Higher Inductive Types without Recursive Higher Constructors

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Inductive types in Martin-Löf type theory

natural numbers

$\mathbb{N}$ is a type with constructors

$$\mathsf{z} : \mathbb{N}$$

$$\mathsf{S} : \mathbb{N} \to \mathbb{N}$$

propositional equality

$x = y$ is a type (for $x, y : A$).
Constructor:

$$\mathsf{refl} : \forall x.\, x = x$$

$\mathbb{N}$-induction

for a family $P : \mathbb{N} \to \mathcal{U}$,

$$P(\mathsf{z})$$

$$\forall n.\, P(n) \to P(\mathsf{S}n)$$

$$\overset{ind}{\Rightarrow} \forall n.\, P(n)$$

=-induction

for $P : (\Sigma_{x,y:A} x = y) \to \mathcal{U}$,

$$\forall x.\, P(x, x, \mathsf{refl}_x)$$

$$\overset{ind}{\Rightarrow} \forall xyq.\, P(x, y, q)$$

Propositional equality ("="), examples

symmetry

$$\mathsf{sym} : \forall x y.\ x = y \to y = x$$

Construction: $\mathsf{sym}(x, x, \mathsf{refl}_x) :\equiv \mathsf{refl}_x$.

transitivity

$$\mathsf{trans} : \forall x y z.\ x = y \to y = z \to x = z$$

Construction: $\mathsf{trans}(x, x, x, \mathsf{refl}_x, \mathsf{refl}_x) :\equiv \mathsf{refl}_x$.

"What's the point? Everything is refl anyway..."
⇒ Try to prove *uniqueness of identity proofs* (UIP), i.e.

$$\forall x y.\ \forall (p, q : x = y).\ p = q$$

"Sure, assume (x, y, p) is just (x, x, refl_x), then what we need is $\forall x.\ \forall (q : x = x).\ \mathsf{refl}_x = q$ and then. . . oh, we are stuck."

Fact: **UIP is not derivable** (Hofmann-Streicher 1998).

Why not add UIP as an axiom?

Arguments for homotopy type theory (“alternative” to UIP):

- want: if $A \simeq B$, then $A = B$ (without forgetting how)

  e.g.: we want to substitute different representations of $\mathbb{N}$ for each other!

  ⇒ univalence (more abstract, more convenient to use)
- “types are [behaved like] spaces”: transport intuition **and results** between homotopy theory and type theory; allows other constructions, including **synthetic homotopy theory**
- equalities are paths, and all type-theoretic statements are up to homotopy and continuous transformations – everything automatically “non-evil”, it’s beautiful!
- by the way:

  homotopy (type theory)
  (homotopy type) theory

Higher inductive types (HITs)

Constructors can construct elements, higher constructors can construct equalities. Example:

real numbers

$\mathbb{R}$ is a type with constructors

$$\mathsf{rat} : \mathbb{Q} \to \mathbb{R}$$

$$\mathsf{lim} : (f : \mathbb{N} \to \mathbb{R}) \to \mathsf{isCauchy}(f) \to \mathbb{R}$$

$$\mathsf{quot} : (u, v : \mathbb{R}) \to u \sim v \to u = v$$

isCauchy and $\sim$ need to be defined at the same time (not shown here).

This is better behaved than a quotient – our $\mathbb{R}$ is complete!

Higher inductive types (HITs)

circle

$\mathbb{S}^1$ is given by the constructors

$$\mathsf{base} : \mathbb{S}^1$$

$$\mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$$

$\mathbb{S}^1$ behaves as one expects. E.g., its fundamental group is equivalent to $\mathbb{Z}$. The fundamental group is essentially $\mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$.

Higher inductive types (HITs)

Propositional Truncation ("Squash", "bracket types")

For a type A, the type $\|A\|$ is given by

$$|-| : A \to \|A\|$$

$$\mathsf{h} : (x, y : \|A\|) \to x =_{\|A\|} y$$

- all elements of $\|A\|$ are equal
- $\|A\|$ is "the proposition that A holds"
- can make "non-continuous" statement, e.g.

$$\Pi_{x:\mathbb{S}^1} \; x = \mathsf{base} \quad - \quad \text{contradiction}$$

$$\Pi_{x:\mathbb{S}^1} \; \|x = \mathsf{base}\| \quad - \quad \text{provable}$$

Recursive versus Non-Recursive HITs

Propositional Truncation $\|A\|$

$$|-| : A \to \|A\|$$

$$\mathsf{h} : (x, y : \|A\|) \to x =_{\|A\|} y$$

Pseudo-truncation $\langle\!\langle A \rangle\!\rangle$

$$\langle - \rangle : A \to \langle\!\langle A \rangle\!\rangle$$

$$\mathsf{t} : (x, y : A) \to \langle x \rangle =_{\langle A \rangle} \langle y \rangle$$

universal property $\|A\|$

$$\frac{\|A\| \to B}{A \to B}$$

if B is propositional

universal property $\langle\!\langle A \rangle\!\rangle$

$$\frac{\langle\!\langle A \rangle\!\rangle \to B}{\Sigma\,(f : A \to B)\,.\,\mathsf{wconst}(f)}$$

for any B

note: $\mathsf{wconst}(f) :\equiv \Pi_{x,y:A}\ f\,a = f\,b$

$\langle\!\langle A \rangle\!\rangle$ has has several names:

- Altenkirch: “constant map classifier” (see u.p.)
- Coquand-Escardó: “generalised circle” ($\langle\!\langle \mathbf{1} \rangle\!\rangle \simeq \mathbb{S}^1$)
- van Doorn: “one-step truncation” (later)

Recursive versus Non-Recursive HITs

Propositional Truncation $\|A\|$

$$|-| : A \to \|A\|$$

$$\mathsf{h} : (x, y : \|A\|) \to x =_{\|A\|} y$$

universal property $\|A\|$

$$\frac{\|A\| \to B}{A \to B}$$

if B is propositional

Pseudo-truncation $⟪A⟫$

$$\langle - \rangle : A \to ⟪A⟫$$

$$\mathsf{t} : (x, y : A) \to x =_{⟪A⟫} y$$

universal property $⟪A⟫$

$$\frac{⟪A⟫ \to B}{\Sigma\,(f : A \to B)\,.\,\mathsf{wconst}(f)}$$

for any B

note: $\mathsf{wconst}(f) :\equiv \Pi_{x,y:A}\ f\,a = f\,b$

Recursion in path constructors makes elimination principles difficult to use! Do we actually need it?

Another HIT

Sequential colimit

Given an ω-chain

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots,$$

its *sequential colimit* A_ω is given by the constructors

$$\mathsf{in} : (n : \mathbb{N}) \to A_n \to A_\omega$$
$$\mathsf{glue} : (n : \mathbb{N}) \to (a : A_n) \to \mathsf{in}_n(a) =_{A_\omega} \mathsf{in}_{n+1}(f_n a)$$

This is a non-recursive HIT.

Higher inductive types (HITs)

Theorem (van Doorn)

The sequential colimit of

$$A \xrightarrow{\langle - \rangle} \langle\!\langle A \rangle\!\rangle \xrightarrow{\langle - \rangle} \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle \xrightarrow{\langle - \rangle} \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle \rangle\!\rangle \xrightarrow{\langle - \rangle} \ldots$$

is propositional (and has the elimination principle of $\|A\|$).

Note: $\langle\!\langle \ldots \langle\!\langle A \rangle\!\rangle \ldots \rangle\!\rangle$ is **very** complicated (homotopically).

Theorem (generalisation)

Given a chain $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots$. If every f_i is weakly constant, then A_ω is propositional (i.e. all its elements are equal).

Higher truncations

- For every $n \geq -2$, we can define the n-truncation $\|A\|_n$ as a HIT which trivialises all levels above n
- to be precise: $\|A\|$ is the case $n \equiv -1$ (index omitted)
- let's exchange the recursive constructor for a non-recursive one: we get general "pseudo-truncations" $\langle\!\langle A \rangle\!\rangle_n$, one constructor is $\langle - \rangle_n : A \to \langle\!\langle A \rangle\!\rangle_n$
- $\langle - \rangle_n$ is **not** weakly constant in general

Consider the chain

$$A \xrightarrow{\langle - \rangle_{-1}} \langle\!\langle A \rangle\!\rangle_{-1} \xrightarrow{\langle - \rangle_0} \langle\!\langle \langle\!\langle A \rangle\!\rangle_{-1} \rangle\!\rangle_0 \xrightarrow{\langle - \rangle_1} \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle_{-1} \rangle\!\rangle_0 \rangle\!\rangle_1 \xrightarrow{\langle - \rangle_2} \ldots$$

Theorem

Every $\langle - \rangle_n$ **in the above chain** is weakly constant (for inhabited A); and the sequential colimit of this chain has all the properties of $\|A\|$.

Chain of higher pseudo-truncations

- the constructed chain converges: after n steps, the first n levels are “correct” (i.e. “conditionally $(n-1)$-connected”
- implies finite elimination principles for truncated types
- our chain is more minimalistic than van Doorn’s – we get a strict natural transformation

$$
\begin{array}{ccccccc}
A & \xrightarrow{\langle - \rangle_{-1}} & \langle\!\langle A \rangle\!\rangle_{-1} & \xrightarrow{\langle - \rangle_{0}} & \langle\!\langle \langle\!\langle A \rangle\!\rangle_{-1} \rangle\!\rangle_{0} & \xrightarrow{\langle - \rangle_{1}} & \cdots \\
\vdots\downarrow & & \vdots\downarrow & & \vdots\downarrow & & \\
\langle\!\langle A \rangle\!\rangle & \xrightarrow{\langle - \rangle} & \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle & \xrightarrow{\langle - \rangle} & \langle\!\langle \langle\!\langle \langle\!\langle A \rangle\!\rangle \rangle\!\rangle \rangle\!\rangle & \xrightarrow{\langle - \rangle} & \cdots
\end{array}
$$

- can derive the finite elimination principles for the chaotic van Doorn chain (any cocone of the second chain gives one of the first).

Conclusion

Conjecture

Every HIT can be represented without recursive path-constructors.

- The general case is expected to be far more difficult.
- The conjecture is currently not even a precise statement: what is “every HIT”? – but that’s another topic. . .

Thank you for your attention!