Type-Based Structural Analysis for Modular Systems of Equations

FPL Away Days 2009

Henrik Nilsson

School of Computer Science

University of Nottingham

The Problem (1)

- A core aspect of equation-based modelling: modular description of models through composition of equation system fragments.
- Central questions:
  - *Does a system of equations have a (unique) solution?*
  - *Does an individual fragment "makes sense"?*
- Desirable to detect problematic fragments and compositions as early as possible.

The Problem (2)

- Consider:

$$x + y + z = 0 \qquad (1)$$

  - Does not have a (unique) solution.
  - Could be part of a system that does have a (unique) solution.
- The same holds for:

$$\begin{aligned} x - y + z &= 1 \\ z &= 2 \end{aligned} \qquad (2)$$

The Problem (3)

- Composing (1) and (2):

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 1 \\ z &= 2 \end{aligned} \qquad (3)$$

  Does have a solution.
- However, the following fragment is over-constrained:

$$\begin{aligned} x &= 1 \\ x &= 2 \end{aligned} \qquad (4)$$

The Problem (4)

- Cannot answer questions regarding solvability comprehensively before we have a ***complete*** system. ***Not very modular!***
- However, maybe violations of certain ***necessary*** conditions for solvability can be checked modularly?
  - ***Variable-equation balance***
  - ***Structural singularity***
- This talk: preliminary investigation into modular checking of structural singularity. (Paper at EOOLT'08)

Modular Systems of Equations (1)

Need notation. Observations:

- a system of equations specifies a ***relation*** among a set of variables
- specifically, our interest is relations on time-varying values or ***signals***
- an equation system fragment needs an ***interface*** to distinguish between local variables and variables used for composition with other fragments.

Modular Systems of Equations (2)

These ideas can be captured through a notion of ***typed signal relations***:

$$
\begin{aligned}
&\mathit{foo} :: \mathit{SR}\ (\mathit{Real}, \mathit{Real}, \mathit{Real})\\
&\mathit{foo} = \mathbf{sigrel}\ (x_1, x_2, x_3)\ \mathbf{where}\\
&\quad f_1\ x_1\ x_2\ x_3 = 0\\
&\quad f_2\ x_2\ x_3 = 0
\end{aligned}
$$

A signal relation is an ***encapsulated equation system fragment.***
Of course, the ideas are general and not limited to equations over signals.

Modular Systems of Equations (3)

Composition can by expressed through ***signal relation application***:

$$
\begin{aligned}
&\mathit{foo} \diamond (u, v, w)\\
&\mathit{foo} \diamond (w, u + x, v + y)
\end{aligned}
$$

yields

$$
\begin{aligned}
f_1\ u\ v\ w &= 0\\
f_2\ v\ w &= 0\\
f_1\ w\ (u + x)\ (v + y) &= 0\\
f_2\ (u + x)\ (v + y) &= 0
\end{aligned}
$$

Modular Systems of Equations (4)

Signal relations are ***first class entities*** at the functional layer. Offers way to parametrise the relations:

$$
\begin{aligned}
&\mathit{foo2} :: \mathit{Int} \rightarrow \mathit{Real} \rightarrow \mathit{SR}\ (\mathit{Real}, \mathit{Real}, \mathit{Real})\\
&\mathit{foo2}\ n\ k = \mathbf{sigrel}\ (x_1, x_2, x_3)\ \mathbf{where}\\
&\quad f_1\ n\ x_1\ x_2\ x_3 = 0\\
&\quad f_2\ x_2\ x_3 \qquad\ = k
\end{aligned}
$$

Example: Resistor Model

$$
\begin{aligned}
&\mathit{twoPin} :: \mathit{SR}\ (\mathit{Pin}, \mathit{Pin}, \mathit{Voltage})\\
&\mathit{twoPin} = \mathbf{sigrel}\ (p, n, u)\ \mathbf{where}\\
&\quad u = p.v - n.v\\
&\quad p.i + n.i = 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{resistor} :: \mathit{Resistance} \rightarrow \mathit{SR}\ (\mathit{Pin}, \mathit{Pin})\\
&\mathit{resistor}\ r = \mathbf{sigrel}\ (p, n)\ \mathbf{where}\\
&\quad \mathit{twoPin} \diamond (p, n, u)\\
&\quad r * p.i = u
\end{aligned}
$$

Tracking Variable/Equation Balance?

Equal number of equations and variables is a necessary condition for solvability. For a modular analysis, one might keep track of the ***balance*** in the signal relation ***type***:

$$\mathit{SR}\ (\ldots)\ \mathbf{n}$$

But very weak assurances:

$$
\begin{aligned}
f(x, y, z) &= 0\\
g(z) &= 0\\
h(z) &= 0
\end{aligned}
$$

A Possible Refinement (1)

A system of equations is ***structurally singular*** iff not possible to put variables and equations in a one-to-one correspondence such that each variable occurs in the equation it is related to.

Structural singlularities are typically indicative of problems.

A Possible Refinement (2)

Structural singularities can be discovered by studying the ***incidence matrix***:

Equations | Incidence Matrix

$$\begin{aligned} f_1(x,y,z) &= 0 \\ f_2(z) &= 0 \\ f_3(z) &= 0 \end{aligned} \qquad \begin{matrix} x & y & z \end{matrix} \atop \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

A Possible Refinement (3)

So maybe we can index signal relations with incidence matrices?

$$foo :: SR\ (Real, Real, Real) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} &foo = \mathbf{sigrel}\ (x_1, x_2, x_3)\ \mathbf{where} \\ &\quad f_1\ x_1\ x_2\ x_3 = 0 \\ &\quad f_2\ x_2\ x_3 \quad\ = 0 \end{aligned}$$

Structural Type (1)

- The ***Structural Type*** represents information about which variables occur in which equations.
- Denoted by an incidence matrix.
- Two interrelated instances:
  - Structural type of a ***system of equations***
  - Structural type of a ***signal relation***

Structural Type (2)

- Structural type for composition of signal relations: ***Straightforward***.
- The structural type of signal ***relation*** obtained by ***abstraction*** over the structural type of a system of equations: ***Less straightforward***.

Composition of Structural Types (1)

Recall

$$foo :: SR\ (Real, Real, Real) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Consider

$$foo \diamond (u, v, w)$$
$$foo \diamond (w, u + x, v + y)$$

in a context with five variables u, v, w, x, y.

Composition of Structural Types (2)

The structural type for the equations obtained by instantiating foo is simply obtained by Boolean matrix multiplication. For $foo \diamond (u, v, w)$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} =$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Composition of Structural Types (3)

For $foo \diamond (w, u + x, v + y)$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} =$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Composition of Structural Types (4)

Complete incidence matrix and corresponding equations:

$$\begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{ll} f_1\ u\ v\ w & = 0 \\ f_2\ v\ w & = 0 \\ f_1\ w\ (u + x)\ (v + y) & = 0 \\ f_2\ (u + x)\ (v + y) & = 0 \end{array}$$

Abstraction over Structural Types (1)

Now consider encapsulating the equations:

$$\begin{aligned} &bar = \textbf{sigrel}\ (u, y)\ \textbf{where} \\ &\quad foo \diamond (u, v, w) \\ &\quad foo \diamond (w, u + x, v + y) \end{aligned}$$

The equations of the body of bar needs to be partitioned into

- ***Local Equations***: equations used to solve for the local variables
- ***Interface Equations***: equations contributed to the outside

Abstraction over Structural Types (2)

How to partition?

- ***A priori local equations***: equations over local variables only.
- ***A priori interface equations***: equations over interface variables only.
- ***Mixed equations***: equations over local and interface variables.

Note: too few or too many local equations gives an opportunity to catch ***locally underdetermined*** or ***overdeteremined*** systems of equations.

Abstraction over Structural Types (3)

In our case:

- We have 1 a priori local equation, 3 mixed equations
- We need to choose 3 local equations and 1 interface equation
- Consequently, ***3*** possibilities, yielding the following possible structural types for bar:

$$\begin{matrix} u & y \\ \end{matrix}\quad \begin{matrix} u & y \end{matrix}\quad \begin{matrix} u & y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\ \begin{pmatrix} 1 & 1 \end{pmatrix}\ \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Abstraction over Structural Types (4)

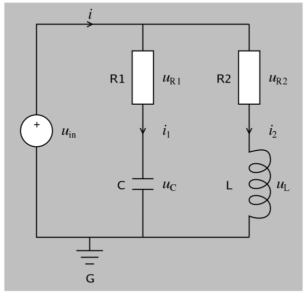
The two last possibilities are equivalent. But still leaves two distinct possibilities. How to choose?

- Assume the choice is free
- Note that a type with more variable occurrences is “better” as it gives more freedom when pairing equations and variables. Thus discard choices that are subsumed by better choices.
- As a last resort, approximate.

Details in the EOOLT’08 paper.

Also in the Paper

- A more realistic modelling example:



- Structural types for components of this model
- Examples of errors caught by the proposed method, but that would not have been found by just counting equations and variables.

Problems

- Structural types not at all intuitive.
- The matrix notation is potentially cumbersome.
- User would likely often have to provide declarations of structural type explicitly.
- Type-checking is (currently) expensive.
- Sensible meta theory?

Questions

- How much do structural types buy over variable-equation balance in practice? Worth the complexity?
- Is there some sensible middle ground between structural types and variable-equation balance that provides most of the benefits of structural types, but in a simpler way?