MGS 2007: ADV Lecture 3

Arrows and Functional Reactive Programming

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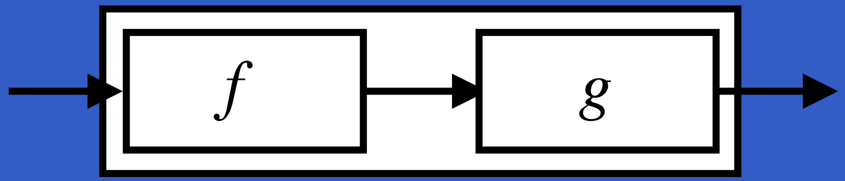
Arrows (1)

System descriptions in the form of block diagrams are very common. Blocks have inputs and outputs and can be combined into larger blocks. For example, serial composition:



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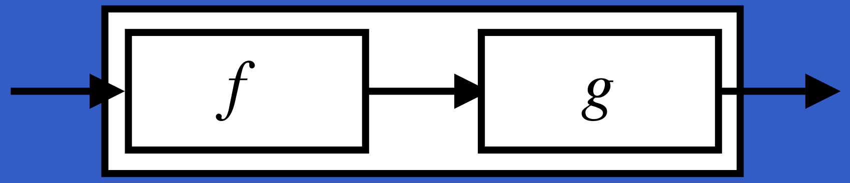
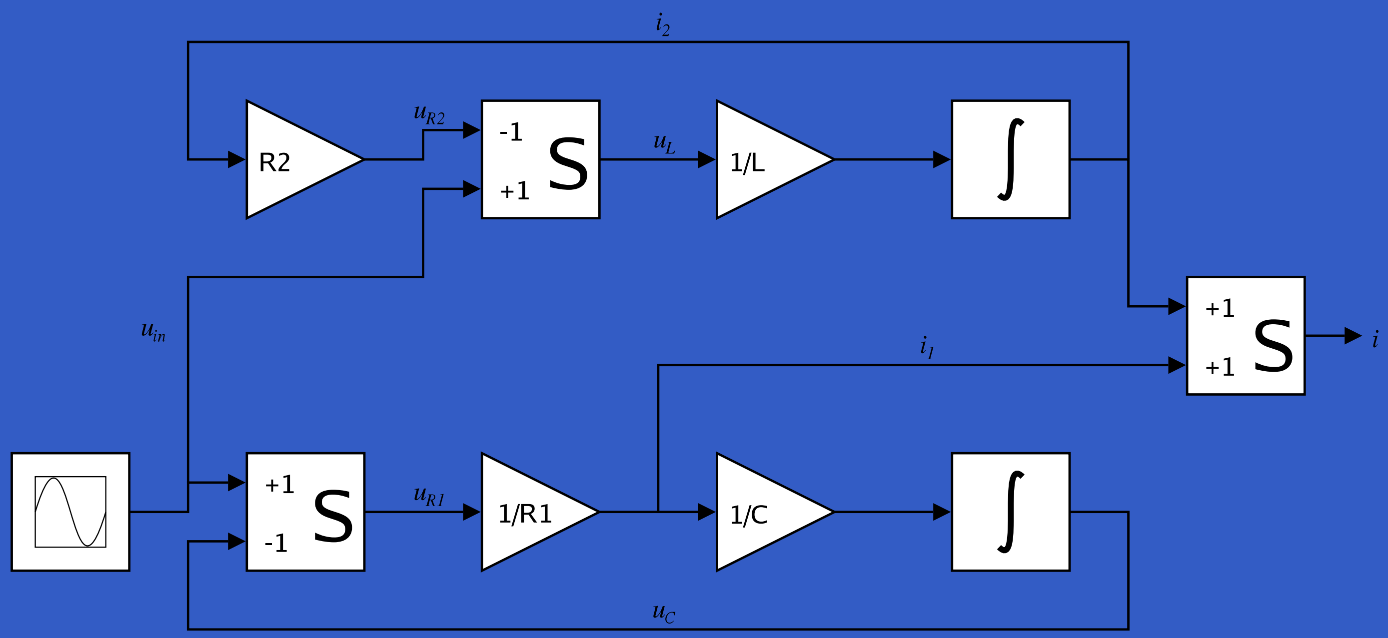


A *combinator* can be defined that captures this idea:

```
(>>>) :: B a b -> B b c -> B a c
```

Arrows (2)

But systems can be complex:



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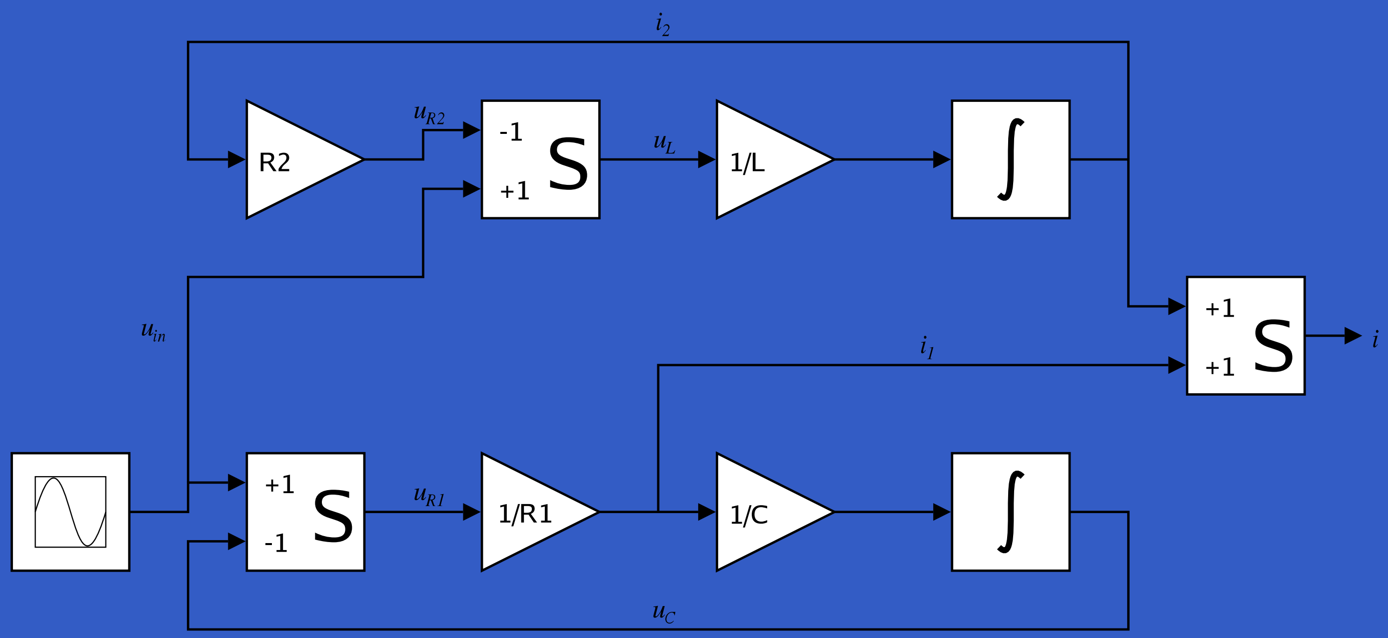

How many and what combinators do we need to be able to describe arbitrary systems?

Arrows (3)

John Hughes' ***arrow*** framework:

- Abstract data type interface for function-like types (or "blocks", if you prefer).

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John Hughes' ***arrow*** framework:

- Abstract data type interface for function-like types (or "blocks", if you prefer).
- Particularly suitable for types representing process-like computations.
- Related to ***monads***, since arrows are computations, but more general.
- Provides a minimal set of "wiring" combinators.

What is an arrow? (1)

- A ***type constructor*** a of arity two.

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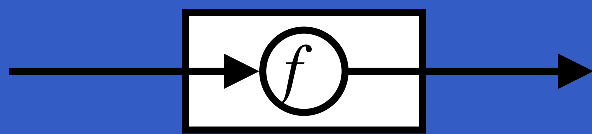
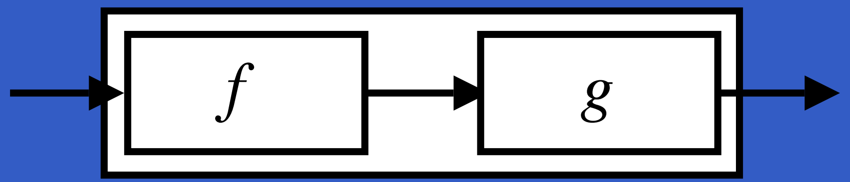
    `first :: a b c -> a (b,d) (c,d)`
- A set of ***algebraic laws*** that must hold.

What is an arrow? (2)

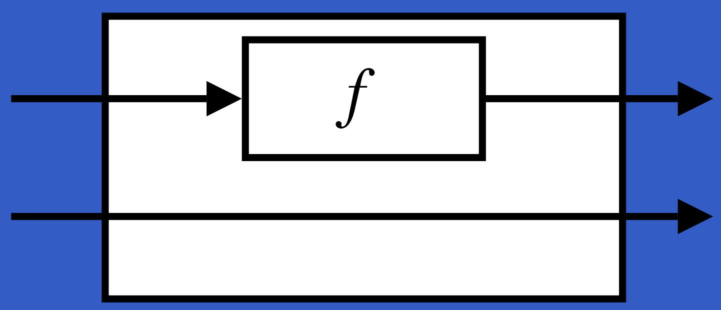
These diagrams convey the general idea:



`arr` f



f `>>>` g



`first` f

The `Arrow` class

In Haskell, a ***type class*** is used to capture these ideas (except for the laws):

```
class Arrow a where
    arr   :: (b -> c) -> a b c
    (>>>) :: a b c -> a c d -> a b d
    first :: a b c -> a (b,d) (c,d)
```

Functions are arrows (1)

Functions are a simple example of arrows, with `(->)` as the arrow type constructor.

Exercise 1: Suggest suitable definitions of

- `arr`
- `(>>>)`
- `first`

for this case!

(We have not looked at what the laws are yet, but they are “natural”.)

Functions are arrows (2)

Solution:

- `arr = id`

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  To see this, recall

  ```
  id :: t -> t
  arr :: (b->c) -> a b c
  ```

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  To see this, recall

  ```
  id :: t -> t
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  ```

  Instantiate with

  $$\texttt{a} = \texttt{(->)}$$
  $$\texttt{t} = \texttt{b->c} = \texttt{(->) b c}$$

Functions are arrows (3)

- `f >>> g = \a -> g (f a)`

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Functions are arrows (3)

- `f >>> g = \a -> g (f a)` ***or***
- `f >>> g = g . f` ***or even***
- `(>>>) = flip (.)`
- `first f = \(b,d) -> (f b,d)`

Functions are arrows (4)

`Arrow` instance declaration for functions:

```
instance Arrow (->) where
    arr     = id
    (>>>)   = flip (.)
    first f = \(b,d) -> (f b,d)
```

Some arrow laws

```
(f >>> g) >>> h = f >>> (g >>> h)
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(f >>> g) >>> h = f >>> (g >>> h)
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  arr (f >>> g)  =  arr f >>> arr g
   arr id >>> f  =  f
              f  =  f >>> arr id
  first (arr f)  =  arr (first f)
first (f >>> g)  =  first f >>> first g
```
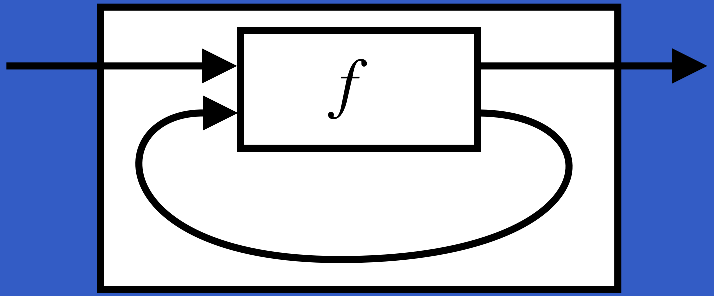

Exercise 2: Draw diagrams illustrating the first and last law!

The `loop` combinator (1)

Another important operator is `loop`: a fixed-point operator used to express recursive arrows or ***feedback***:



`loop` f

The `loop` combinator (2)

Not all arrow instances support `loop`. It is thus a method of a separate class:

```
class Arrow a => ArrowLoop a where
    loop :: a (b, d) (c, d) -> a b c
```

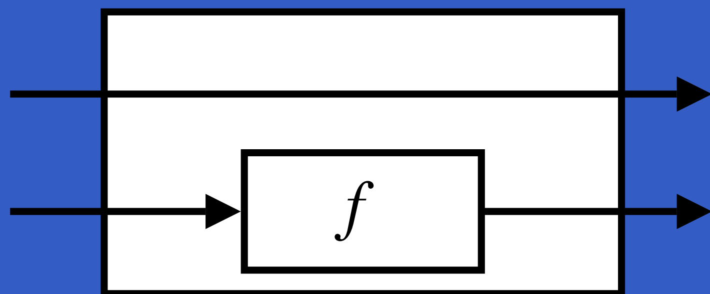
Remarkably, the four combinators `arr`, `>>>`, `first`, and `loop` are sufficient to express any conceivable wiring!

Some more arrow combinators (1)

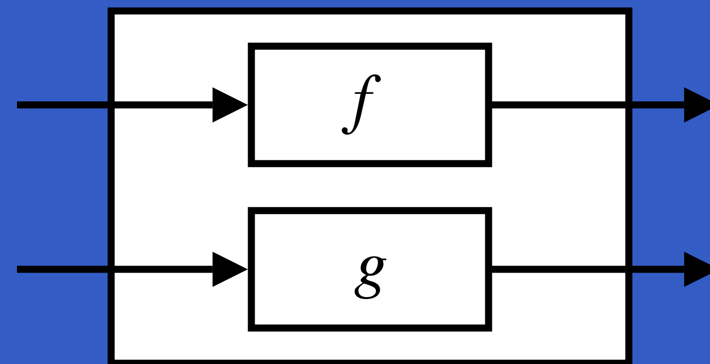
```
second :: Arrow a =>
    a b c -> a (d,b) (d,c)

(***) :: Arrow a =>
    a b c -> a d e -> a (b,d) (c,e)

(&&&) :: Arrow a =>
    a b c -> a b d -> a b (c,d)
```

Some more arrow combinators (2)

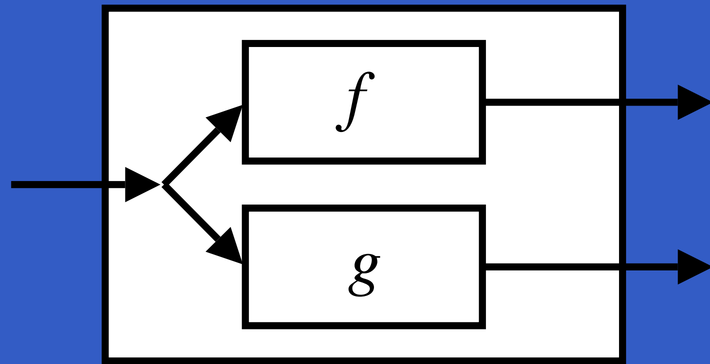
As diagrams:



second f



f *** g



f &&& g

Some more arrow combinators (3)

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```
second :: Arrow a => a b c -> a (d,b) (d,c)
second f = arr swap >>> first f >>> arr swap
swap (x,y) = (y,x)
```

Some more arrow combinators (3)

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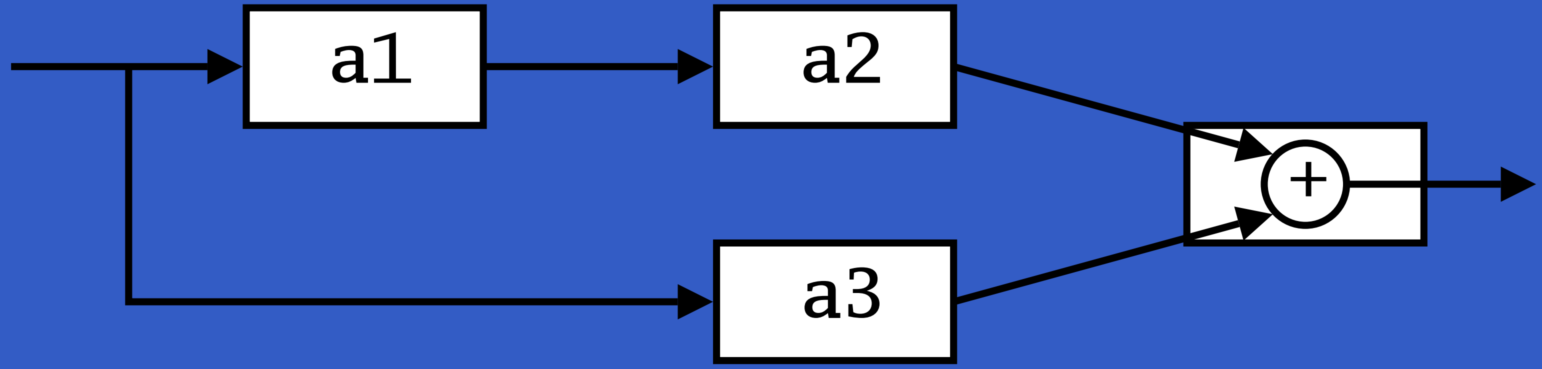
(&&&) :: Arrow a => a b c -> a b d -> a b (c,d)
f &&& g = arr (\x->(x,x)) >>> (f *** g)
```

Exercise 3

Describe the following circuit using arrow combinators:




```
a1, a2, a3 :: A Double Double
```

Exercise 3: One solution

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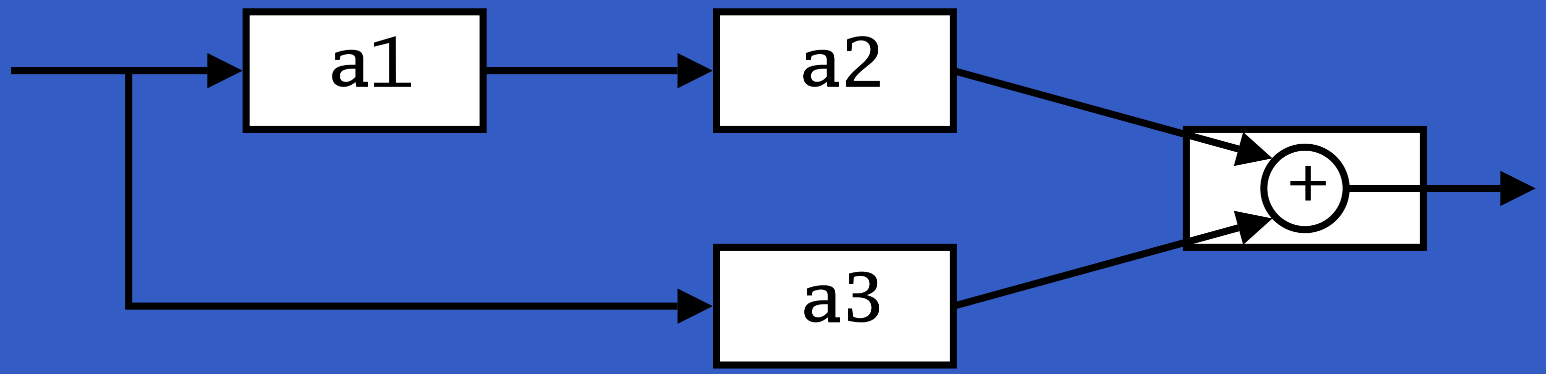


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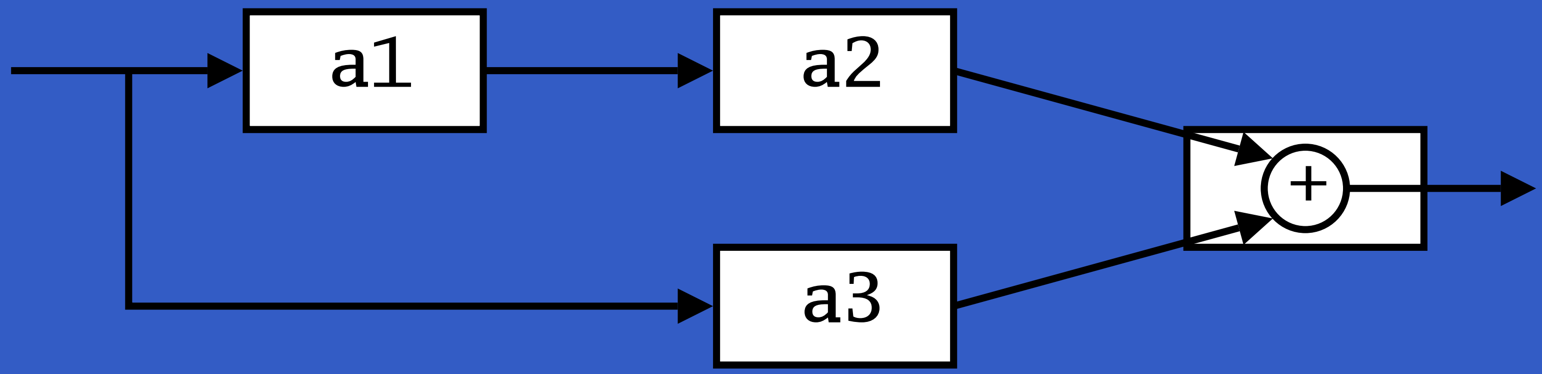


```
a1, a2, a3 :: A Double Double

circuit_v1 :: A Double Double
circuit_v1 = (a1 &&& arr id)
             >>> (a2 *** a3)
             >>> arr (uncurry (+))
```

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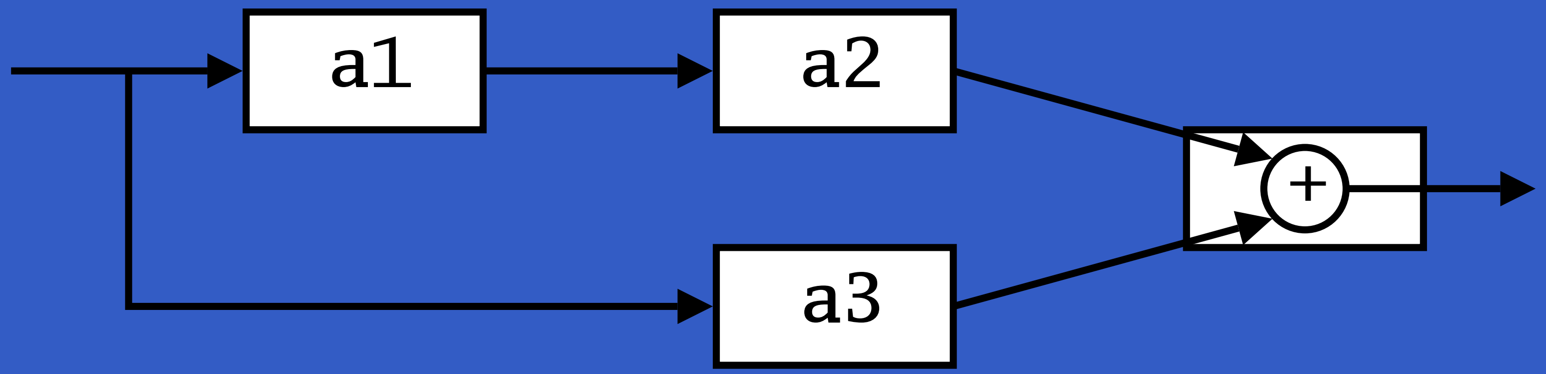


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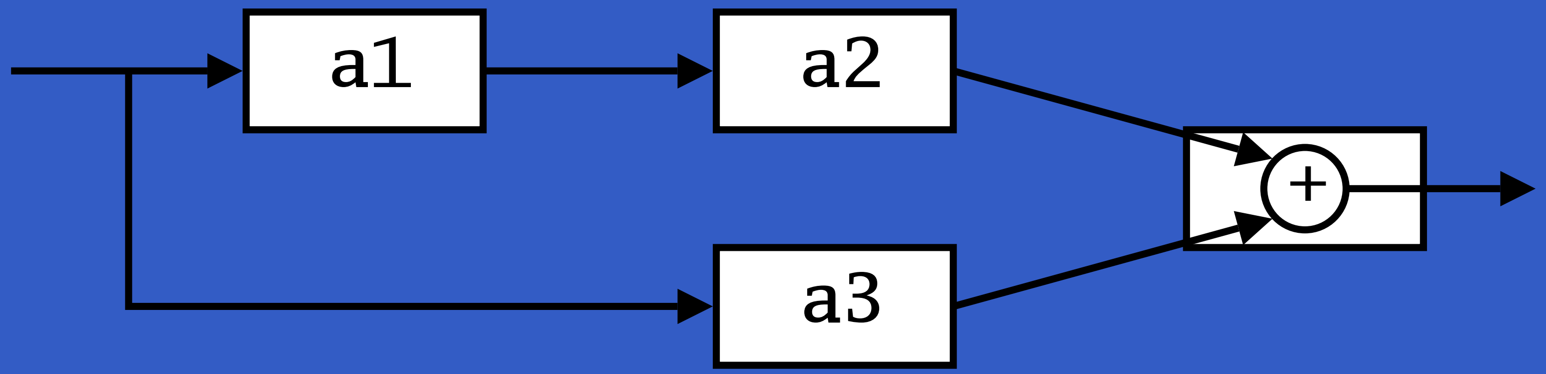


```
a1, a2, a3 :: A Double Double

circuit_v2 :: A Double Double
circuit_v2 = arr (\x -> (x,x))
             >>> first a1
             >>> (a2 *** a3)
             >>> arr (uncurry (+))
```

Note on the definition of (***) (1)

Are the following two definitions of (***) equivalent?

- `f *** g = first f >>> second g`
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Note on the definition of (***) (1)

Are the following two definitions of (`***`) equivalent?

- `f *** g = first f >>> second g`
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No, in general

$$\texttt{first}\ f \ \texttt{>>>}\ \texttt{second}\ g \neq \texttt{second}\ g\ \texttt{>>>}\ \texttt{first}\ f$$

since the ***order*** of the two possibly effectful computations f and g are different.

Note on the definition of (***) (2)

Similarly

$$(f *** g) >>> (h *** k) \neq (f >>> h) *** (g >>> k)$$

since the order of f and g differs.

Note on the definition of (***) (2)

Similarly

$$(f \; \texttt{***} \; g) \ggg (h \; \texttt{***} \; k) \;\neq\; (f \ggg h) \; \texttt{***} \; (g \ggg k)$$

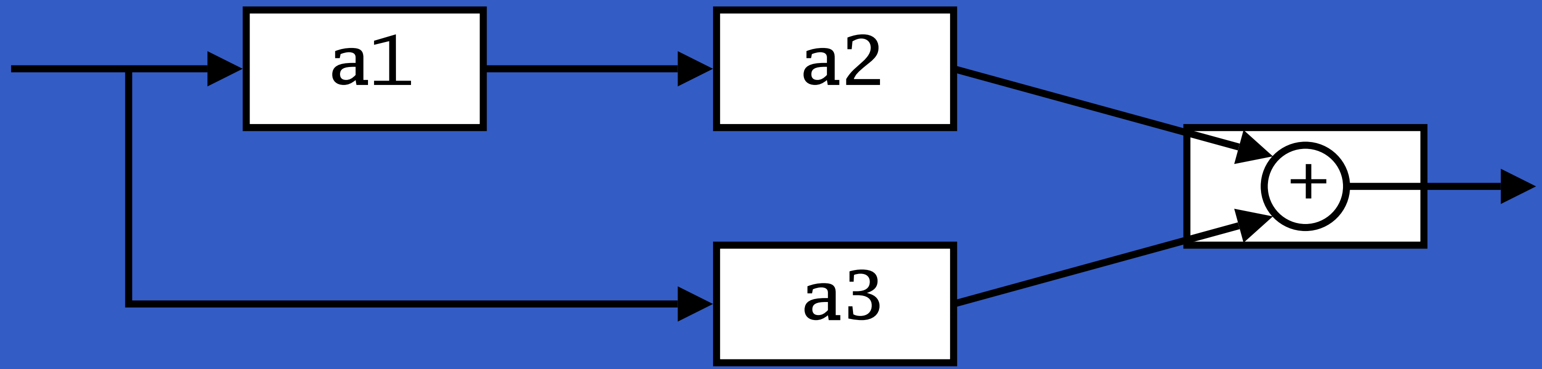
since the order of f and g differs.

However, the following ***is*** true (an additional law):

$$\begin{aligned} & \texttt{first}\, f \ggg \texttt{second}\,(\texttt{arr}\; g) \\ = \;\; & \texttt{second}\,(\texttt{arr}\; g) \ggg \texttt{first}\, f \end{aligned}$$

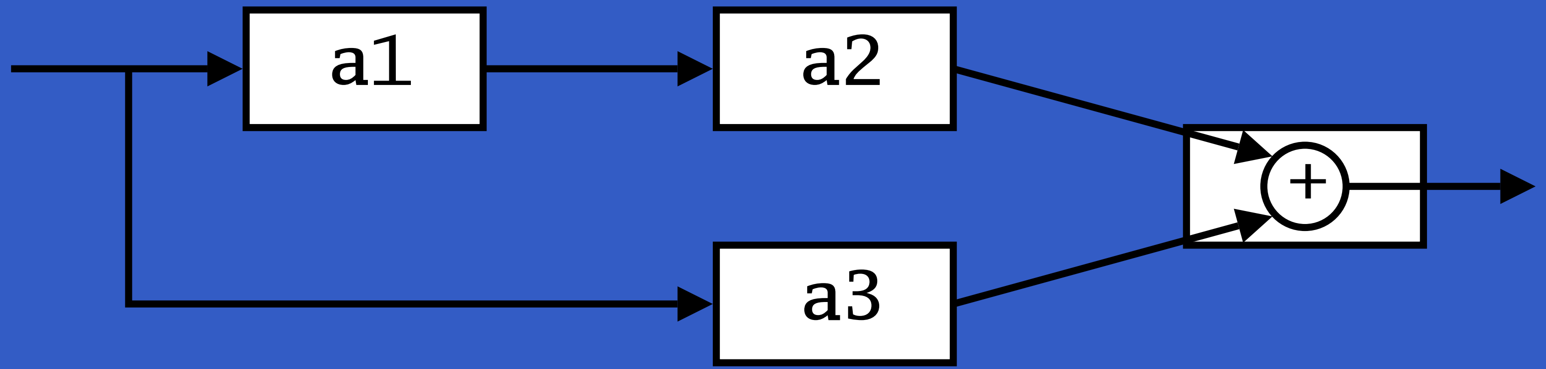
However, for certain ***arrow instances*** equalites like the ones above do hold.

Yet an attempt at exercise 3




```
circuit_v3 :: A Double Double
circuit_v3 = (a1 &&& a3)
             >>> first a2
             >>> arr (uncurry (+))
```

Yet an attempt at exercise 3




```
circuit_v3 :: A Double Double
circuit_v3 = (a1 &&& a3)
               >>> first a2
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```

Exercise 4: Are `circuit_v1`, `circuit_v2`, and `circuit_v3` all equivalent?

The arrow do notation (1)

Ross Paterson's do-notation for arrows supports ***pointed*** arrow programming. Only ***syntactic sugar***.

proc pat -> do [rec]

pat_1 <- $sfexp_1$ -< exp_1

pat_2 <- $sfexp_2$ -< exp_2

...

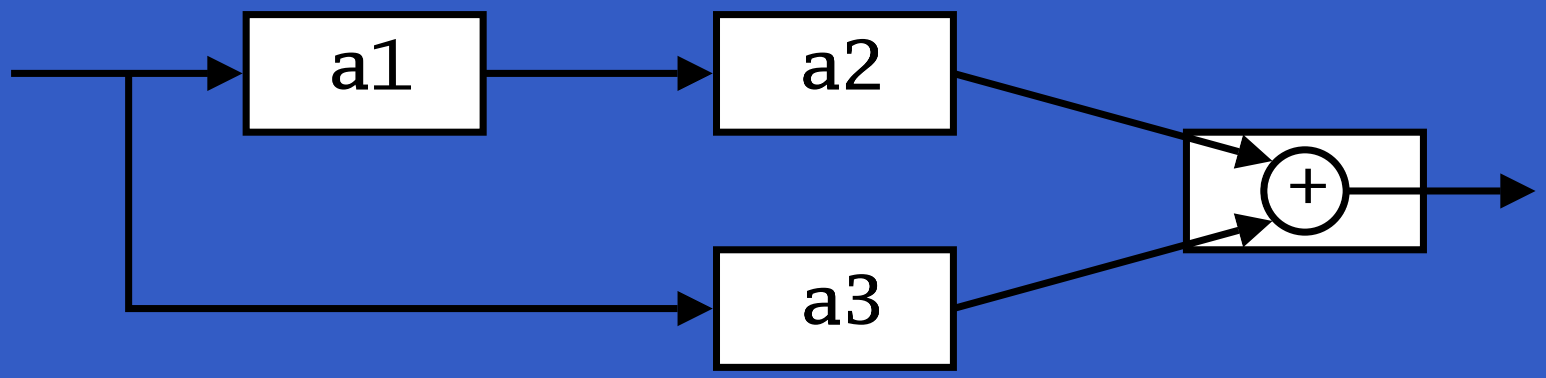
pat_n <- $sfexp_n$ -< exp_n

returnA -< exp

Also: let pat = exp $\equiv$ pat <- arr id -< exp

The arrow do notation (2)

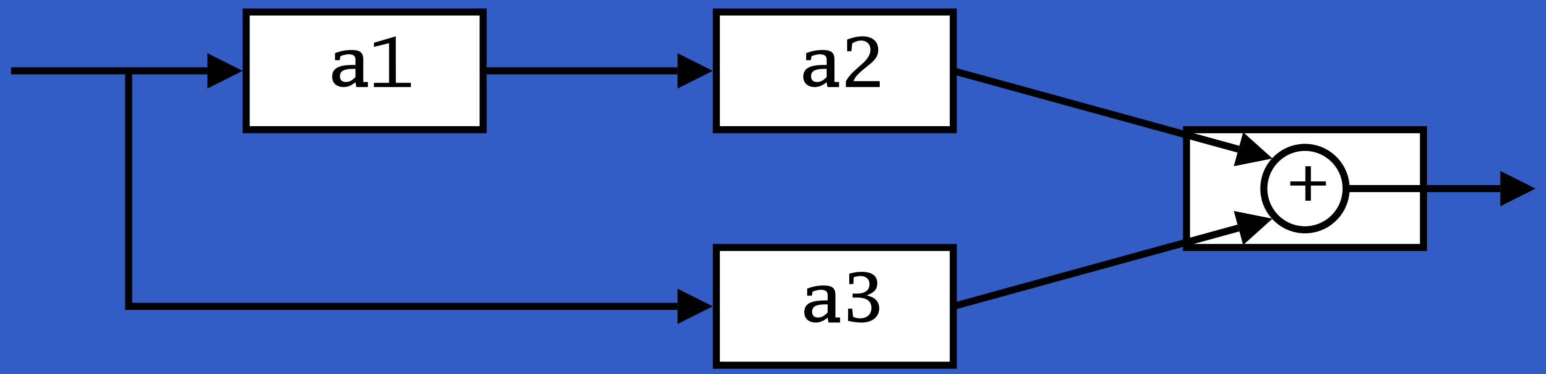
Let us redo exercise 3 using this notation:




```
circuit_v4 :: A Double Double
circuit_v4 = proc x -> do
    y1 <- a1 -< x
    y2 <- a2 -< y1
    y3 <- a3 -< x
    returnA -< y2 + y3
```

The arrow do notation (3)

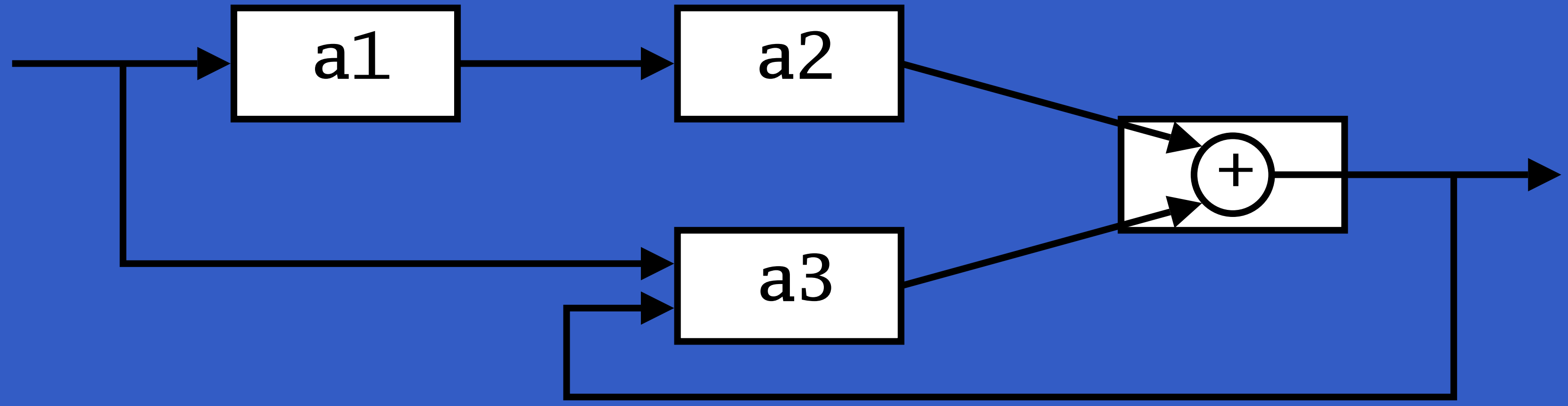
We can also mix and match:




```
circuit_v5 :: A Double Double
circuit_v5 = proc x -> do
    y2 <- a2 <<< a1 -< x
    y3 <- a3        -< x
    returnA -< y2 + y3
```

The arrow do notation (4)

Recursive networks: do-notation:




```
a1, a2 :: A Double Double
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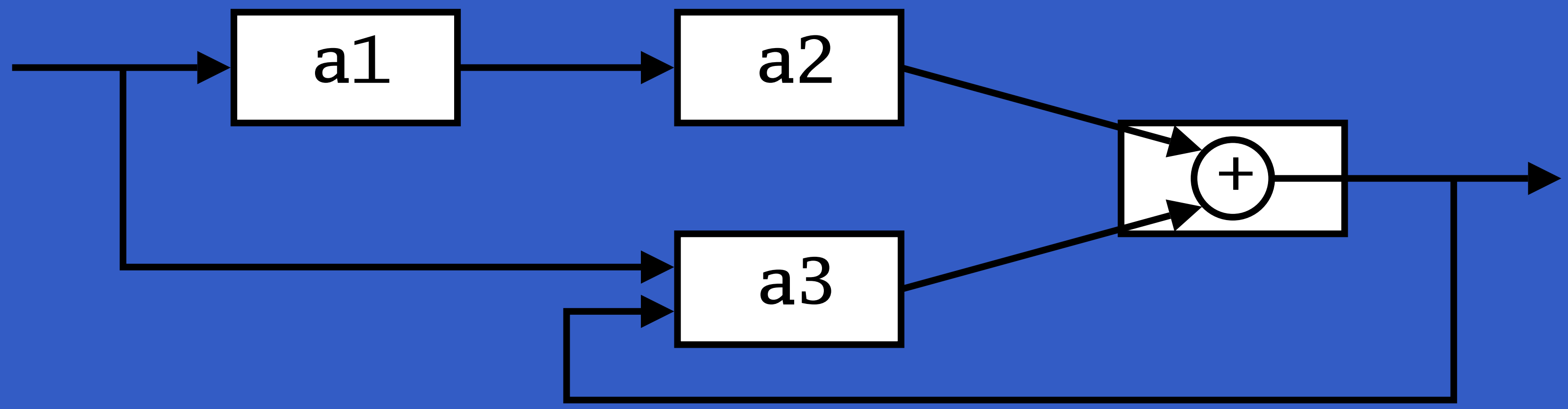
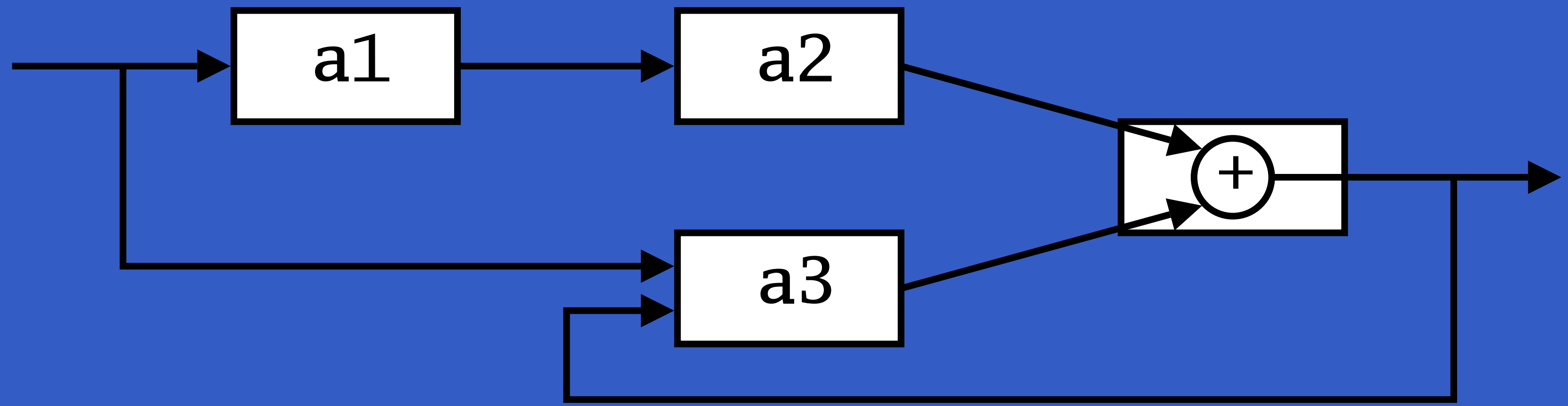



```
a1, a2 :: A Double Double
a3 :: A (Double,Double) Double
```

Exercise 5: Describe this using only the arrow combinators.

The arrow do notation (5)




```
circuit = proc x -> do
    rec
        y1 <- a1 -< x
        y2 <- a2 -< y1
        y3 <- a3 -< (x, y)
        let y = y2 + y3
    returnA -< y
```

Arrows and Monads (1)

Arrows generalize monads: for every monad type there is an arrow, the ***Kleisli category*** for the monad:

```
newtype Kleisli m a b = K (a -> m b)

instance Monad m => Arrow (Kleisli m) where
    arr f       = K (\b -> return (f b))
    K f >>> K g = K (\b -> f b >>= g)
```

Arrows and Monads (2)

But not every arrow is a monad. However, arrows that support an additional `apply` operation ***are*** effectively monads:

```
apply :: Arrow a => a (a b c, b) c
```

Exercise 6: Verify that

```
newtype M b = M (A () b)
```

is a monad if `A` is an arrow supporting `apply`; i.e., define `return` and `bind` in terms of the arrow operations (and verify that the monad laws hold).

An application: FRP

Functional Reactive Programming (FRP):

- Paradigm for ***reactive programming*** in a functional setting:
  - Input arrives ***incrementally*** while system is running.
  - Output is generated in response to input in an interleaved and ***timely*** fashion.

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- Originated from Functional Reactive Animation (Fran) (Elliott & Hudak).

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- Originated from Functional Reactive Animation (Fran) (Elliott & Hudak).
- Has evolved in a number of directions and into different concrete implementations.

Yampa

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- The most recent Yale FRP implementation.

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- ***Embedding*** in Haskell (a Haskell library).

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- Discrete-time signals modelled by continuous-time signals and an option type.

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- The most recent Yale FRP implementation.
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- ***Arrows*** used as the basic structuring framework.
- ***Continuous time***.
- Discrete-time signals modelled by continuous-time signals and an option type.
- Advanced ***switching constructs*** allows for highly dynamic system structure.

Related languages

FRP related to:

- Synchronous languages, like Esterel, Lucid Synchrone.
- Modeling languages, like Simulink.

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Distinguishing features of FRP:

- First class reactive components.
- Allows highly dynamic system structure.
- Supports hybrid (mixed continuous and discrete) systems.

FRP applications

Some domains where FRP has been used:

- Graphical Animation (Fran: Elliott, Hudak)
- Robotics (Frob: Peterson, Hager, Hudak, Elliott, Pembeci, Nilsson)
- Vision (FVision: Peterson, Hudak, Reid, Hager)
- GUIs (Fruit: Courtney)
- Hybrid modeling (Nilsson, Hudak, Peterson)

Yampa?

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Another
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???

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???

No . . .

Yampa?

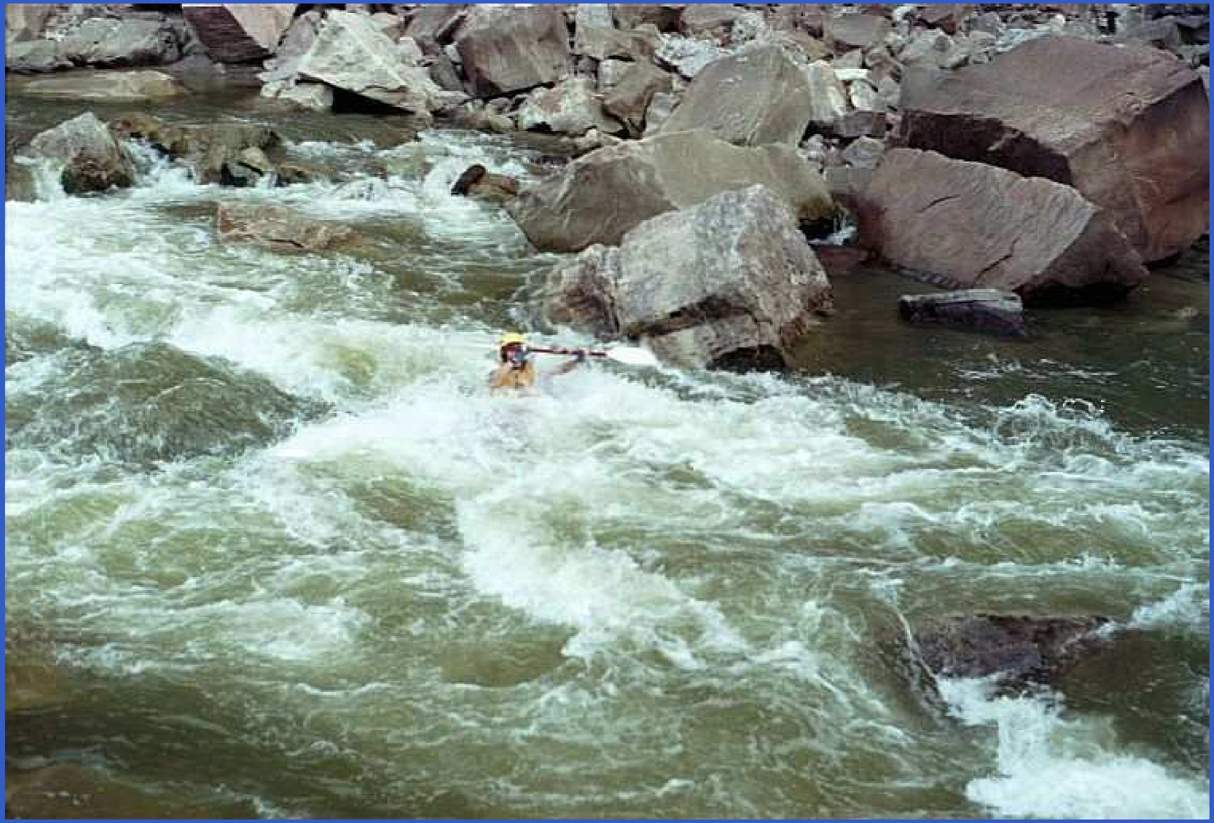
Yampa is a river . . .

Yampa?

. . . with long calmly flowing sections . . .

Yampa?

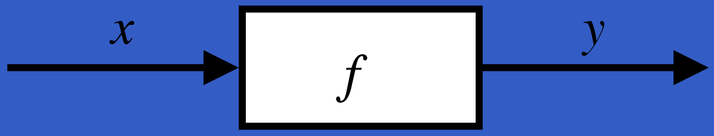
. . . and abrupt whitewater transitions in between.



A good metaphor for hybrid systems!

Signal functions

Key concept: ***functions on signals***.



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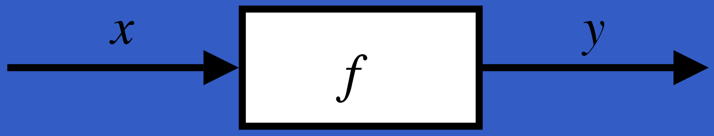

Intuition:

$$\texttt{Signal}\ \alpha \approx \texttt{Time}{\rightarrow}\alpha$$
$$x :: \texttt{Signal T1}$$
$$y :: \texttt{Signal T2}$$
$$\texttt{SF}\ \alpha\ \beta \approx \texttt{Signal}\ \alpha\ {\rightarrow}\texttt{Signal}\ \beta$$
$$f :: \texttt{SF T1 T2}$$

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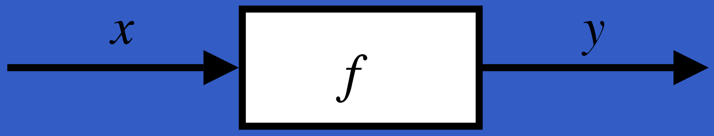


Intuition:

$$\texttt{Signal}\ \alpha \approx \texttt{Time}{\rightarrow}\alpha$$
$$x :: \texttt{Signal T1}$$
$$y :: \texttt{Signal T2}$$
$$\texttt{SF}\ \alpha\ \beta \approx \texttt{Signal}\ \alpha \rightarrow \texttt{Signal}\ \beta$$
$$f :: \texttt{SF T1 T2}$$

Additionally: ***causality*** requirement.

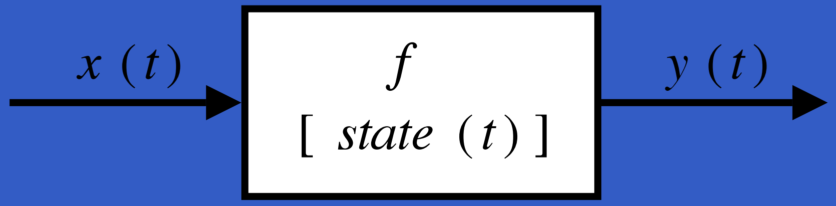
Signal functions and state

Alternative view:

Signal functions and state

Alternative view:

Signal functions can encapsulate ***state***.

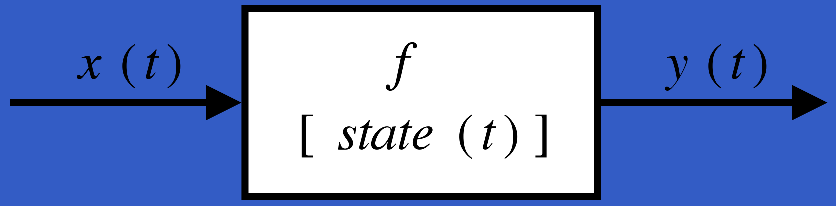


$state(t)$ summarizes input history $x(t')$, $t' \in [0, t]$.

Signal functions and state

Alternative view:

Signal functions can encapsulate ***state***.



$state(t)$ summarizes input history $x(t')$, $t' \in [0, t]$.

Functions on signals are either:

- ***Stateful***: $y(t)$ depends on $x(t)$ and $state(t)$
- ***Stateless***: $y(t)$ depends only on $x(t)$

Yampa and Arrows

SF is an arrow. Signal function instances of core combinators:

- `arr :: (a -> b) -> SF a b`
- `>>> :: SF a b -> SF b c -> SF a c`
- `first :: SF a b -> SF (a,c) (b,c)`
- `loop :: SF (a,c) (b,c) -> SF a b`

But `apply` has no useful meaning. Hence SF is ***not*** a monad.

Some further basic signal functions

- ```
  identity :: SF a a
  identity = arr id
  ```
  ```

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- ```
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- ```
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  integral :: VectorSpace a s=>SF a a
  ```
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- ```
  time :: SF a Time
  time = constant 1.0 >>> integral
  ```

Some further basic signal functions

- ```
  identity :: SF a a
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- ```
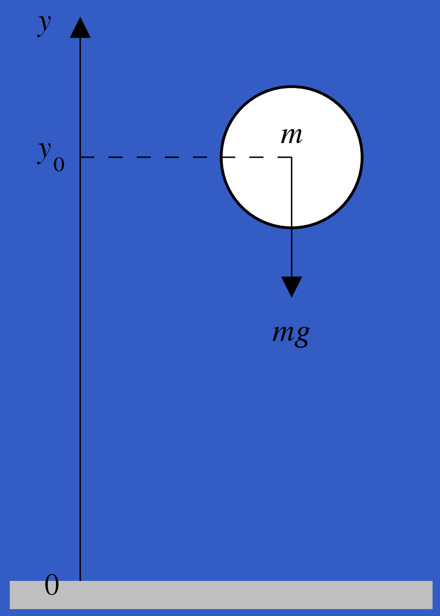
  constant :: b -> SF a b
  constant b = arr (const b)
  ```
- ```
  integral :: VectorSpace a s=>SF a a
  ```
- ```
  time :: SF a Time
  time = constant 1.0 >>> integral
  ```
- ```
  (^<<) :: (b->c) -> SF a b -> SF a c
  f (^<<) sf = sf >>> arr f
  ```
  ```

Example: A bouncing ball



$$
\begin{aligned}
y &= y_0 + \int v \, \mathrm{d}t \\
v &= v_0 + \int -9.81
\end{aligned}
$$

On impact:

$$v = -v(t-)$$

(fully elastic collision)

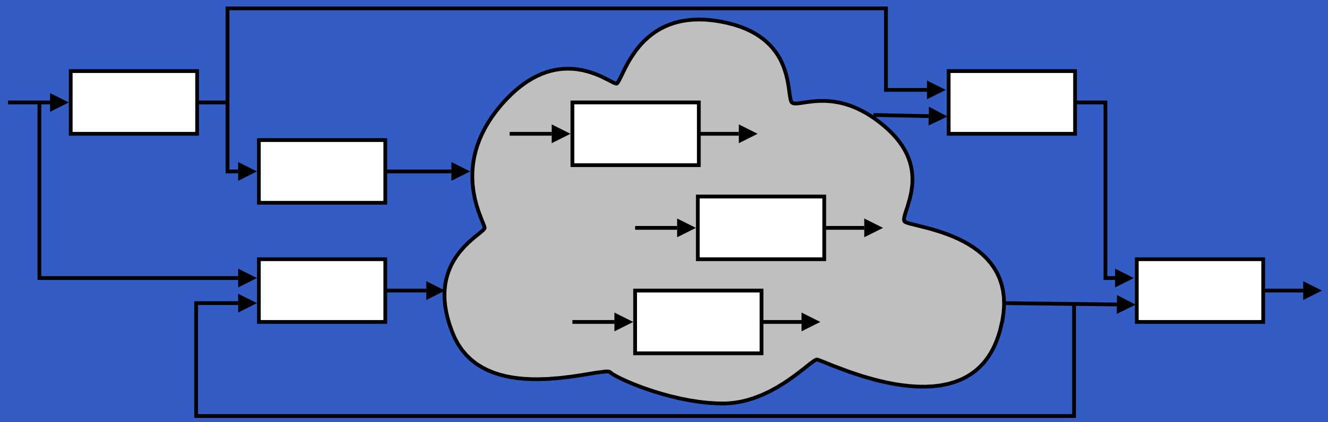
Part of a model of the bouncing ball

Free-falling ball:

```
type Pos = Double
type Vel = Double

fallingBall ::
    Pos -> Vel -> SF () (Pos, Vel)
fallingBall y0 v0 = proc () -> do
    v <- (v0 +) ^<< integral -< -9.81
    y <- (y0 +) ^<< integral -< v
    returnA -< (y, v)
```

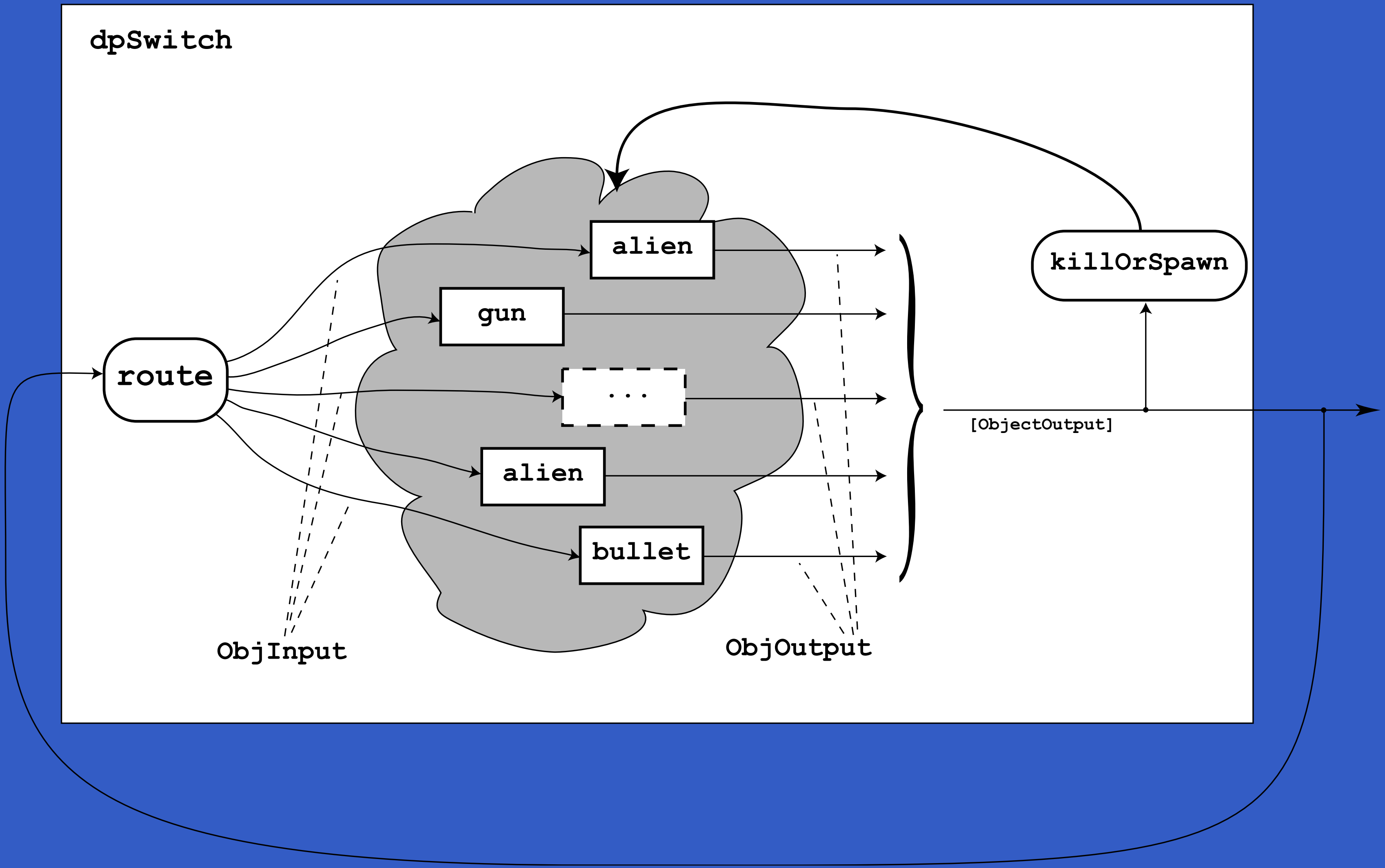
Dynamic system structure

Switching allows the structure of the system to evolve over time:



Example: Space Invaders

Overall game structure



Reading

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