Decidability and Semidecidability via Ordinals

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What are ordinal numbers?

One answer: **Numbers** for counting/ordering:

$$0, \quad 1, \quad 2, \quad 3, \quad \ldots \quad \omega, \quad \omega+1, \quad \omega+2, \quad \ldots \quad \omega\cdot 2+19, \ldots$$

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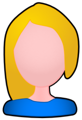

Alice
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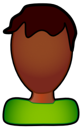

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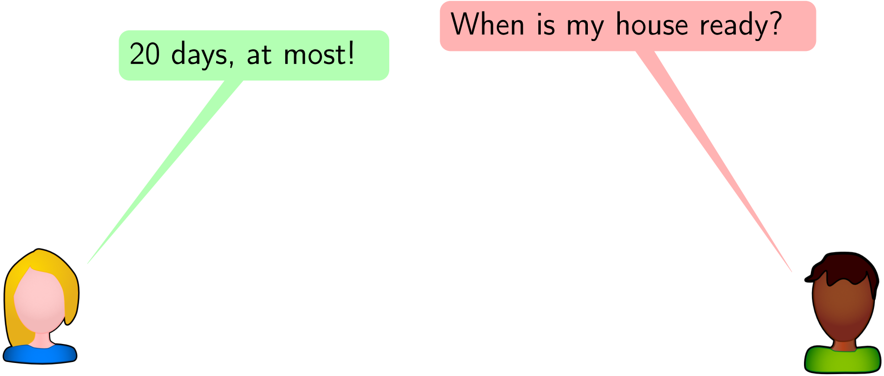

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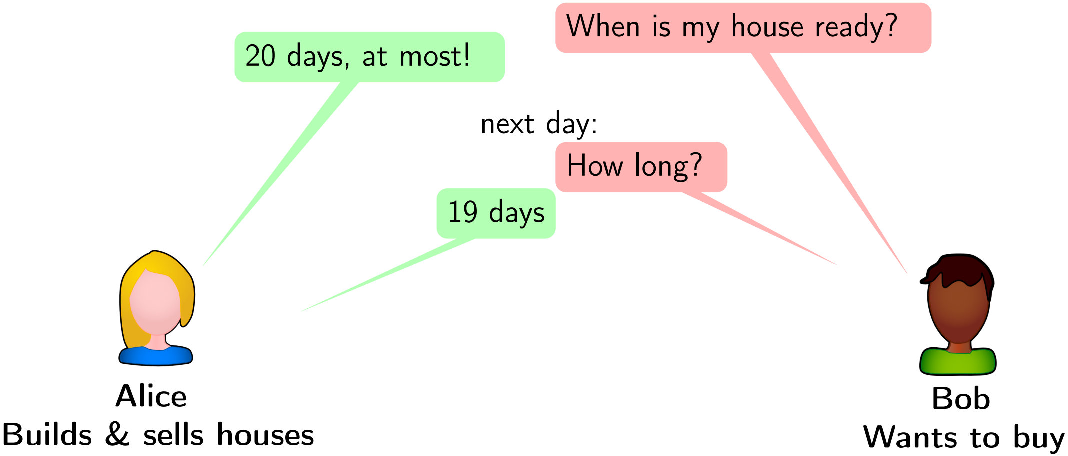

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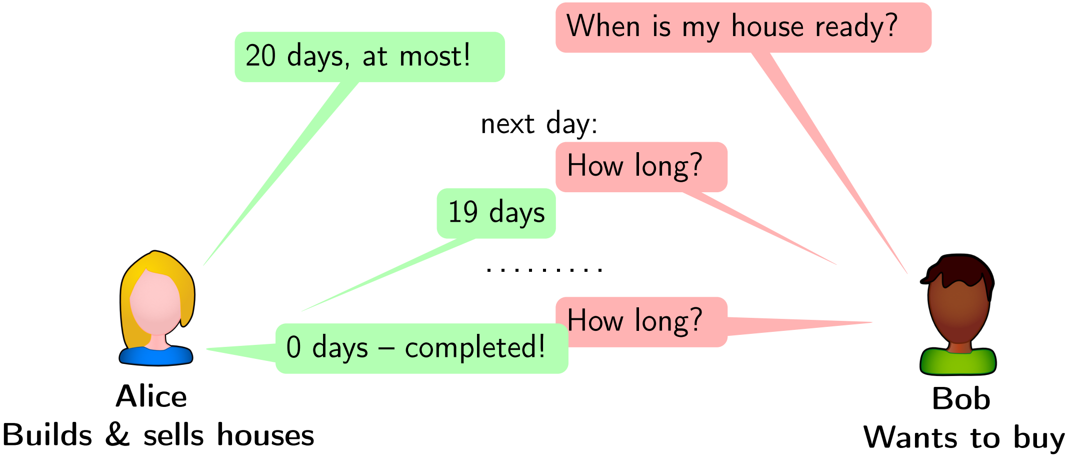

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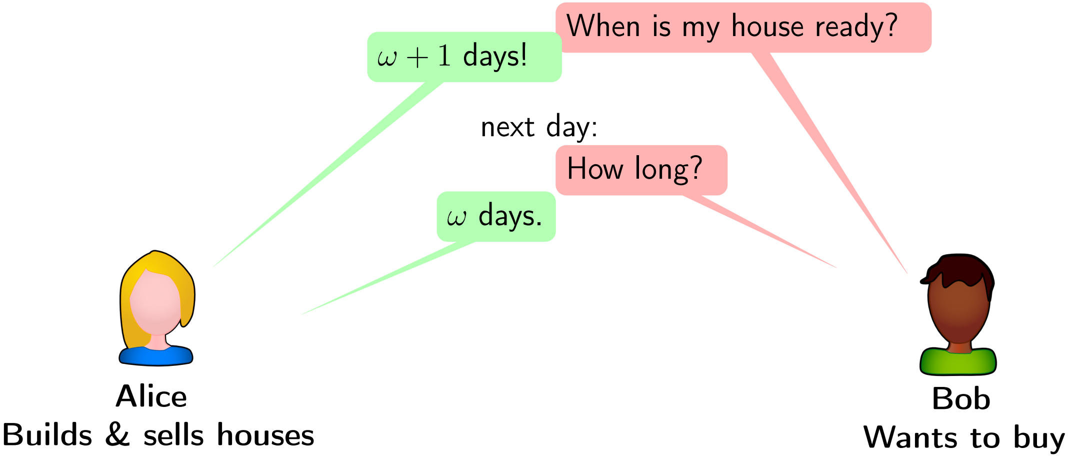

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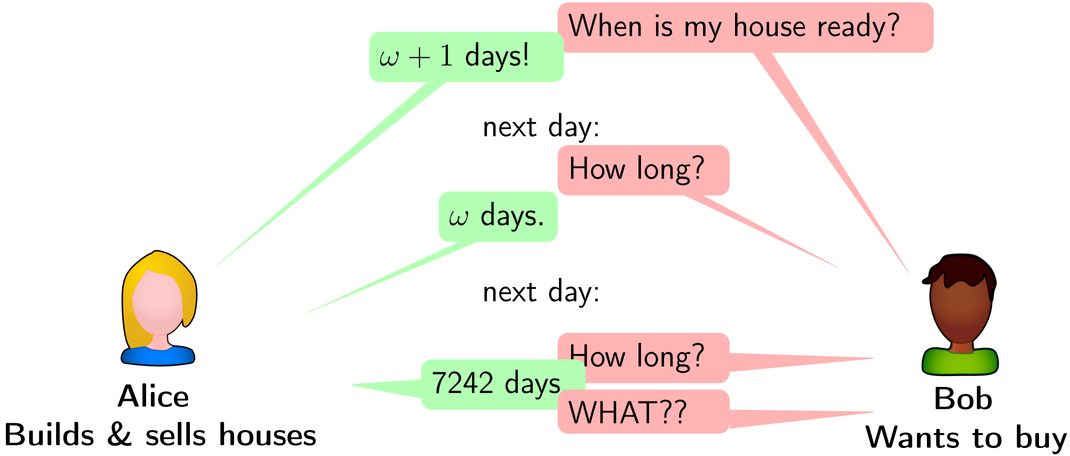

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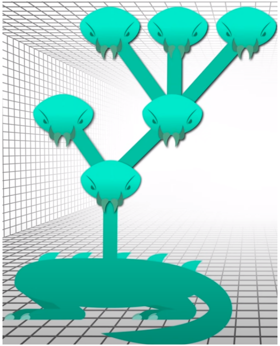

Hydra by Kirby and Paris 1982, and pictures by PBS Infinite Series, `https://youtu.be/uWwUpEY4c8o`

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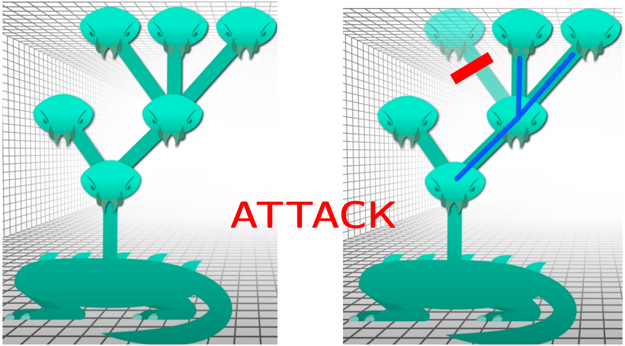

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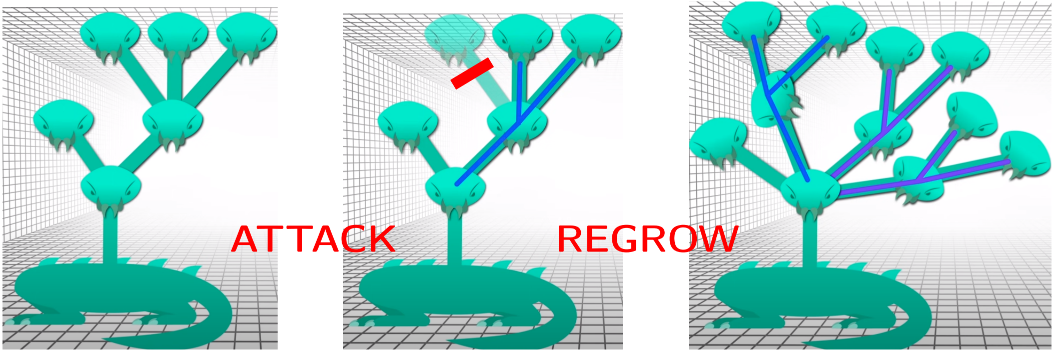

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Brouwer ordinal trees in constructive type theory

Inductive type $\mathcal{B}$ of Brouwer trees:

data $\mathcal{B}$ where

zero : $\mathcal{B}$

succ : $\mathcal{B} \to \mathcal{B}$

limit : $(\mathbb{N} \to \mathcal{B}) \to \mathcal{B}$

Then: Define $\omega := \mathsf{limit}(0, 1, 2, 3, \ldots)$

$\omega \cdot 2 := \mathsf{limit}(\omega, \omega + 1, \omega + 2, \ldots)$

and so on (addition, multiplication, exponentiation are standard).

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One problem (for our application): limit$(0, 1, 2, 3, \ldots) \neq$ limit$(1, 2, 3, \ldots)$

Our approach: induction-induction and path constructors, ensuring:

- Limits can only be taken of strictly increasing sequences;
- Bisimilar sequences have equal limits.

In cubical Agda:

```
data Brw where
  zero  : Brw
  succ  : Brw → Brw
  limit : (f : ℕ → Brw) → {f↑ : increasing f} → Brw
  bisim : ∀ f {f↑} g {g↑} →
          f ≈ g →
          limit f {f↑} ≡ limit g {g↑}
  trunc : isSet Brw

data _≤_ where
  ≤-zero      : ∀ {x} → zero ≤ x
  ≤-trans     : ∀ {x y z} → x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : ∀ {x y} → x ≤ y → succ x ≤ succ y
  ≤-cocone    : ∀ {x} f {f↑ k} → (x ≤ f k) → (x ≤ limit f {f↑})
  ≤-limiting  : ∀ f {f↑ x} → ((k : ℕ) → f k ≤ x) → limit f {f↑} ≤ x
  ≤-trunc     : ∀ {x y} → isProp (x ≤ y)
```

note: $x < y$
means succ $x \leq y$

Everything that one can "reasonably expect" works: $<$ is wellfounded, $\leq$ is anti-symmetric, limits are actually limits, arithmetic operations work, and so on.

Decidability properties

P is *decidable* if we can prove $P \uplus \neg P$.

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   Sure: No for zero and limits; for succ y, check whether $y = 16$.

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4. $x > \omega$?

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4. $x > \omega$?

   Can decide it for zero and succ, but: limit$(x_0, x_1, x_2, \ldots) > \omega$?

When is $\mathsf{limit}(x_0, x_1, x_2, \ldots) > \omega$?

- For any i, we can check whether x_i is finite.
- As soon as we discover an infinite x_i, the question is decided positively.
- Only if all x_i are finite, the answer is negative.

$\Rightarrow$ *Semidecidable*.

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$\Rightarrow$ *Semidecidable*.

Definition (Bauer 2006): P is semidecidable if

$$\exists (s : \mathbb{N} \to \mathsf{Bool}).\ P \leftrightarrow \exists k. s_k = \mathsf{true}$$

(Note: $\exists (x : A). B(x)$ means $\| \Sigma (x : A). B(x) \|$.)

Fact: For any x, the question $x > \omega$ is semidecidable.

The other direction

Given $s : \mathbb{N} \to \mathsf{Bool}$, we can construct an increasing sequence f by:

$$f\,0 :\equiv \mathsf{zero}$$

$$f\,(n+1) = \begin{cases} (f\,n) + \omega & \textit{if } n \textit{ is [minimal] such that } s_n = \mathsf{true} \\ \mathsf{succ}(f\,n) & \textit{else.} \end{cases}$$

Then: $(\mathsf{limit}\, f > \omega) \leftrightarrow (\exists k. s_k = \mathsf{true})$.

Semidecidability via ordinals

Via these translations: For any proposition P,

$\exists(y : \mathcal{B}).P \leftrightarrow (y > \omega)$ $\longleftrightarrow$ $\exists(s : \mathbb{N} \to \mathsf{Bool}).\ P \leftrightarrow \exists k.s_k = \mathsf{true}$

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What if we swap ω for another ordinal α?

$\exists(y:\mathcal{B}).P \leftrightarrow (y > \alpha)$ *"decidable in α steps"*

(or $y \geq \alpha$, $y = \alpha$, any $Q(y)$, $\ldots$)

Fewer than ω steps

Let n be a natural number. Then:

$\exists (y : \mathcal{B}). P \leftrightarrow (y > n)$	$\longleftrightarrow$	$P \uplus \neg P$
"P decidable in n steps"		"P decidable"

More than ω steps – an example

Twin prime conjecture (TPC):

There are arbitrarily large numbers p such that p and $p + 2$ are both prime.

It's clearly semidecidable whether there is a twin pair $> 10^{1,000,000}$, but TPC doesn't seem to be semidecidable.

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However, one can show:

$$\exists (y : \mathcal{B}). \text{TPC} \leftrightarrow (y = \omega^2)$$

"TPC is decidable in ω^2 steps."
(Note: $=$ can be replaced by $>$ or $\geq$.)

TPC's ordinal

Define a sequence $f : \mathbb{N} \to \mathcal{B}$ by:

$$f\,0 :\equiv \mathsf{zero}$$
$$f\,(n+1) = \begin{cases} (f\,n) + \omega & \textit{if } n \textit{ and } n+2 \textit{ are prime} \\ (f\,n) + 1 & \textit{else.} \end{cases}$$

Claim: TPC $\leftrightarrow$ $\mathsf{limit}\, f = \omega^2$ $[\leftrightarrow$ $\mathsf{succ}(\mathsf{limit}\, f) > \omega^2]$

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Claim: TPC $\leftrightarrow$ limit $f = \omega^2$ $[\leftrightarrow$ succ(limit $f) > \omega^2]$

Sketch TPC $\to$ (limit $f = \omega^2$):

For any n, we find k s.t. there are at least n twin prime pairs below k, thus $f_k \geq \omega \cdot k$, thus limit $f \geq \omega \cdot \omega$.
At the same time, f never exceeds ω^2.

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Sketch (limit $f \geq \omega^2) \to$ TPC:

$$\begin{aligned} (\text{limit } f \geq \omega^2) \quad &\Rightarrow \quad \exists k.f_k \geq \omega \cdot n \\ &\Rightarrow \quad \exists k.\neg\neg(\text{There are at least } n \text{ twin primes } \leq k) \\ &\Rightarrow \quad \exists k.\text{There are at least } n \text{ twin primes } \leq k \\ &\Rightarrow \quad \Sigma k : \mathbb{N}.\text{There are at least } n \text{ twin primes } \leq k \\ &\Rightarrow \quad \text{There is a twin prime pair above } n \end{aligned}$$

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Claim: TPC $\leftrightarrow$ limit $f = \omega^2$ $[\leftrightarrow$ succ(limit $f) > \omega^2]$

Thanks for your attention!

(limit $f \geq \omega^2$) $\Rightarrow$ $\exists k. f_k \geq \omega \cdot n$
$\Rightarrow$ $\exists k. \neg\neg$(There are at least n twin primes $\leq k$)
$\Rightarrow$ $\exists k$.There are at least n twin primes $\leq k$
$\Rightarrow$ $\Sigma k : \mathbb{N}$.There are at least n twin primes $\leq k$
$\Rightarrow$ There is a twin prime pair above n