On Hedberg's Theorem: Proving and Painting

Nicolai Kraus

25/05/12

Reminder: Equality

Definitional Equality

"Real" decidable equality for typechecking, computation; e. g.

$$(\lambda a.b)x =_\beta b[x/a]$$

Reminder: Equality

Definitional Equality

“Real” decidable equality for typechecking, computation; e. g.

$$(\lambda a.b)x =_\beta b[x/a]$$

Propositional Equality

Equality needing a proof, i. e. a term of the identity type

Reminder: Identity Types

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Formation

$$\frac{a, b \ : \ A}{a \equiv b \ : \ \mathit{type}}$$

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Introduction

$$\frac{a \;:\; A}{refl_a \;:\; a \equiv a}$$

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$$\frac{a \; : \; A}{refl_a \; : \; a \equiv a}$$

Elimination (J)

$$\frac{P \; : \; (a, b : A) \to a \equiv b \to Set \qquad m : \forall a. P(a, a, refl_a)}{J_{(a,b,q)} \; : \; P(a, b, q)}$$

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Computation (β)

$$J_{(a,a,refl_a)} =_\beta ma$$

Elimination: J versus K

Eliminator J

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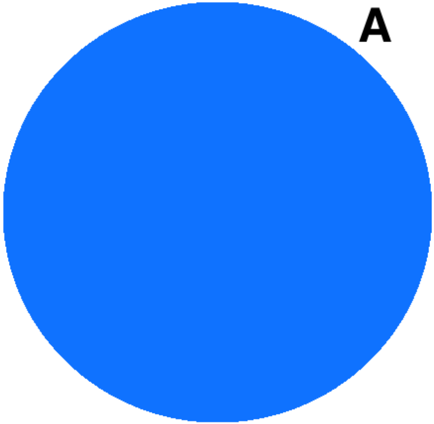
Eliminator K

$$\frac{\begin{array}{c} a \;:\; A \\ P \;:\; a \equiv a \to Set \\ P(refl_a) \end{array}}{K_q \;:\; P(q)}$$

Homotopic Model

$A : type$



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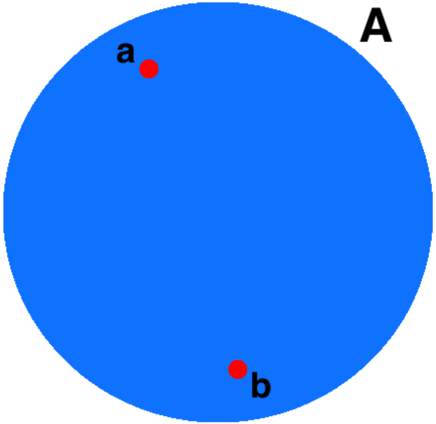

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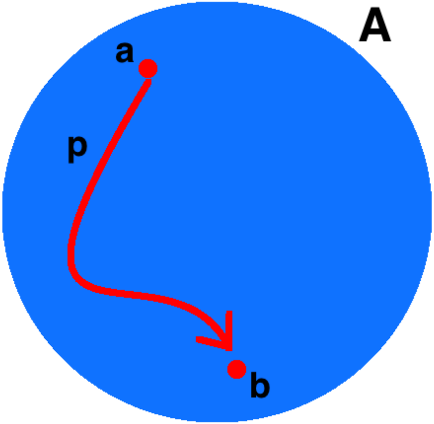

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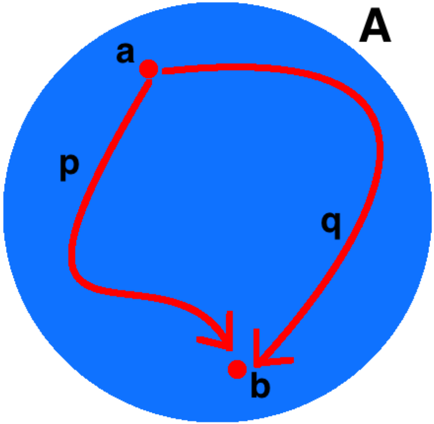

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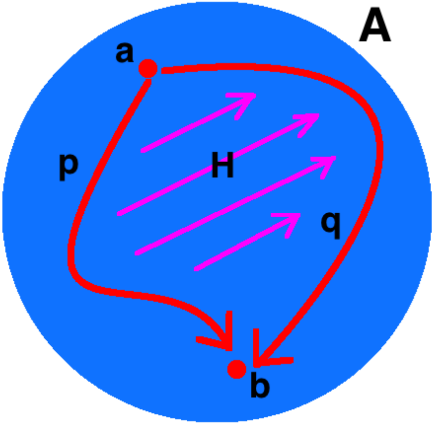
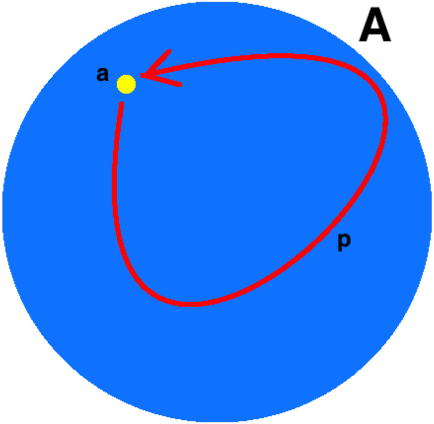
H : $p \equiv q$



K revisited

Want to prove:
$\forall p\,.\,p \equiv refl_a$



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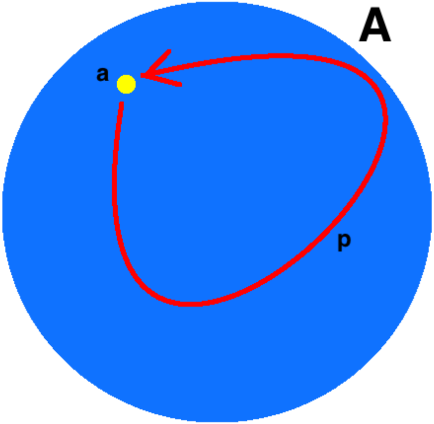

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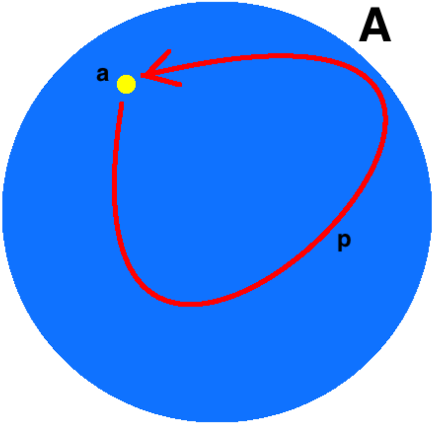

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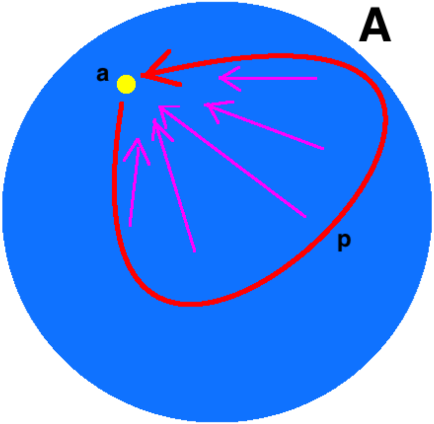

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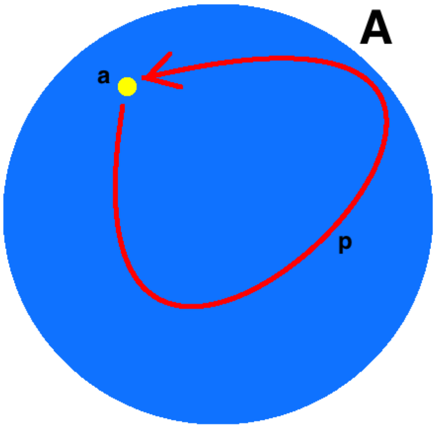

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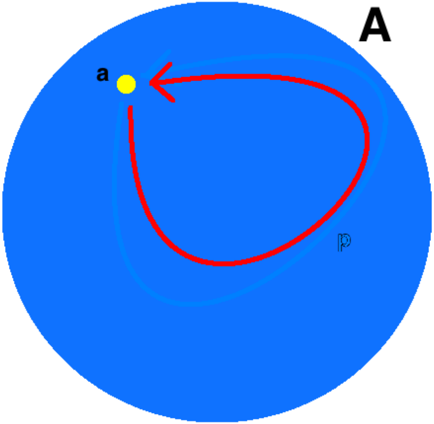

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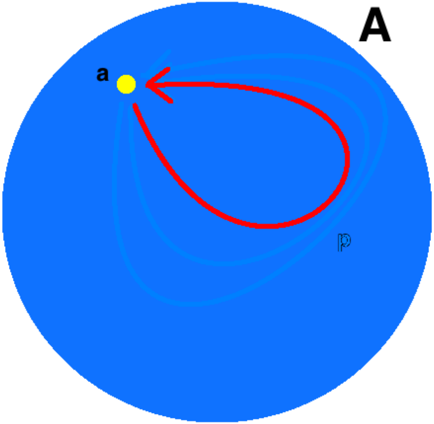

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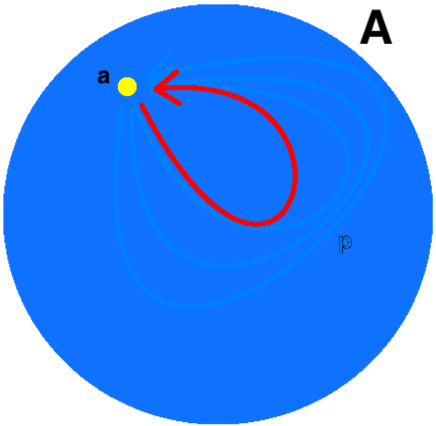

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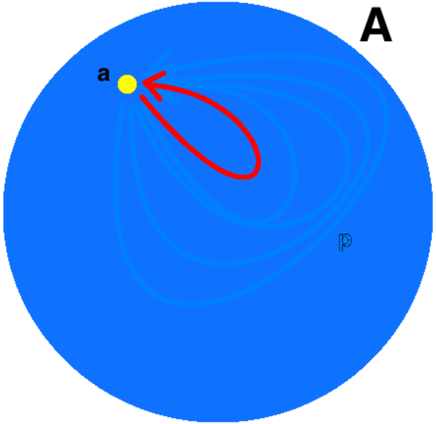

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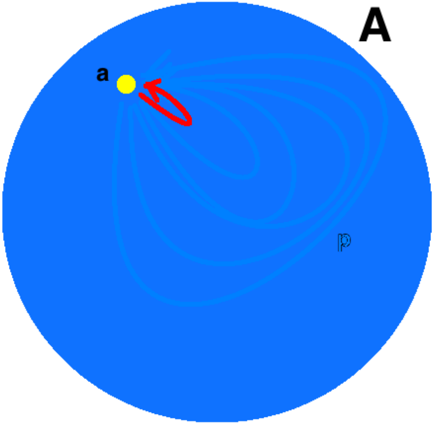

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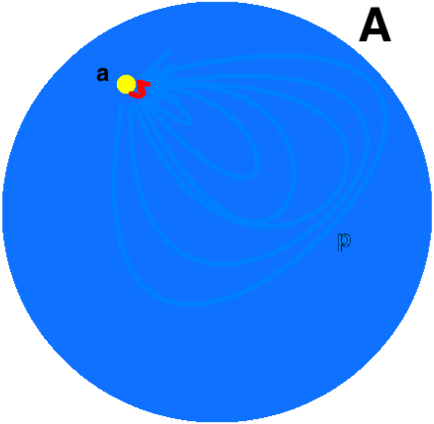
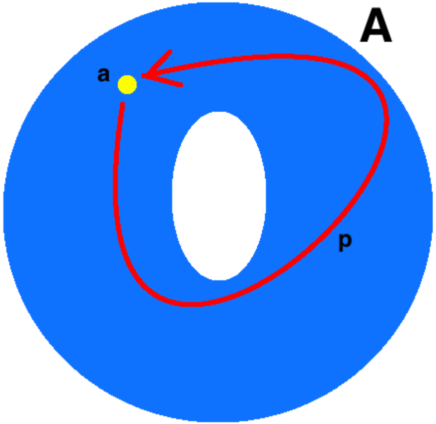


K revisited

Okay, but what now?



J revisited

Want: $(a, a, p) \equiv (a, a, refl_a)$

J shows:

$\forall (a, b, q)_{:\Sigma(a,b:A).a \equiv b},$

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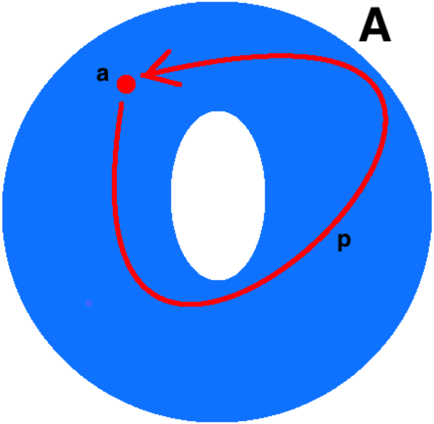

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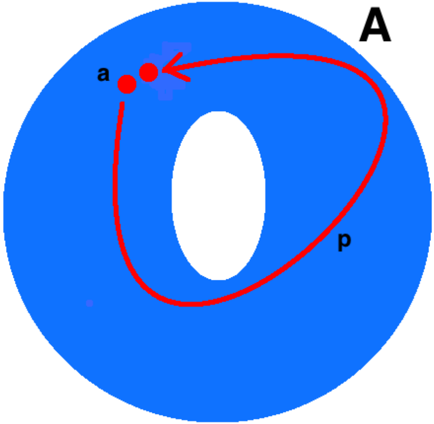

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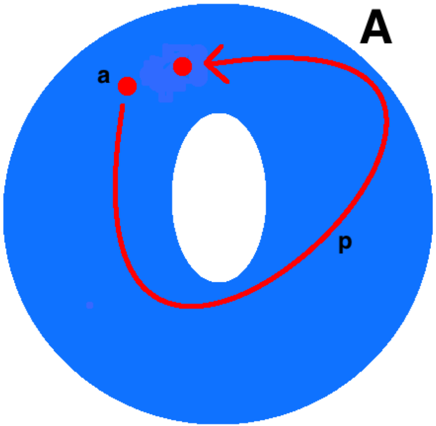

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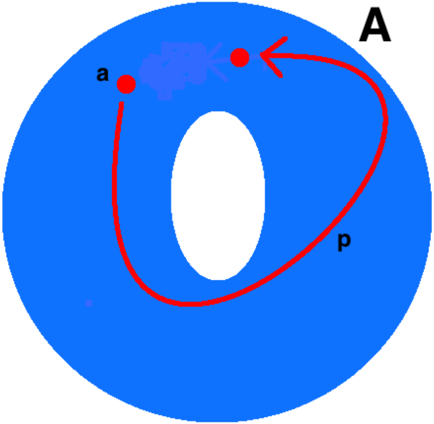

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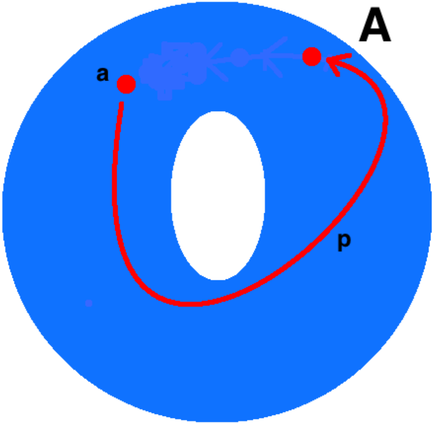

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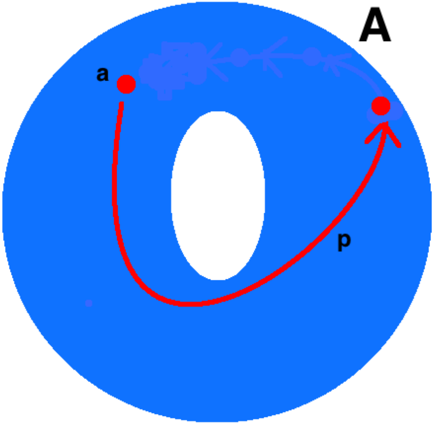

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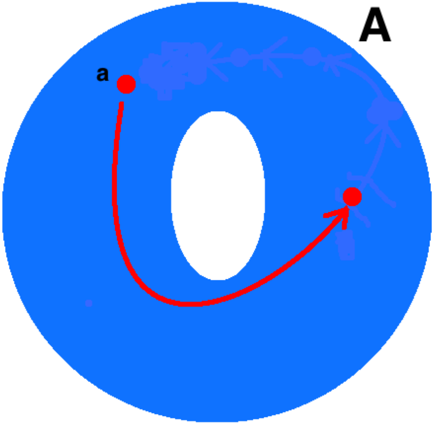

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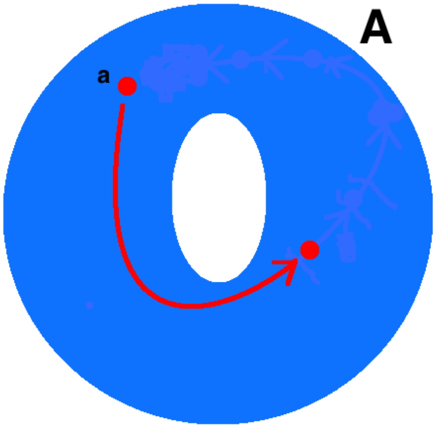

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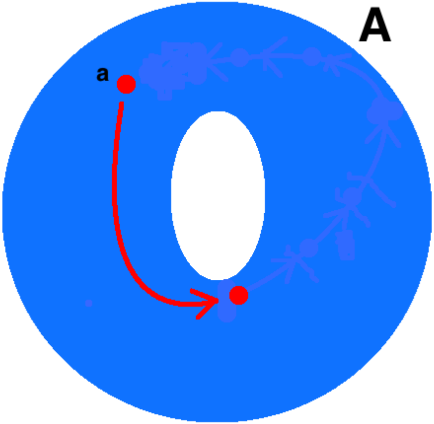

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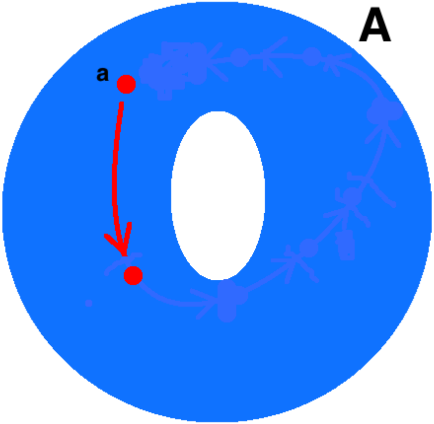

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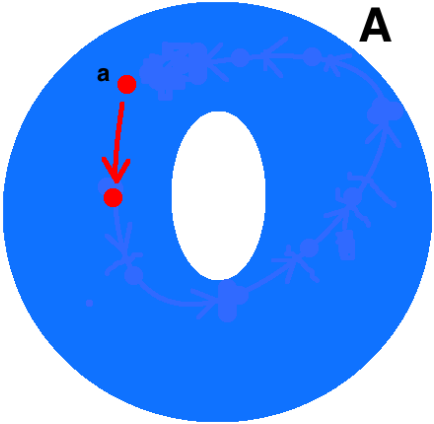

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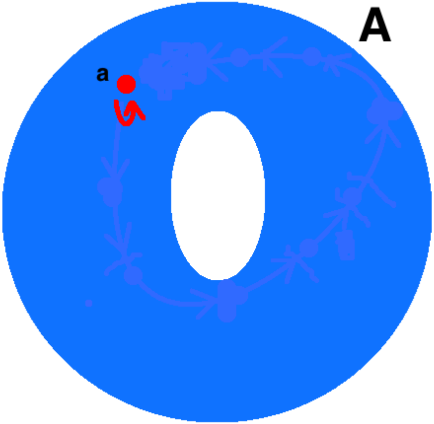

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J versus K revisited

Fix $a : A$.

J (variant of Paulin-Mohring)

To show $\forall(b, q)_{:\Sigma(b:A).a \equiv b} \,.\, P(b, q)$, just prove $P(a, \mathit{refl}_a)$.

K (reformulated)

$\forall q_{:a \equiv a} \,.\, q \equiv \mathit{refl}_a$

Hedberg's theorem

Fix a type A.

Decidable Equality

$$DecidableEquality := \forall a, b \ \ (a \equiv b + \neg a \equiv b)$$

Hedberg's theorem

$$DecidableEquality \longrightarrow K$$

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$$DecidableEquality \longrightarrow K$$

Proof.

- Given $dec : (a, b : A) \to (a \equiv b + \neg\, a \equiv b)$.

Hedberg's theorem

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$$\mathit{DecidableEquality} \longrightarrow K$$

Proof.

- Given $dec : (a, b : A) \rightarrow (a \equiv b + \neg\, a \equiv b)$.
- Given any $a, b : A$ and $p : a \equiv b$.

Hedberg's theorem

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$$DecidableEquality \longrightarrow K$$

Proof.

- Given $dec : (a, b : A) \rightarrow (a \equiv b + \neg\, a \equiv b)$.
- Given any $a, b : A$ and $p : a \equiv b$.
- $dec\, a\, b = inl\, q_1$ and $dec\, a\, a = inl\, q_2$ for some q_1, q_2.

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- Claim: $p \equiv q_1 \circ q_2^{-1}$

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- Proof with J: Just show $refl_a \equiv q_2 \circ q_2^{-1}$. That's true!

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$DecidableEquality \longrightarrow K$

Proof.

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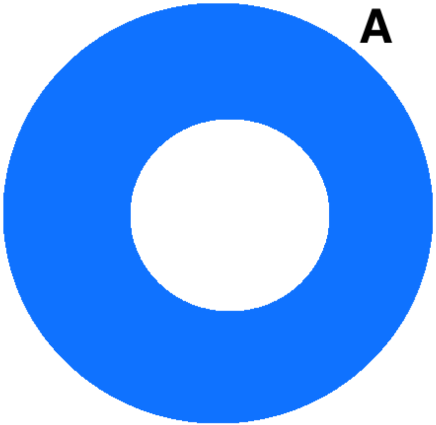
- Claim: $p \equiv q_1 \circ q_2^{-1}$
- Proof with J: Just show $refl_a \equiv q_2 \circ q_2^{-1}$. That's true!
- Special case: If $p : a \equiv a$, then $p \equiv refl_a$. □

Corollary[1]: The Circle type does not have decidable equality

$dec : (a, b : A) \to$
$(a \equiv b + \neg a \equiv b)$



[1] If you believe me that K does not hold for the Circle type

Nearly uncountable many things to be done . . .

- Higher Inductive Types (see Mike Shulman's work)
- Model construction with modern abstract (not point-set) homotopy theory
- Constructive Simplicial Sets (the combinatorial version of what I have shown; see Thierry Coquand's / Simon Huber's work)
- Univalent foundations / Univalence ("alternative" to K) in general (see Voevodsky)
- . . . and possible computational properties (Thorsten?)

THANK YOU!