newtype** can "inherit" any of the instances of the representation type:

$$\textbf{newtype}\ Time = Time\ Int\ \textbf{deriving}\ Num$$

With the extension `-XStandaloneDeriving`, instances can be derived separately from a type definition (even in a separate module):

$$\textbf{deriving instance}\ Eq\ Time$$
$$\textbf{deriving instance}\ Eq\ a \Rightarrow Eq\ (Tree\ a)$$

Class Hierarchy

Type classes form a hierarchy. E.g.:

$$\begin{aligned}&\textbf{class } Eq\ a \Rightarrow Ord\ a \textbf{ where}\\ &\quad (<=) :: a \to a \to Bool\\ &\quad \ldots\end{aligned}$$

Eq is a superclass of Ord; i.e., any type in Ord must also be in Eq.

Haskell vs. OO Overloading (1)

A method, or overloaded function, may thus be understood as a family of functions where the right one is chosen depending on the types.

A bit like OO languages like Java. But the underlying mechanism is quite different and much more general. Consider `read`:

$$read :: (Read\ a) \Rightarrow String \rightarrow a$$

Note: overloaded on the ***result*** type! A method that converts from a string to ***any*** other type in class `Read`!

Haskell vs. OO Overloading (2)

$$> \textbf{let}\ xs = [1,2,3] :: [Int]$$
$$> \textbf{let}\ ys = [1,2,3] :: [Double]$$
$$> xs$$
$$[1,2,3]$$
$$> ys$$
$$[1.0,2.0,3.0]$$
$$> (read\ \texttt{"42"} : xs)$$
$$[42,1,2,3]$$
$$> (read\ \texttt{"42"} : ys)$$
$$[42.0,1.0,2.0,3.0]$$

Haskell vs. OO Overloading (3)

Taking Java as a typical OO example:

- ***Classes*** and ***interfaces*** define sets of methods that elements of a type must support.

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Taking Java as a typical OO example:

- ***Classes*** and ***interfaces*** define sets of methods that elements of a type must support.
- Through ***generics***, classes can be parametrised on types that can be bounded by classes and interfaces, a little like constraints in Haskell's class/instance declarations.
- However, the overloading is always on the ***object***; i.e. effectively the ***first argument*** to a method:

```
object.method(arg1, arg2, ...)
```

Implementation (1)

The class constraints represent extra implicit arguments that are filled in by the compiler. These arguments are (roughly) the functions to use.

Thus, internally $(==)$ is a ***higher order function*** with ***three*** arguments:

$$(==)\ eqF\ x\ y = eqF\ x\ y$$

Implementation (2)

An expression like

$$1 == 2$$

is essentially translated into

$$(==)\ primEqInt\ 1\ 2$$

Implementation (3)

So one way of understanding a type like

$$(==) :: Eq\ a \Rightarrow a \rightarrow a \rightarrow Bool$$

is that $Eq\ a$ corresponds to an extra implicit argument.

The implicit argument corresponds to a so called directory, or tuple/record of functions, one for each method of the type class in question.

Implementation (4)

A rough illustration of the idea:

$$\begin{array}{l}\textbf{class } Foo\ a \textbf{ where} \\ \quad fie :: a \to Bool \\ \quad fum :: a \to Int\end{array}$$

The types of methods fie and fum:

$$\begin{array}{l}fie :: Foo\ a \Rightarrow a \to Bool \\ fum :: Foo\ a \Rightarrow a \to Int\end{array}$$

Implementation (5)

As Foo have two methods, the dictionary needs to carry two functions. If a pair were to be used for this purpose, the actual implementations would be something along the lines:

$$
\begin{aligned}
&\mathit{fie} :: (a \to \mathit{Bool}, a \to \mathit{Int}) \to a \to \mathit{Bool} \\
&\mathit{fie}\ \mathit{dict}\ x = (\mathit{fst}\ \mathit{dict})\ x \\
&\mathit{fie} :: (a \to \mathit{Bool}, a \to \mathit{Int}) \to a \to \mathit{Bool} \\
&\mathit{fie}\ \mathit{dict}\ x = (\mathit{snd}\ \mathit{dict})\ x
\end{aligned}
$$

Some Basic Haskell Classes (1)

$$\textbf{class } Eq\ a \textbf{ where}$$

$$(==), (/=) :: a \to a \to Bool$$

$$\textbf{class } (Eq\ a) \Rightarrow Ord\ a \textbf{ where}$$

$$compare\ \ :: a \to a \to Ordering$$

$$(<), (<=), (>=), (>) :: a \to a \to Bool$$

$$max, min :: a \to a \to a$$

$$\textbf{class } Show\ a \textbf{ where}$$

$$show :: a \to String$$

Some Basic Haskell Classes (2)

$$
\begin{aligned}
&\textbf{class } \mathit{Num}\ a \textbf{ where} \\
&\quad (+), (-), (*) :: a \to a \to a \\
&\quad \mathit{negate} \qquad\quad :: a \to a \\
&\quad \mathit{abs}, \mathit{signum} :: a \to a \\
&\quad \mathit{fromInteger} :: \mathit{Integer} \to a
\end{aligned}
$$

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$$
\begin{aligned}
&\textbf{class } \mathit{Num}\ a \Rightarrow \mathit{Fractional}\ a \textbf{ where} \\
&\quad (/) \quad :: a \to a \to a \\
&\quad \mathit{recip} :: a \to a \\
&\quad \mathit{fromRational} :: \mathit{Rational} \to a
\end{aligned}
$$

Some Basic Haskell Classes (3)

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Haskell's numeric literals are overloaded:

- 42 means $fromInteger\ 42$
- 1.23 means $fromRational\ (133 \% 100)$

Some Basic Haskell Classes (3)

Quiz: What is the type of a numeric literal like 42? What about 1.23? Why?

Haskell's numeric literals are overloaded:

- 42 means $fromInteger\ 42$
- 1.23 means $fromRational\ (133 \,\%\, 100)$

Thus:

$$42 \quad :: Num\ a \Rightarrow a$$
$$1.23 :: Fractional\ a \Rightarrow a$$

A Typing Conundrum (1)

Overloaded (numeric) literals can lead to some surprises.

What is the type of the following list? Is it even well-typed???

$$[1, [2, 3]]$$

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What is the type of the following list? Is it even well-typed???

$$[1, [2, 3]]$$

Surprisingly, it is well-typed:

$$> :\mathbf{type}\ [1, [2, 3]]$$
$$[1, [2, 3]] :: (Num\ [t], Num\ t) \Rightarrow [[t]]$$

Why?

A Typing Conundrum (2)

The list is expanded into:

$$[fromInteger\ 1,\\ [fromInteger\ 2, fromInteger\ 3]]$$

Thus, if there were some type t for which $[t]$ were an instance of Num, the 1 would be an overloaded literal of that type, matching the type of the second element of the list.

A Typing Conundrum (2)

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$$[fromInteger\ 1,\\ [fromInteger\ 2, fromInteger\ 3]]$$

Thus, if there were some type t for which $[t]$ were an instance of Num, the 1 would be an overloaded literal of that type, matching the type of the second element of the list.

Normally there are no such instances, so what almost certainly is a mistake will be caught. But the error message is rather confusing.

Multi-parameter Type Classes

GHC supports an extension to allow a class to have more than one parameter; i.e., definining a ***relation*** on types rather than just a predicate:

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This often lead to type inference ambiguities. Can be addressed through ***functional dependencies***:

$$\mathbf{class}\ StateMonad\ s\ m \mid m \rightarrow s\ \mathbf{where}\ \ldots$$

This enforces that all instances will be such that m uniquely determines s.

Application: Automatic Differentiation

- ***Automatic Differentiation***: method for augmenting code so that derivative(s) computed along with main result.
- Purely algebraic method: arbitrary code can be handled
- Exact results
- But no separate, self-contained representation of the derivative.

Automatic Differentiation: Key Idea

Consider a code fragment:

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Suppose x' and y' are the derivatives of x and y w.r.t. a common variable. Then the code can be augmented to compute the derivatives of $z1$ and $z2$:

$$z1 = x + y$$
$$z1' = x' + y'$$
$$z2 = x * z1$$
$$z2' = x' * z1 + x * z1'$$

Approaches

- Source-to-source translation
- Overloading of arithmetic operators and mathematical functions

The following variation is due to Jerzy Karczmarczuk. Infinite list of derivatives allows derivatives of ***arbitrary*** order to be computed.

Functional Automatic Differentiation (1)

Introduce a new numeric type C: value of a continuously differentiable function at a point along with ***all*** derivatives at that point:

$$\mathbf{data}\ C = C\ Double\ C$$

$$valC\ (C\ \ a\ \_) = a$$

$$derC\ (C\ \_\ x') = x'$$

Functional Automatic Differentiation (2)

Constants and the variable of differentiation:

$$zeroC :: C$$
$$zeroC = C\ 0.0\ zeroC$$
$$constC :: Double \to C$$
$$constC\ a = C\ a\ zeroC$$
$$dVarC :: Double \to C$$
$$dVarC\ a = C\ a\ (constC\ 1.0)$$

Functional Automatic Differentiation (3)

Part of numerical instance:

$$\begin{aligned}
&\textbf{instance } Num\ C \textbf{ where}\\
&\quad (C\ a\ x') + (C\ b\ y') = C\ (a+b)\ (x'+y')\\
&\quad (C\ a\ x') - (C\ b\ y') = C\ (a-b)\ (x'-y')\\
&\quad x@(C\ a\ x') * y@(C\ b\ y') =\\
&\qquad C\ (a*b)\ (x'*y + x*y')\\
&\quad fromInteger\ n = constC\ (fromInteger\ n)
\end{aligned}$$

Functional Automatic Differentiation (4)

Computation of $y = 3t^2 + 7$ at $t = 2$:

$$t = dVarC\ 2$$
$$y = 3 * t * t + 7$$

We can now get whichever derivatives we need:

$$\begin{array}{lcl} valC\ y & \Rightarrow & 19.0 \\ valC\ (derC\ y) & \Rightarrow & 12.0 \\ valC\ (derC\ (derC\ y)) & \Rightarrow & 6.0 \\ valC\ (derC\ (derC\ (derC\ y))) & \Rightarrow & 0.0 \end{array}$$

Functional Automatic Differentiation (5)

Of course, we're not limited to picking just one point. Let $tvals$ be a list of points of interest:

$$[3 * t * t + 7 \mid tval \leftarrow tvals, \mathbf{let}\ t = dVarC\ tval]$$

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$$[3 * t * t + 7 \mid tval \leftarrow tvals, \mathbf{let}\ t = dVarC\ tval]$$

Or we can define a function:

$$\begin{aligned} &y :: Double \rightarrow C \\ &y\ tval = 3 * t * t + 7 \\ &\quad \mathbf{where} \\ &\qquad t = dVarC\ tval \end{aligned}$$

Reading

- Jerzy Karczmarczuk. Functional differentiation of computer programs. *Higher-Order and Symbolic Computation*, 14(1):35–57, March 2001.