MGS 2012: FUN Lecture 3 Monads

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"everything is explicit."

Can add a lot of clutter, make it hard to maintain code

Conundrum

"Shall I be pure or impure?" (Wadler, 1992)

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- Absence of effects:
 - facilitates understanding and reasoning
 - makes lazy evaluation viable
 - allows choice of reduction order, e.g. parallel
 - enhances modularity and reuse.

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- Absence of effects:
 - facilitates understanding and reasoning
 - makes lazy evaluation viable
 - allows choice of reduction order, e.g. parallel
 - enhances modularity and reuse.
- Disciplined use of effects (state, exceptions,
 - ...) can:
 - help making code concise
 - facilitate maintenance
 - improve the efficiency.

Example: A Compiler Fragment (1)

Identification is the task of relating each applied identifier occurrence to its declaration or definition:

public class C {
 int x, n;
 void set(int n) { x = n; }
}

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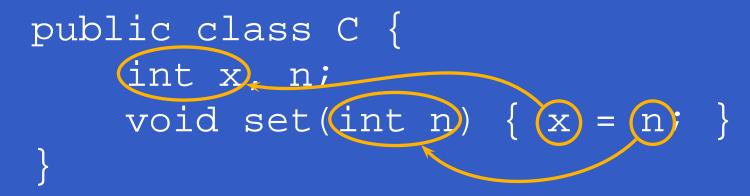
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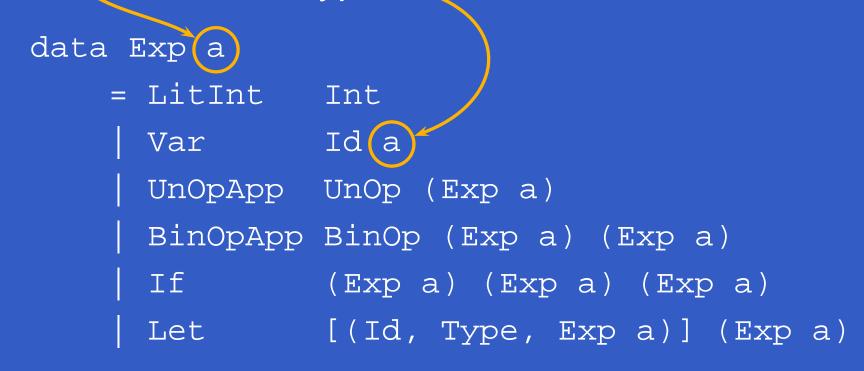


In the body of set, the one applied occurrence of

- x refers to the instance variable x
- n refers to the argument n.

Example: A Compiler Fragment (2)

Consider an AST Exp for a simple expression language. Exp is a parameterized type: the type parameter a allows variables to be annotated with an attribute of type a.



Example: A Compiler Fragment (3)

Example: The following code fragment

let int x = 7 in x + 35
would be represented like this (before
identification):

Let [("x", IntType, LitInt 7)] (BinOpApp Plus (Var "x" ()) (LitInt 35))

Example: A Compiler Fragment (4)

Goals of the *identification* phase:

 Annotate each applied identifier occurrence with attributes of the corresponding variable declaration.

I.e., map unannotated AST Exp () to annotated AST Exp Attr.

 Report conflicting variable definitions and undefined variables.

identification :: Exp () -> (Exp Attr, [ErrorMsg])

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Example: Before Identification

Let [("x", IntType, LitInt 7)] (BinOpApp Plus (Var "x" ()) (LitInt 35))

After identification:

Example: A Compiler Fragment (6)

enterVar inserts a variable at the given scope level and of the given type into an environment.

- Check that no variable with same name has been defined at the same scope level.
- If not, the new variable is entered, and the resulting environment is returned.
- Otherwise an error message is returned.

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Example: A Compiler Fragment (7)

Functions that do the real work:

```
identAux ::
    Int -> Env -> Exp ()
    -> (Exp Attr, [ErrorMsg])
identDefs ::
    Int -> Env -> [(Id, Type, Exp ())]
    -> ([(Id, Type, Exp Attr)],
        Env,
        [ErrorMsg])
```

Example: A Compiler Fragment (8)

identDefs l env [] = ([], env, []) identDefs l env ((i,t,e) : ds) = ((i,t,e') : ds', env'', ms1++ms2++ms3) where (e', ms1) = identAux l env e (env', ms2) =case enterVar i l t env of Left env' -> (env', []) Right $m \rightarrow (env, [m])$ (ds', env'', ms3) =identDefs l env' ds

Example: A Compiler Fragment (9)

Error checking and collection of error messages arguably added a lot of *clutter*. And repetitive. The *core* of the algorithm is this:

identDefs l env [] = ([], env)
identDefs l env ((i,t,e) : ds) =
 ((i,t,e') : ds', env'')

where

e' = identAux l env e
env' = enterVar i l t env
(ds', env'') = identDefs l env' ds
Errors are just a side effect.

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- Key idea: Computational types: an object of type MA denotes a computation of an object of type A.
- Thus we shall be both pure and impure, whatever takes our fancy!
- Monads originated in Category Theory.
- Adapted by
 - Moggi for structuring denotational semantics
 - Wadler for structuring functional programs

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- promote disciplined use of effects since the type reflects which effects can occur;
- allow great flexibility in tailoring the effect structure to precise needs;
- support changes to the effect structure with minimal impact on the overall program structure;
- allow integration into a pure setting of real effects such as
 - I/O
 - mutable state.

This Lecture

Pragmatic introduction to monads:

- Effectful computations
- Identifying a common pattern
- Monads as a design pattern

Example 1: A Simple Evaluator

data Exp = Lit Integer

Add Exp Exp

Sub Exp Exp

Mul Exp Exp

Div Exp Exp

eval :: Exp -> Integer eval (Lit n) = n eval (Add e1 e2) = eval e1 + eval e2 eval (Sub e1 e2) = eval e1 - eval e2 eval (Mul e1 e2) = eval e1 * eval e2 eval (Div e1 e2) = eval e1 * div * eval e2

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Making the Evaluator Safe (1)

data Maybe a = Nothing Just a

safeEval :: Exp -> Maybe Integer safeEval (Lit n) = Just n safeEval (Add e1 e2) = case safeEval e1 of Nothing -> Nothing Just n1 -> case safeEval e2 of Nothing -> Nothing Just $n2 \rightarrow Just (n1 + n2)$

Making the Evaluator Safe (2)

safeEval (Sub e1 e2) =
 case safeEval e1 of
 Nothing -> Nothing
 Just n1 ->
 case safeEval e2 of
 Nothing -> Nothing
 Just n2 -> Just (n1 - n2)

Making the Evaluator Safe (3)

safeEval (Mul e1 e2) =
 case safeEval e1 of
 Nothing -> Nothing
 Just n1 ->
 case safeEval e2 of
 Nothing -> Nothing
 Just n2 -> Just (n1 * n2)

Making the Evaluator Safe (4)

safeEval (Div e1 e2) = case safeEval e1 of Nothing -> Nothing Just n1 -> case safeEval e2 of Nothing -> Nothing Just n2 ->if n2 == 0then Nothing else Just (n1 'div' n2)

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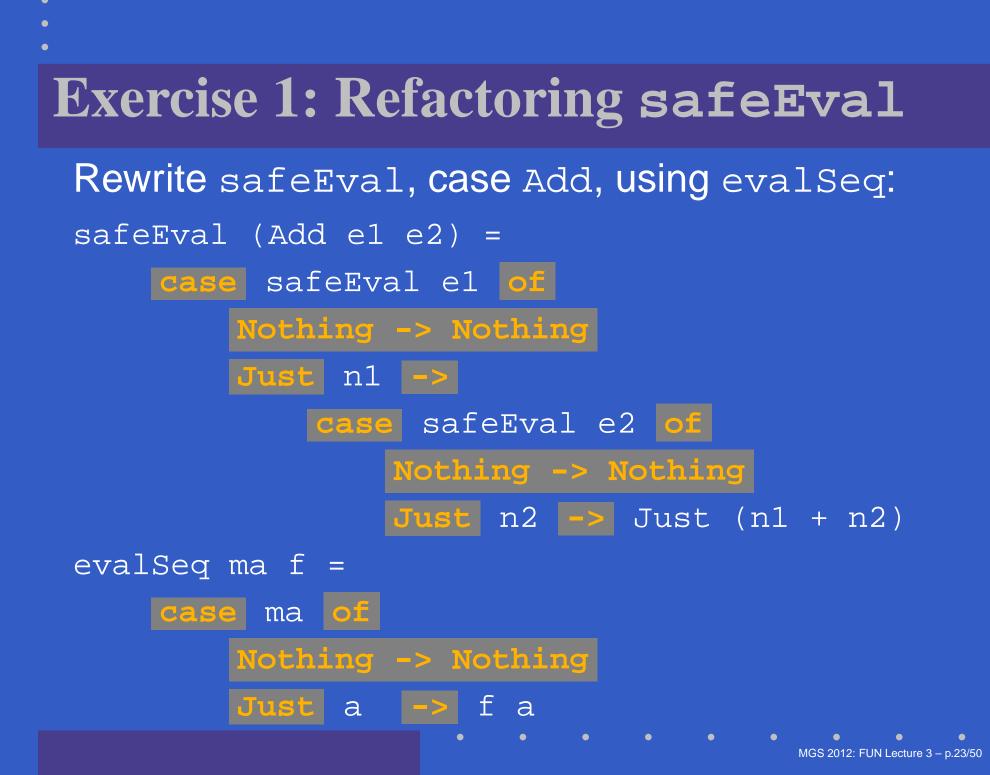
We note:

- Sequencing of evaluations (or computations).
- If one evaluation fails, fail overall.
- Otherwise, make result available to following evaluations.

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Sequencing Evaluations

evalSeq :: Maybe Integer -> (Integer -> Maybe Integer) -> Maybe Integer evalSeq ma f = case ma of Nothing -> Nothing Just a -> f a



Exercise 1: Solution

safeEval :: Exp -> Maybe Integer
safeEval (Add e1 e2) =
 evalSeq (safeEval e1)
 (\n1 -> evalSeq (safeEval e2)
 (\n2 -> Just (n1+n2))

or

safeEval :: Exp -> Maybe Integer
safeEval (Add e1 e2) =
 safeEval e1 'evalSeq' (\n1 ->
 safeEval e2 'evalSeq' (\n2 ->
 Just (n1 + n2)))

Aside: Scope Rules of λ -abstractions

The scope rules of λ-abstractions are such that parentheses can be omitted: safeEval :: Exp -> Maybe Integer ... safeEval (Add e1 e2) = safeEval e1 'evalSeq' \n1 -> safeEval e2 'evalSeq' \n2 -> Just (n1 + n2)

Refactored Safe Evaluator (1)

safeEval :: Exp -> Maybe Integer safeEval (Lit n) = Just n safeEval (Add e1 e2) = safeEval e1 'evalSeq' \n1 -> safeEval e2 'evalSeq' n2 ->Just (n1 + n2)safeEval (Sub e1 e2) = safeEval e1 'evalSeq' \n1 -> safeEval e2 'evalSeq' n2 ->Just (n1 - n2)

Refactored Safe Evaluator (2)

safeEval (Mul e1 e2) = safeEval e1 'evalSeq' \n1 -> safeEval e2 'evalSeq' n2 ->Just (n1 * n2) safeEval (Div e1 e2) = safeEval e1 'evalSeq' \n1 -> safeEval e2 'evalSeq' n2 ->if n2 == 0then Nothing else Just (n1 'div' n2)

Inlining evalseq (1)

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safeEval (Add e1 e2) =
  case (safeEval e1) of
   Nothing -> Nothing
   Just a -> (\n1 -> safeEval e2 ...) a
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Inlining evalSeq (2)

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   Nothing -> Nothing
    Just n1 -> case safeEval e2 of
                 Nothing -> Nothing
                 Just a -> (\n2 -> ...) a
```

Inlining evalseq (3)

safeEval (Add e1 e2) = case (safeEval e1) of Nothing -> Nothing Just n1 -> case safeEval e2 of Nothing -> Nothing Just n2 -> (Just n1 + n2) Good excercise: verify the other cases.

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- When sequencing possibly failing computations, a natural choice is to fail overall once a subcomputation fails.
- I.e. failure is an effect, implicitly affecting subsequent computations.
- Let's generalize and adopt names reflecting our intentions.

Successful computation of a value:

mbReturn :: a -> Maybe a

mbReturn = Just

Sequencing of possibly failing computations:

mbSeq :: Maybe a -> (a -> Maybe b) -> Maybe b
mbSeq ma f =

case ma of

Nothing -> Nothing

Just a -> f a

Failing computation:

mbFail :: Maybe a
mbFail = Nothing

The Safe Evaluator Revisited

safeEval :: Exp -> Maybe Integer
safeEval (Lit n) = mbReturn n
safeEval (Add e1 e2) =
 safeEval e1 `mbSeq` \n1 ->
 safeEval e2 `mbSeq` \n2 ->
 mbReturn (n1 + n2)

safeEval (Div e1 e2) =
 safeEval e1 `mbSeq` \n1 ->
 safeEval e2 `mbSeq` \n2 ->
 if n2 == 0 then mbFail
 else mbReturn (n1 `div` n2)))

• • •

Example 2: Numbering Trees

data Tree a = Leaf a | Node (Tree a) (Tree a)

numberTree :: Tree a -> Tree Int numberTree t = fst (ntAux t 0) where ntAux :: Tree a -> Int -> (Tree Int, Int) $ntAux (Leaf _) = n = (Leaf n, n+1)$ ntAux (Node t1 t2) n =let (t1', n') = ntAux t1 nin let (t2', n'') = ntAux t2 n'in (Node t1' t2', n'')

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 Repetitive pattern: threading a counter through a sequence of tree numbering computations.

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Can we do better?

Stateful Computations (1)

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type S a = Int -> (a, Int)
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- The following type synonym captures this idea:

type S a = Int -> (a, Int)
(Only Int state for the sake of simplicity.)

 A value (function) of type S a can now be viewed as denoting a stateful computation computing a value of type a.

Stateful Computations (2)

 When sequencing stateful computations, the resulting state should be passed on to the next computation.

- When sequencing stateful computations, the resulting state should be passed on to the next computation.
- I.e. state updating is an effect, implicitly affecting subsequent computations. (As we would expect.)

Computation of a value without changing the state (For ref.: S a = Int -> (a, Int)):

sReturn :: a -> S a

sReturn a = ???

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Sequencing of stateful computations:

sSeq :: S a -> (a -> S b) -> S b sSeq sa f = ???

Computation of a value without changing the state (For ref.: S a = Int -> (a, Int)):

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Sequencing of stateful computations:

sSeq :: S a -> (a -> S b) -> S b
sSeq sa f = \n ->
 let (a, n') = sa n
 in f a n'

Reading and incrementing the state (For ref.: S a = Int -> (a, Int)):

- sInc :: S Int
- sInc = ???

Reading and incrementing the state (For ref.: S a = Int -> (a, Int)):

sInc :: S Int sInc = $\langle n - \rangle$ (n, n + 1)

Numbering trees revisited

data Tree a = Leaf a Node (Tree a) (Tree a)

numberTree :: Tree a -> Tree Int numberTree t = fst (ntAux t 0) where ntAux :: Tree a -> S (Tree Int) ntAux (Leaf) = sInc 'sSeq' $\langle n - \rangle$ sReturn (Leaf n) ntAux (Node t1 t2) = ntAux t1 'sSeq' \t1' -> ntAux t2 'sSeq' $\langle t2' - \rangle$ sReturn (Node t1' t2')



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Observations

- The "plumbing" has been captured by the abstractions.
- In particular:
 - counter no longer manipulated directly
 - no longer any risk of "passing on" the wrong version of the counter!

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 In fact, both examples are instances of the general notion of a MONAD.

Monads in Functional Programming

A monad is represented by:

A type constructor

M :: * -> *

M T represents computations of a value of type **T**.

A polymorphic function

return :: a -> M a

for lifting a value to a computation.

A polymorphic function

(>>=) :: M a -> (a -> M b) -> M b for sequencing computations.

Exercise 2: join and fmap

Equivalently, the notion of a monad can be captured through the following functions:

return :: a -> M a
join :: (M (M a)) -> M a
fmap :: (a -> b) -> (M a -> M b)

join "flattens" a computation, fmap "lifts" a function to map computations to computations.

Define join and fmap in terms of >>= (and return), and >>= in terms of join and fmap.

(>>=) :: M a -> $(a \rightarrow M b) \rightarrow M b$

Exercise 2: Solution

join :: M (M a) -> M a
join mm = mm >>= id

fmap :: (a -> b) -> M a -> M b
fmap f m = m >>= \a -> return (f a)
OT:
fmap :: (a -> b) -> M a -> M b
fmap f m = m >>= return . f
(>>=) :: M a -> (a -> M b) -> M b
m >>= f = join (fmap f m)

Monad laws

Additionally, the following *laws* must be satisfied:

 $\operatorname{return} x \mathrel{>>=} f = f x$

m >>= return = m

 $(m >>= f) >>= g = m >>= (\lambda x \to f x >>= g)$

I.e., return is the right and left identity for >>=, and >>= is associative.

Monads in Category Theory (1)

The notion of a monad originated in Category Theory. There are several equivalent definitions (Benton, Hughes, Moggi 2000):

 Kleisli triple/triple in extension form: Most closely related to the >>= version:

A Klesili triple over a category C is a triple $(T, \eta, _^*)$, where $T : |C| \to |C|$, $\eta_A : A \to TA$ for $A \in |C|$, $f^* : TA \to TB$ for $f : A \to TB$.

(Additionally, some laws must be satisfied.)

Monads in Category Theory (2)

 Monad/triple in monoid form: More akin to the join/fmap version:

A monad over a category C is a triple (T, η, μ) , where $T : C \to C$ is a functor, $\eta : \mathrm{id}_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are natural transformations.

(Additionally, some commuting diagrams must be satisfied.)

Reading

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