Syntax for two-level type theory

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1 Introduction

In homotopy type theory [13] (HoTT), properties that are not invariant under homotopy cannot be expressed internally. An important case is the concept of semisimplicial types, whose definition is so far elusive in HoTT. Voevodsky defined a special Homotopy Type System [14] (HTS) as a formal theory which allows constructions that require access to non-homotopy-invariant notions. Two-level type theory [2] (2LTT) is envisioned to be a variant of HTS, and is composed of two separate levels of types: the outer level is Martin-Löf type theory plus the uniqueness of identity proofs [12] (UIP); the inner level is HoTT. These levels are related by a conversion function from the inner to the outer level that preserves context extensions.

The paper [2] proposes a semantics for 2LTT based on categories with families [7], which justifies reasoning *inside* the inner system with the full power of HoTT, and reasoning *about* the inner system within the outer system to circumvent a number of expressive limits of the former. With this approach it is possible to study properties of HoTT syntactically in the two-level system, and, by conservativity [4], to reflect them back in the HoTT world. Among the applications of this approach are results on Reedy fibrant diagrams [2], the Univalence Principle [1], and internal ∞-categories with families [8], which have been suggested as a way to overcome known difficulties one encounters when formalising type theory in type theory. In summary, despite the intrinsic expressive and proving power of HoTT, a wide range of results rely on meta-reasoning and meta-principles, which cannot entirely be formalised within the theory. The two-level approach formalises these meta-principles in a theory which is compatible both technically and philosophically with HoTT, allowing for their mechanisation. However, the syntax of 2LTT is just sketched in [2].

2 Syntax

In this contribution, we propose a system of inference rules for 2LTT with an infinite hierarchy of Tarski-style universes as uniform constructions [10]; the rules allow us to define the syntax in detail, clearly illustrating the behaviour of the two levels, and how they interact. In contrast to [2], we pay particular attention to the definition of Tarski-style universes, following the guidelines of [10]: other than the function El_i, which maps the codes $A : \mathcal{U}_i$ into types $\mathsf{El}_i(A)\ \mathsf{type}$ and is present in [2], we introduce a function lift_i mapping terms of one universe $A : \mathcal{U}_i$ into terms of the next one, $\mathsf{lift}_i(A) : \mathcal{U}_{i+1}$. In [2], the lift operation is not present, and the universes are *cumulative*. In our system those two functions commute:

$$\frac{\Gamma \vdash A : \mathcal{U}_i}{\Gamma \vdash \mathsf{El}_{i+1}(\mathsf{lift}_i(A)) \equiv \mathsf{El}_i(A)\ \mathsf{type}}\ \mathcal{U}\text{-}\mathsf{lift}$$

The same happens for inner types; indeed, $A\,\mathsf{type}$ means that A is an outer type, while $A\,\mathsf{type}^o$ means that A is an inner type. This emphasises another difference between our approach and the 2LTT paper: we do not have a *size* for types; on the contrary, in [2] it is specified as $A\,\mathsf{type}_i$ or $A\,\mathsf{type}_i^o$: if $A : \mathcal{U}_i$, then $\mathsf{El}_i(A)\,\mathsf{type}_i$. Moreover, besides the conversion function c from inner to outer types introduced in [2], we define a conversion function c' from inner to outer codes, i.e., terms of the universes: if $A : \mathcal{U}_i^o$, then $c'(A) : \mathcal{U}_i$. It is required that $\mathsf{El}, \mathsf{lift}, c$ and c' commute. We formalise the fact that the conversion function preserves context extension by introducing a notion of equivalence between contexts together with the rule

$$\frac{\Gamma \vdash A\,\mathsf{type}^o}{\Gamma, x : A \equiv \Gamma, y : c(A)\,\mathsf{ctx}}\ {\equiv}{-}\mathsf{ctx}{-}\mathsf{EXT}$$

Then, we define a generalisation of the notion of category with families which allows us to interpret our formalisation of the two levels and the Tarski-style universes, called *two-level model*, together with a notion of morphism between models. We plan to show the compatibility of our system with the (almost) standard semantics for 2LTT by proving an initiality result; this will essentially extend recent work for Martin-Löf type theory by Brunerie, de Boer, Lumsdaine, and Mörtberg [3, 9, 6]. We define the syntactical two-level model by quotienting the syntax, similar to [11, 5], and prove that it is the initial object in the category of models.

Our long term goal is to develop the basis for a proof assistant that implements 2LTT and allows one to use additional inner and outer axioms, some of which have been already suggested [2], to formalise in parallel the inner and outer levels, and their relations.

3 Open questions

There are some open issues, which we hope to understand better in the future:

1. Can the conversion c as well as the operators lift and El be made "silent" in order to make a potential proof assistant more convenient to use?

2. We aim to avoid cumulativity, which can create difficulties with typing. However, with the rule $\mathcal{U}{-}\mathsf{lift}$, we aim to recover the main benefits or cumulativity. What models can we hope for?

3. In the current version, we use judgmental equality of contexts in the rule $\equiv{-}\mathsf{ctx}-\mathsf{EXT}$; is this too strict for the purpose of construction of models? What are the proof-theoretic consequences?

References

[1] Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. The Univalence Principle, 2021. https://arxiv.org/abs/2102.06275.

[2] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler. Two-level type theory and applications. *arXiv:1705.03307v3*, 2019.

[3] Guillaume Brunerie, Menno de Boer, Peter LeFanu Lumsdaine, and Anders Mörtberg. A formalization of the initiality conjecture in Agda. Talk given by Brunerie at the HoTT 2019 conference, slides available at https://guillaumebrunerie.github.io/pdf/initiality.pdf, 2019.

[4] Paolo Capriotti. *Models of type theory with strict equality.* PhD thesis, University of Nottingham, 2016. http://eprints.nottingham.ac.uk/39382/1/thesis.pdf.

[5] Thierry Coquand and Simon Castellan. Dependent type theory as the initial category with families. 2014.

[6] Menno de Boer. *A Proof and Formalization of the Initiality Conjecture of Dependent Type Theory*. Licentiate dissertation, Department of Mathematics, Stockholm University, 2020.

[7] Peter Dybjer. Internal type theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs*, pages 120–134, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[8] Nicolai Kraus. Internal ∞-categorical models of dependent type theory: Towards 2LTT eating HoTT. *Logic in Computer Science (LICS'21)*, 2021. Available as arXiv:2009.01883.

[9] Peter LeFanu Lumsdaine and Guillaume Brunerie. Initiality for Martin-Löf type theory. Talk at the Homotopy Type Theory Electronic Seminar Talks (HOTTEST), https://www.youtube.com/watch?v=1ogUFFUfU_M, 2020.

[10] Erik Palmgren. On universes in type theory. In G. Sambin and J. Smith, editors, *Twenty-Five Years of Constructive Type Theory*. Oxford University Press, 1998.

[11] Thomas Streicher. *Semantics of Type Theory*. Progress in Theoretical Computer Science. Springer, 1991.

[12] Thomas Streicher. *Investigations into intensional type theory*. Habilitationsschrift, Ludwig-Maximilians-Universität München, 1993.

[13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[14] Vladimir Voevodsky. A simple type system with two identity types. Unpublished note, 2013.