Generalizations of Hedberg's Theorem

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28/06/13

Views on Martin-Löf Type Theory

MLTT is a formal system
(with dependent types, Σ, Π, inductive types, ...)
can be used for...

Programming

- type system can provide a precise specification
- e. g. Agda code can be compiled to a Haskell program

Mathematics

- foundation of mathematics
- proof assistants (e. g. Coq):
  - help finding proofs
  - allow formalizing (and thereby verifying) results
- e. g. a lot of axiomatic homotopy theory has been formalized in Homotopy Type Theory

Equality in MLTT

Definitional Equality

Decidable equality for typechecking & computation; e. g.

$$(\lambda x.t)a \equiv t[a/x]$$

Propositional Equality

Equality needing a proof, e. g.

$$\forall\, m\, n\, .\, (m + n) = (n + m)$$

Equality in MLTT

Propositional equality

... is "just" an inductive type

Formation

$$\frac{a, b \,:\, A}{a =_A b \,:\, \mathcal{U}}$$

Introduction

$$\frac{a \,:\, A}{refl_a \,:\, a =_A a}$$

Elimination (J - Paulin-Mohring) for any $a : A$

$$\frac{P \,:\, (b : A) \to a =_A b \to \mathcal{U} \qquad m \,:\, P\, a\, refl_a}{J_P m \,:\, \forall (b, q).\, P(b, q)}$$

Computation (β)

$$J_P m\, a\, refl_a \equiv_\beta m$$

Uniqueness of Identity Proofs (UIP)

Given $a : A$.

- Can we show
  $(b, c : A) \to (p : a = b) \to (q : a = c) \to (b, p) = (c, q)$?

  Induction/J/"pattern matching" on (b, p)
  $\Rightarrow \quad (c : A) \to (q : a = c) \to (a, refl_a) = (c, q)$.
  Induction on (c, q) $\quad \Rightarrow \quad (a, refl_a) = (a, refl_a)$.

- Can we show $(b : A) \to (p, q : a = b) \to p = q$?

  Induction on (b, p)
  $\Rightarrow \quad (q : a = a) \to (refl_a = q)$.

???

Uniqueness of Identity Proofs (UIP)

[potential] Axiom UIP, aka K

$$\frac{p,\ q\ :\ a = b}{\mathrm{UIP}\ :\ p = q}$$

Advantages

- simple
- more powerful pattern matching

Disadvantages

- if $A \simeq B$, we want to treat A and B as equal $\Rightarrow$ the isomorphism matters (UIP incompatible with univalence)
- nontrivial equality structure can be useful (Homotopy Type Theory uses it to formalize axiomatic homotopy theory)

Hedberg's Theorem

Which types satisfy UIP naturally?

$\text{DecidableEquality}_A$, i. e.
$\forall\, a\, b\, .\, (a = b\, +\, \neg\, a = b)$

$\Downarrow$

$\forall xy.\, f(x) = f(y)$

there is a family $g_{ab} : a = b \to a = b$ of **constant** endofunctions

$\Updownarrow$

UIP_A, i. e.
$\forall (p, q : a = b).\, p = q$

Strengthening Hedberg's Theorem

DecidableEquality is a very strong property.
How about something weaker? For example:

Separated (¬¬-stable equality)

$$\forall\, a\, b\, .\, \neg\neg(a = b) \;\to\; a = b$$

With function extensionality,

$$\mathrm{separated}_A \to \mathrm{UIP}_A$$

Truncation

$\neg\neg A$ can be seen as "anonymous existence".

A better way to say that A is "anonymously" inhabited is *truncation* $\|A\|$, aka *squash types* or *bracket types* (Awodey / Bauer).

Properties:

- In $\|A\|$, we cannot distinguish the different inhabitants, i. e. $\|A\|$ is a *proposition*
- $A \to \|A\|$
- If $A \to P$ and P is a proposition, then $\|A\| \to P$

Generalizations

h-separated$_A$, i. e.
$\|a = b\| \to a = b$

$\Updownarrow$

there is a family
$g_{ab} : a = b \to a = b$ of
constant endofunctions

$\Updownarrow$

UIP$_A$, i. e.
$(p, q : a = b) \to p = q$

h-stable$_X$, i. e.
$\|X\| \to X$

$\Downarrow$ (easy) $\Uparrow$ (hard)

there is a **constant**
$g : X \to X$

$\Uparrow$

X is a proposition, i. e.
$(p, q : X) \to p = q$

Applications I

Define $\langle\langle X \rangle\rangle$ as
"every constant endofunction on X has a fixed point".

$\langle\langle X \rangle\rangle$ is a new notion of anonymous existence, similar to $\|X\|$, but definable in basic MLTT.

$$X \Rightarrow \|X\| \Rightarrow \langle\langle X \rangle\rangle \Rightarrow \neg\neg X$$
and all implications are strict

Applications II

Assume

every type has a constant endofunction.

What is this statement's status?

- It follows from "excluded middle", $\forall A.\, A + \neg A$
- (We think) it does not imply $\forall A.\, A + \neg A$
- consequence: UIP
- stronger consequence: all equalities are decidable

Surprise?

We know:	How about:
there is **constant** function $X \to X$	there is **constant** function $X \to Y$
$\Updownarrow$	$\Uparrow$ (trivial) $\quad \not\Downarrow$
$\lVert X \rVert \to X$	$\lVert X \rVert \to Y$

$\Downarrow$ seems to fail due to a homotopical problem. Apparently, we need an infinite tower of coherence conditions (c. f. defining semi-simplicial types, open problem of the Princeton special year program on UF/HoTT).

Questions?

Thank you!