Truncation Levels in Homotopy Type Theory

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My time as a PhD student

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Thank you!

I am grateful to many people for sharing their knowledge with me, especially **Christian Sattler** and **Paolo Capriotti**

Type theory

Formal systems for programming, proving, formalising, foundation of mathematics

Central: $\mathbf{x} : \mathbf{A}$, "x is a term of type A" (in some context)

→ interpretations:

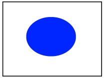
- ⋆ A is a set and x an element [Russel, 1903]
- ⋆ A is a problem and x a solution [Brouwer–Heyting, Kolmogorov, 1932]
- ⋆ A is a proposition and x a proof [Curry-Howard, 1969]
- ⋆ A is a space and x a point (in case of MLTT)
  early form: Hofmann–Streicher 1996;
  Voevodsky (from 2006/10);
  Awodey–Warren 2009; . . .

⇒ *Homotopy type theory / Univalent foundations*

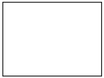
I have adapted this list from Pelayo-Warren, "Homotopy type theory and Voevodsky's univalent foundations".

Truncation levels (Voevodsky: h-levels)

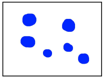
⋆ Truncation levels express (an upper bound of) the homotopical complexity of types, starting as follows:



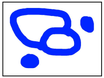
level -2: “contractible”, equivalent to **Unit**



level -1: “propositional”, contractible equality types



level 0: a “set”, propositional equality types



level 1: a “groupoid”, equality types are sets

Non-truncated types (1)

Well-known fact: In Martin-Löf type theory with the univalence axiom, the lowest universe $\mathcal{U}_0$ is not a set.

Proof: Bool is equivalent to itself in two different ways (identity and negation), thus univalence gives two different elements of Bool = Bool.

Open problem of the special year in Princeton (2012): Given a hierarchy $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \ldots$ of univalent universes, can we construct types that are provably not n-truncated?

This is indeed the case (I presented a proof in Princeton in April 2013).

Non-truncated types (2)

Extended answer (K. and Sattler 2013/2015):

- ⋆ The universe $\mathcal{U}_n$ is not n-truncated.
- ⋆ $\mathcal{U}_n$, restricted to n-truncated types, is a “strict” $(n+1)$-type.
- ⋆ With some additional effort, we get a strict n-type which has trivial homotopy groups on all levels except n.
- ⋆ Note: It is consistent to assume that $\mathcal{U}_n$ is $(n+1)$-truncated, i.e. the first two results are optimal. The third “wastes” one universe level.

$\mathcal{U}_1$ is not 1-truncated, proof

(0) Assume $\mathcal{U}_1$ is 1-truncated.

(1) Set $L :\equiv \Sigma(X : \mathcal{U}_0).(X = X)$.

(2) If $\mathcal{U}_1$ is 1-truncated, then $L = L$ is a set.

(3) Then, $\mathsf{refl}_L = \mathsf{refl}_L$ is a proposition.

(4) Univalence-translated: $(\mathsf{id}_L, \mathsf{e}_{\mathsf{id}}) = (\mathsf{id}_L, \mathsf{e}_{\mathsf{id}})$ is a proposition.

(5) Simplifies to: $\mathsf{id}_L = \mathsf{id}_L$ is a proposition.

(6) By function extensionality: $\Pi_{x:L}(x = x)$ is a proposition.

(7) By unfolding and currying: $\Pi_{A:\mathcal{U}_0}\Pi_{p:A=A}\,(A, p) = (A, p)$ is a proposition.

(8) Rewrite with standard lemmas: $\Pi_{A:\mathcal{U}_0}\Pi_{p:A=A}\,\Sigma(q : A = A).(p \cdot q = q \cdot p)$ is a proposition.

(9) ... but this type has multiple elements, e.g. $\lambda A.\lambda p.(\mathsf{refl}_A, \_)$ and $\lambda A.\lambda p.(p, \_)$.

$\mathcal{U}_n$ is not n-truncated, some ideas

Recall: $\mathcal{U}_n$ is n-truncated $\leftrightarrow$ $\Omega^{n+1}(\mathcal{U}_n, X)$ is contractible.

- ⋆ By induction on n.
- ⋆ Consider $(n+1)$-loops in $\mathcal{U}_n^n$, i.e.:

  $$\Sigma(A : \mathcal{U}_n^n).\Omega^{n+1}(\mathcal{U}_n^n, A).$$

  Here, $\mathcal{U}_n^n$ is $\mathcal{U}_n$ restricted to n-truncated types (crucial trick!).
- ⋆ We can "move between universes" with our *local-global looping principle*:

  $$\Omega^{n+2}(\mathcal{U}, A) \simeq \Pi_{a:A}\Omega^{n+1}(A, a)$$

  (this is simple, essentially function extensionality).

New topic: Propositional truncation

- ⋆ In HoTT: we consider an operation $\|-\|$ which turns a type into a propositional type. Roughly: reflector of the subcategory of propositional types.
- ⋆ We only know how to construct a function $\|A\| \to B$ if B is propositional.
- ⋆ The (in my opinion) main result of my thesis is:

$$(\|A\| \to B) \quad \simeq \quad \mathcal{U}^{\Delta_+^{op}}(\mathcal{T}A, \mathcal{E}B)$$

where $\mathcal{E}B$ is the Reedy fibrant replacement of (const) B and $\mathcal{T}A$ the [0]-coskeleton of A.

Very much related to 6.2.3.4 in Lurie's *Higher Topos Theory* and 7.8 in Rezk's *Toposes and Homotopy Toposes*.

- ⋆ I will not talk about this today. Instead, I conclude with a fun result.

A "mysterious puzzle"

Consider the function $|-| : \mathbb{N} \to \|\mathbb{N}\|$.
There is a term myst such that $\Pi_{n:\mathbb{N}}\, \mathsf{myst}(|n|) = n$.

- Consequence: $0 = \mathsf{myst}(|0|) \neq \mathsf{myst}(|1|) = 1$ How?
- Solution: the type of myst is **not** just $\|N\| \to \mathbb{N}$. In fact, $\mathsf{myst} : \Pi_{x:\|\mathbb{N}\|} C(x)$ with a **very** complicated C. It just happens that $C(|n|) \equiv \mathbb{N}$!
- Here's how to do it:

  Observe that $(\mathbb{N}, 0) = (\mathbb{N}, n)$ as pointed types. Define

  $$f : \mathbb{N} \to \Sigma(Y : \mathcal{U}_\bullet).((\mathbb{N}, 0) = Y)$$
  $$n \mapsto ((\mathbb{N}, n), \_)$$
- $f' : \|\mathbb{N}\| \to \Sigma(Y : \mathcal{U}_\bullet).((\mathbb{N}, 0) = Y)$
- define $\mathsf{myst} :\equiv \mathsf{snd} \circ \mathsf{fst} \circ f'$.

Conclusions

I have done some stuff about truncation levels in type theory, and I really enjoyed my time as a PhD student.

Thank you!