MGS 2012: FUN Lecture 3

Monads

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A Blessing and a Curse

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Conundrum

“Shall I be pure or impure?” (Wadler, 1992)

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  - facilitates understanding and reasoning
  - makes lazy evaluation viable
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- Absence of effects:
  - facilitates understanding and reasoning
  - makes lazy evaluation viable
  - allows choice of reduction order, e.g. parallel
  - enhances modularity and reuse.
- Disciplined use of effects (state, exceptions, . . .) can:
  - help making code concise
  - facilitate maintenance
  - improve the efficiency.

Example: A Compiler Fragment (1)

Identification is the task of relating each applied identifier occurrence to its declaration or definition:

```
public class C {
    int x, n;
    void set(int n) { x = n; }
}
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In the body of `set`, the one applied occurrence of

- `x` refers to the ***instance variable*** `x`
- `n` refers to the ***argument*** `n`.

Example: A Compiler Fragment (2)

Consider an AST `Exp` for a simple expression language. `Exp` is a parameterized type: the ***type parameter a*** allows variables to be annotated with an attribute of type **a**.

```
data Exp a
    = LitInt    Int
    | Var       Id a
    | UnOpApp   UnOp (Exp a)
    | BinOpApp  BinOp (Exp a) (Exp a)
    | If        (Exp a) (Exp a) (Exp a)
    | Let       [(Id, Type, Exp a)] (Exp a)
```

Example: A Compiler Fragment (3)

Example: The following code fragment

```
let int x = 7 in x + 35
```

would be represented like this (before identification):

```
Let [("x", IntType, LitInt 7)]
    (BinOpApp Plus
              (Var "x" ())
              (LitInt 35))
```

Example: A Compiler Fragment (4)

Goals of the ***identification*** phase:

- Annotate each applied identifier occurrence with attributes of the corresponding variable declaration.
  I.e., map unannotated AST **`Exp ()`** to annotated AST **`Exp Attr`**.
- Report conflicting variable definitions and undefined variables.

```
identification ::
    Exp () -> (Exp Attr,   [ErrorMsg])
```

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Example: A Compiler Fragment (5)

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Example: Before Identification

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Let [("x", IntType, LitInt 7)]
    (BinOpApp Plus
              (Var "x" ())
              (LitInt 35))
```

After identification:

```
Let [("x", IntType, LitInt 7)]
    (BinOpApp Plus
              (Var "x" (1, IntType))
              (LitInt 35))
```

Example: A Compiler Fragment (6)

`enterVar` inserts a variable at the given scope level and of the given type into an environment.

- Check that no variable with same name has been defined at the same scope level.
- If not, the new variable is entered, and the ***resulting environment*** is returned.
- Otherwise an ***error message*** is returned.

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enterVar :: Id -> Int -> Type -> Env
               -> Either   Env   ErrorMsg
```

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enterVar :: Id -> Int -> Type -> Env
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```

Example: A Compiler Fragment (7)

Functions that do the real work:

```
identAux ::
    Int -> Env -> Exp ()
    -> (Exp Attr, [ErrorMsg])

identDefs ::
    Int -> Env -> [(Id, Type, Exp ())]
    -> ([(Id, Type, Exp Attr)],
        Env,
        [ErrorMsg])
```

Example: A Compiler Fragment (8)

```
identDefs l env [] = ([], env, [])
identDefs l env ((i,t,e) : ds) =
  ((i,t,e') : ds', env'', ms1++ms2++ms3)
  where
    (e', ms1) = identAux l env e
    (env', ms2) =
         case enterVar i l t env of
            Left env' -> (env', [])
            Right m   -> (env,  [m])
    (ds', env'', ms3) =
       identDefs l env' ds
```

Example: A Compiler Fragment (9)

Error checking and collection of error messages arguably added a lot of ***clutter***. And repetitive. The ***core*** of the algorithm is this:

```
identDefs l env [] = ([], env)
identDefs l env ((i,t,e) : ds) =
  ((i,t,e') : ds', env'')
  where
    e'            = identAux l env e
    env'          = enterVar i l t env
    (ds', env'') = identDefs l env' ds
```

Errors are just a ***side effect***.

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- Key idea: ***Computational types***: an object of type $M\,A$ denotes a ***computation*** of an object of type A.
- ***Thus we shall be both pure and impure, whatever takes our fancy!***
- Monads originated in Category Theory.
- Adapted by
  - Moggi for structuring denotational semantics
  - Wadler for structuring functional programs

Answer to Conundrum: Monads (2)

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- support changes to the effect structure with minimal impact on the overall program structure;

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Monads

- promote ***disciplined*** use of effects since the type reflects which effects can occur;
- allow great flexibility in tailoring the effect structure to precise needs;
- support changes to the effect structure with minimal impact on the overall program structure;
- allow integration into a pure setting of ***real*** effects such as
  - I/O
  - mutable state.

This Lecture

Pragmatic introduction to monads:

- Effectful computations
- Identifying a common pattern
- Monads as a ***design pattern***

Example 1: A Simple Evaluator

```
data Exp = Lit Integer
         | Add Exp Exp
         | Sub Exp Exp
         | Mul Exp Exp
         | Div Exp Exp

eval :: Exp -> Integer
eval (Lit n)      = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Sub e1 e2) = eval e1 - eval e2
eval (Mul e1 e2) = eval e1 * eval e2
eval (Div e1 e2) = eval e1 `div` eval e2
```

Making the Evaluator Safe (1)

```
data Maybe a = Nothing | Just a

safeEval :: Exp -> Maybe Integer
safeEval (Lit n) = Just n
safeEval (Add e1 e2) =
    case safeEval e1 of
        Nothing -> Nothing
        Just n1 ->
            case safeEval e2 of
                Nothing -> Nothing
                Just n2 -> Just (n1 + n2)
```

Making the Evaluator Safe (2)

```
safeEval (Sub e1 e2) =
    case safeEval e1 of
        Nothing -> Nothing
        Just n1 ->
            case safeEval e2 of
                Nothing -> Nothing
                Just n2 -> Just (n1 - n2)
```

Making the Evaluator Safe (3)

```
safeEval (Mul e1 e2) =
    case safeEval e1 of
        Nothing -> Nothing
        Just n1 ->
            case safeEval e2 of
                Nothing -> Nothing
                Just n2 -> Just (n1 * n2)
```

Making the Evaluator Safe (4)

```
safeEval (Div e1 e2) =
    case safeEval e1 of
        Nothing -> Nothing
        Just n1 ->
            case safeEval e2 of
                Nothing -> Nothing
                Just n2 ->
                    if n2 == 0
                    then Nothing
                    else Just (n1 `div` n2)
```

Any Common Pattern?

Clearly a lot of code duplication!
Can we factor out a common pattern?

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We note:

- ***Sequencing*** of evaluations (or ***computations***).

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We note:

- ***Sequencing*** of evaluations (or ***computations***).
- If one evaluation fails, fail overall.
- Otherwise, make result available to following evaluations.

Sequencing Evaluations

```
evalSeq :: Maybe Integer
           -> (Integer -> Maybe Integer)
           -> Maybe Integer
evalSeq ma f =
    case ma of
        Nothing -> Nothing
        Just a  -> f a
```

Exercise 1: Refactoring `safeEval`

Rewrite `safeEval`, case `Add`, using `evalSeq`:

```
safeEval (Add e1 e2) =
    case safeEval e1 of
        Nothing -> Nothing
        Just n1 ->
            case safeEval e2 of
                Nothing -> Nothing
                Just n2 -> Just (n1 + n2)
evalSeq ma f =
    case ma of
        Nothing -> Nothing
        Just a  -> f a
```

Exercise 1: Solution

```
safeEval :: Exp -> Maybe Integer
safeEval (Add e1 e2) =
    evalSeq (safeEval e1)
            (\n1 -> evalSeq (safeEval e2)
                            (\n2 -> Just (n1+n2)
```

or

```
safeEval :: Exp -> Maybe Integer
safeEval (Add e1 e2) =
    safeEval e1 `evalSeq` (\n1 ->
    safeEval e2 `evalSeq` (\n2 ->
    Just (n1 + n2)))
```

Aside: Scope Rules of λ-abstractions

The scope rules of λ-abstractions are such that parentheses can be omitted:

```
safeEval :: Exp -> Maybe Integer
...
safeEval (Add e1 e2) =
    safeEval e1 `evalSeq` \n1 ->
    safeEval e2 `evalSeq` \n2 ->
    Just (n1 + n2)
...
```

Refactored Safe Evaluator (1)

```
safeEval :: Exp -> Maybe Integer
safeEval (Lit n) = Just n
safeEval (Add e1 e2) =
    safeEval e1 ‘evalSeq‘ \n1 ->
    safeEval e2 ‘evalSeq‘ \n2 ->
    Just (n1 + n2)
safeEval (Sub e1 e2) =
    safeEval e1 ‘evalSeq‘ \n1 ->
    safeEval e2 ‘evalSeq‘ \n2 ->
    Just (n1 - n2)
```

Refactored Safe Evaluator (2)

```
safeEval (Mul e1 e2) =
    safeEval e1 `evalSeq` \n1 ->
    safeEval e2 `evalSeq` \n2 ->
    Just (n1 * n2)
safeEval (Div e1 e2) =
    safeEval e1 `evalSeq` \n1 ->
    safeEval e2 `evalSeq` \n2 ->
    if n2 == 0
    then Nothing
    else Just (n1 `div` n2)
```

Inlining `evalSeq` (1)

```
safeEval (Add e1 e2) =
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safeEval (Add e1 e2) =
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```

=

```
safeEval (Add e1 e2) =
  case (safeEval e1) of
    Nothing -> Nothing
    Just a -> (\n1 -> safeEval e2 ...) a
```

Inlining `evalSeq` (2)

=

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Inlining `evalSeq` (3)

=

```
safeEval (Add e1 e2) =
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                 Nothing -> Nothing
                 Just n2 -> (Just n1 + n2)
```

Good excercise: verify the other cases.

`Maybe` Viewed as a Computation (1)

- Consider a value of type `Maybe a` as denoting a ***computation*** of a value of type `a` that ***may fail***.

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- I.e. ***failure is an effect***, implicitly affecting subsequent computations.

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- Consider a value of type `Maybe a` as denoting a ***computation*** of a value of type `a` that ***may fail***.
- When sequencing possibly failing computations, a natural choice is to fail overall once a subcomputation fails.
- I.e. ***failure is an effect***, implicitly affecting subsequent computations.
- Let's generalize and adopt names reflecting our intentions.

Maybe Viewed as a Computation (2)

Successful computation of a value:

```
mbReturn :: a -> Maybe a
mbReturn = Just
```

Sequencing of possibly failing computations:

```
mbSeq :: Maybe a -> (a -> Maybe b) -> Maybe b
mbSeq ma f =
    case ma of
        Nothing -> Nothing
        Just a  -> f a
```

Maybe Viewed as a Computation (3)

Failing computation:

```
mbFail :: Maybe a
mbFail = Nothing
```

The Safe Evaluator Revisited

```
safeEval :: Exp -> Maybe Integer
safeEval (Lit n) = mbReturn n
safeEval (Add e1 e2) =
    safeEval e1 ‘mbSeq‘ \n1 ->
    safeEval e2 ‘mbSeq‘ \n2 ->
    mbReturn (n1 + n2)
...
safeEval (Div e1 e2) =
    safeEval e1 ‘mbSeq‘ \n1 ->
    safeEval e2 ‘mbSeq‘ \n2 ->
    if n2 == 0 then mbFail
    else mbReturn (n1 ‘div‘ n2)))
```

Example 2: Numbering Trees

```
data Tree a = Leaf a | Node (Tree a) (Tree a)

numberTree :: Tree a -> Tree Int
numberTree t = fst (ntAux t 0)
    where
        ntAux :: Tree a -> Int -> (Tree Int,Int)
        ntAux (Leaf _)      n = (Leaf n, n+1)
        ntAux (Node t1 t2) n =
            let (t1', n')  = ntAux t1 n
            in let (t2', n'') = ntAux t2 n'
               in (Node t1' t2', n'')
```

Observations

- Repetitive pattern: threading a counter through a ***sequence*** of tree numbering ***computations***.

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Can we do better?

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- A ***stateful computation*** consumes a state and returns a result along with a possibly updated state.

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- The following type synonym captures this idea:

```
type S a = Int -> (a, Int)
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(Only `Int` state for the sake of simplicity.)

Stateful Computations (1)

- A ***stateful computation*** consumes a state and returns a result along with a possibly updated state.
- The following type synonym captures this idea:

```
type S a = Int -> (a, Int)
```

  (Only `Int` state for the sake of simplicity.)
- A value (function) of type `S a` can now be viewed as denoting a stateful computation computing a value of type `a`.

Stateful Computations (2)

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- I.e. ***state updating is an effect***, implicitly affecting subsequent computations.
  (As we would expect.)

Stateful Computations (3)

Computation of a value without changing the state (For ref.: `S a = Int -> (a, Int)`):

```
sReturn :: a -> S a
sReturn a = ???
```

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Sequencing of stateful computations:

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Sequencing of stateful computations:

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sSeq sa f = \n ->
    let (a, n') = sa n
    in f a n'
```

Stateful Computations (4)

Reading and incrementing the state
(For ref.: `S a = Int -> (a, Int)`):

```
sInc :: S Int
sInc = ???
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Stateful Computations (4)

Reading and incrementing the state
(For ref.: `S a = Int -> (a, Int)`):

```
sInc :: S Int
sInc = \n -> (n, n + 1)
```

Numbering trees revisited

```
data Tree a = Leaf a | Node (Tree a) (Tree a)

numberTree :: Tree a -> Tree Int
numberTree t = fst (ntAux t 0)
    where
        ntAux :: Tree a -> S (Tree Int)
        ntAux (Leaf _)  =
            sInc `sSeq` \n -> sReturn (Leaf n)
        ntAux (Node t1 t2) =
            ntAux t1 `sSeq` \t1' ->
            ntAux t2 `sSeq` \t2' ->
            sReturn (Node t1' t2')
```

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- In particular:
  - counter no longer manipulated directly
  - no longer any risk of “passing on” the wrong version of the counter!

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Comparison of the examples

- Both examples characterized by sequencing of effectful computations.
- Both examples could be neatly structured by introducing:
  - A type denoting computations
  - A function constructing an effect-free computation of a value
  - A function constructing a computation by sequencing computations
- In fact, both examples are instances of the general notion of a ***MONAD***.

Monads in Functional Programming

A monad is represented by:

- A type constructor

  ```
  M :: * -> *
  ```

  `M T` represents computations of a value of type `T`.

- A polymorphic function

  ```
  return :: a -> M a
  ```

  for lifting a value to a computation.

- A polymorphic function

  ```
  (>>=) :: M a -> (a -> M b) -> M b
  ```

  for sequencing computations.

Exercise 2: `join` and `fmap`

Equivalently, the notion of a monad can be captured through the following functions:

```
return :: a -> M a
join :: (M (M a)) -> M a
fmap :: (a -> b) -> (M a -> M b)
```

`join` "flattens" a computation, `fmap` "lifts" a function to map computations to computations.

Define `join` and `fmap` in terms of >>= (and `return`), and >>= in terms of `join` and `fmap`.

```
(>>=) :: M a -> (a -> M b) -> M b
```

Exercise 2: Solution

```
join :: M (M a) -> M a
join mm = mm >>= id
```

```
fmap :: (a -> b) -> M a -> M b
fmap f m = m >>= \a -> return (f a)
```

or:

```
fmap :: (a -> b) -> M a -> M b
fmap f m = m >>= return . f
```

```
(>>=) :: M a -> (a -> M b) -> M b
m >>= f = join (fmap f m)
```

Monad laws

Additionally, the following ***laws*** must be satisfied:

$$
\begin{aligned}
\texttt{return}\ x \mathbin{\texttt{>>=}} f &= f\ x \\
m \mathbin{\texttt{>>=}} \texttt{return} &= m \\
(m \mathbin{\texttt{>>=}} f) \mathbin{\texttt{>>=}} g &= m \mathbin{\texttt{>>=}} (\lambda x \rightarrow f\ x \mathbin{\texttt{>>=}} g)
\end{aligned}
$$

I.e., `return` is the right and left identity for >>=, and >>= is associative.

Monads in Category Theory (1)

The notion of a monad originated in Category Theory. There are several equivalent definitions (Benton, Hughes, Moggi 2000):

- ***Kleisli triple/triple in extension form:*** Most closely related to the >>= version:

  A ***Klesili triple*** over a category $\mathcal{C}$ is a triple $(T, \eta, \_^*)$, where $T : |\mathcal{C}| \to |\mathcal{C}|$, $\eta_A : A \to TA$ for $A \in |\mathcal{C}|$, $f^* : TA \to TB$ for $f : A \to TB$.

  (Additionally, some laws must be satisfied.)

Monads in Category Theory (2)

- ***Monad/triple in monoid form:*** More akin to the `join`/`fmap` version:

  A ***monad*** over a category $\mathcal{C}$ is a triple (T, η, μ), where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\eta : \mathrm{id}_{\mathcal{C}} \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ are natural transformations.

  (Additionally, some commuting diagrams must be satisfied.)

Reading

- Philip Wadler. The Essence of Functional Programming. *Proceedings of the 19th ACM Symposium on Principles of Programming Languages (POPL'92)*, 1992.
- Nick Benton, John Hughes, Eugenio Moggi. Monads and Effects. In *International Summer School on Applied Semantics 2000*, Caminha, Portugal, 2000.
- *All About Monads.* `http://www.haskell.org/haskellwiki/all_about_monads`