CATEGORY THEORY
MIDLANDS GRADUATE SCHOOL 2023

EXERCISE 1 (2 APRIL)

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Reminder

Definition 1 (category)**.** *A* category $\mathcal{C}$ *consists of:*

- *a collection* $\mathcal{C}_0$ *of objects, written* $X, Y, Z, \ldots$
- *for any two objects* X *and* Y*, a collection* $\mathcal{C}(X, Y)$ *of morphisms, written* $f, g, h, \ldots$
- *a composition operation: for* $f \in \mathcal{C}(X, Y)$ *and* $g \in \mathcal{C}(Y, Z)$*, we have* $g \circ f \in \mathcal{C}(X, Z)$
- *for any object* X*, the identity morphism* $\mathsf{id}_X \in \mathcal{C}(X, X)$

such that:

- *Every identity morphism is left- and right-neutral. This means that, for* $f \in \mathcal{C}(X, Y)$*, we have* $f \circ \mathsf{id}_X = f$ *and* $\mathsf{id}_Y \circ f = f$*.*
- *Composition is associative, i.e.* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

Definition 2 (initial and terminal object)**.** *An object* X *in a category* $\mathcal{C}$ *is* initial *if, for every object* Y*, there is exactly one morphism from* X *to* Y*. This means that* $\mathcal{C}(X, Y)$ *is the one-element set. An object* Z *is* terminal *if, for every object* Y*, there is exactly one morphism from* Y *to* Z*, i.e.,* $\mathcal{C}(Y, Z)$ *is the one-element set.*

Definition 3 (isomorphism)**.** *A morphism* $f \in \mathcal{C}(X, Y)$ *is an* isomorphism *if there is a morphism* $g \in \mathcal{C}(Y, X)$ *such that* $g \circ f = \mathsf{id}_X$ *and* $f \circ g = \mathsf{id}_Y$*.*

Exercise 1: The free category on a directed multigraph

A *directed multigraph* (sometimes also called *quiver*) consists of:

- a set V of vertices
- for each pair (a, b) of vertices, a set $E(a, b)$ of edges from a to b.

Note that the edges are directed, i.e., $E(a, b)$ is not necessarily the same as $E(b, a)$, and many parallel edges are allowed.

Let $G = (V, E)$ be a directed multigraph. The *free category on* G, (here) written $\mathcal{F}_G$, has V as objects and a morphism $\mathcal{F}_G(X, Y)$ is a sequence of consecutive edges, starting in X and ending in Y.

a. Show that $\mathcal{F}_G$ is a category.

b. What are the isomorphisms in $\mathcal{F}_G$?

c. For which G does $\mathcal{F}_G$ have an initial object? And when does $\mathcal{F}_G$ have a terminal object?

Exercise 2: The category REL

The objects of the category REL are sets. For sets X and Y, an element of $\mathsf{REL}(X, Y)$ is a *relation* between X and Y, i.e., a subset $R \subseteq (X \times Y)$. The composition of relations $R \subseteq (X \times Y)$ and $S \subseteq (Y \times Z)$ is given by

$$\{ (x, z) \mid \exists y \in Y.(x, y) \in R \wedge (y, z) \in S \}.$$

a. Can you define identities and prove that REL is a category?

b. What are the initial and terminal object of REL?

c. What are the isomorphisms?

Bonus Exercise: The (almost-)category SPAN

The objects of SPAN are sets. A morphism between sets X and Y consists of a set Z together with a function $f : Z \to X$ and a function $g : Z \to Y$.

a. The composition operation works in a similar way as for REL, but we don't need an existential quantifier this time. Can you make this operation precise?

b. Can you prove associativity? Can you define identities and prove that SPAN is a category? What works or fails? (Hint: It only almost works. SPAN is a structure that is slightly weaker than a category, namely a bicategory.)

c. What are the initial and terminal object of SPAN?

d. What are the isomorphisms?

e. How does SPAN compare to the category REL?