Internal ∞-Categorical Models of Dependent Type Theory

Towards 2LTT Eating HoTT

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LICS'21 (Rome/online), 29 June 2021

Why?

Goal: Define what a model of type theory is
– in type theory!
(in particular: intended initial model $\sim$ "syntax")

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```
record CwF : Set₁ where
  field
    Con : Set
    Sub : Con → Con → Set
    Ty  : Con → Set
    Tm  : (Γ : Con) → Ty Γ → Set

    •   : Con
    _▸_ : (Γ : Con) → Ty Γ → Con

    -- (and so on)
```

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    -- (and so on)
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CwF definition as a Generalised Algebraic Theory

$$
\begin{array}{lcl}
\mathsf{Con} & : & \mathsf{Type} \\
\mathsf{Sub} & : & \mathsf{Con} \to \mathsf{Con} \to \mathsf{Type} \\
\_ \diamond \_ & : & \mathsf{Sub}\,\Theta\,\Delta \to \mathsf{Sub}\,\Gamma\,\Theta \to \mathsf{Sub}\,\Gamma\,\Delta \\
\mathsf{assoc} & : & (\sigma \diamond \delta) \diamond \nu = \sigma \diamond (\delta \diamond \nu) \\
\mathsf{id} & : & \mathsf{Sub}\,\Gamma\,\Gamma \\
\mathsf{idl}_\sigma & : & \mathsf{id} \diamond \sigma = \sigma \\
\mathsf{idr}_\sigma & : & \sigma \diamond \mathsf{id} = \sigma \\
\bullet & : & \mathsf{Con} \\
\epsilon & : & \mathsf{Sub}\,\Gamma\,\bullet \\
\bullet\eta & : & \forall(\sigma : \mathsf{Sub}\,\Gamma\,\bullet).\ \sigma = \epsilon \\
\mathsf{Ty} & : & \mathsf{Con} \to \mathsf{Type} \\
\_[\_]^{\mathsf{T}} & : & \mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma \\
[\mathsf{id}]^{\mathsf{T}} & : & A[\mathsf{id}]^{\mathsf{T}} = A \\
[\diamond]^{\mathsf{T}} & : & A[\sigma \diamond \delta]^{\mathsf{T}} = A[\sigma]^{\mathsf{T}}[\delta]^{\mathsf{T}}
\end{array}
$$

$$
\begin{array}{lcll}
\mathsf{Tm} & : & (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type} & \\
\_[\_]^{\mathsf{t}} & : & \mathsf{Tm}\,\Delta\,A \to (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}}) & \\
[\mathsf{id}]^{\mathsf{t}} & : & t[\mathsf{id}]^{\mathsf{t}} = t & \text{over } [\mathsf{id}]^{\mathsf{T}} \\
[\diamond]^{\mathsf{t}} & : & t[\sigma \diamond \delta]^{\mathsf{t}} = t[\sigma]^{\mathsf{t}}[\delta]^{\mathsf{t}} t & \text{over } [\diamond]^{\mathsf{T}} \\
\_ \triangleright \_ & : & (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con} & \\
\mathsf{p} & : & \mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma & \\
\mathsf{q} & : & \mathsf{Tm}\,(\Gamma \triangleright A)\,(A[\mathsf{p}]^{\mathsf{T}}) & \\
\_\,,\,\_ & : & (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}}) \to \mathsf{Sub}\,\Gamma\,(\Delta \triangleright A) & \\
\triangleright\beta_1 & : & \mathsf{p} \diamond (\sigma, t) = \sigma & \\
\triangleright\beta_2 & : & \mathsf{q}[\sigma, t]^{\mathsf{t}} = tt & \text{over } [\diamond]^{\mathsf{T}} \text{ and } \triangleright\beta_1 \\
\triangleright\eta & : & (\mathsf{p}, \mathsf{q}) = \mathsf{id} & \\
,\diamond & : & (\sigma, t) \diamond \nu = (\sigma \diamond \nu, t[\nu]^{\mathsf{t}})t & \text{over } [\diamond]^{\mathsf{T}}
\end{array}
$$

(Good definition in a
type theory with K/UIP)

CwF definition as a Generalised Algebraic Theory

Con	:	Type	
Sub	:	$\mathsf{Con} \to \mathsf{Con} \to \mathsf{Type}$	
$\_ \diamond \_$	:	$\mathsf{Sub}\,\Theta\,\Delta \to \mathsf{Sub}\,\Gamma\,\Theta \to \mathsf{Sub}\,\Gamma\,\Delta$	
assoc	:	$(\sigma \diamond \delta) \diamond \nu = \sigma \diamond (\delta \diamond \nu)$	
id	:	$\mathsf{Sub}\,\Gamma\,\Gamma$	category
idl_σ	:	$\mathsf{id} \diamond \sigma = \sigma$	
idr_σ	:	$\sigma \diamond \mathsf{id} = \sigma$	
$\bullet$	:	Con	
ϵ	:	$\mathsf{Sub}\,\Gamma\,\bullet$	
$\bullet\eta$	:	$\forall(\sigma : \mathsf{Sub}\,\Gamma\,\bullet).\ \sigma = \epsilon$	
Ty	:	$\mathsf{Con} \to \mathsf{Type}$	
$\_[\_]^{\mathsf{T}}$	:	$\mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma$	
$[\mathsf{id}]^{\mathsf{T}}$	:	$A[\mathsf{id}]^{\mathsf{T}} = A$	
$[\diamond]^{\mathsf{T}}$	:	$A[\sigma \diamond \delta]^{\mathsf{T}} = A[\sigma]^{\mathsf{T}}[\delta]^{\mathsf{T}}$	

Tm	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type}$	
$\_[\_]^{\mathsf{t}}$	:	$\mathsf{Tm}\,\Delta\,A \to (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}})$	
$[\mathsf{id}]^{\mathsf{t}}$	:	$t[\mathsf{id}]^{\mathsf{t}} = t$	over $[\mathsf{id}]^{\mathsf{T}}$
$[\diamond]^{\mathsf{t}}$	:	$t[\sigma \diamond \delta]^{\mathsf{t}} = t[\sigma]^{\mathsf{t}}[\delta]^{\mathsf{t}} t$	over $[\diamond]^{\mathsf{T}}$
$\_ \triangleright \_$	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}$	
p	:	$\mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma$	
q	:	$\mathsf{Tm}\,(\Gamma \triangleright A)\,(A[\mathsf{p}]^{\mathsf{T}})$	
$\_ , \_$	:	$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}}) \to \mathsf{Sub}\,\Gamma\,(\Delta \triangleright A)$	
$\triangleright\beta_1$	:	$\mathsf{p} \diamond (\sigma, t) = \sigma$	
$\triangleright\beta_2$	:	$\mathsf{q}[\sigma, t]^{\mathsf{t}} = tt$	over $[\diamond]^{\mathsf{T}}$ and $\triangleright\beta_1$
$\triangleright\eta$	:	$(\mathsf{p}, \mathsf{q}) = \mathsf{id}$	
$, \diamond$	:	$(\sigma, t) \diamond \nu = (\sigma \diamond \nu, t[\nu]^{\mathsf{t}})t$	over $[\diamond]^{\mathsf{T}}$

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category

$\bullet$	:	Con
ϵ	:	$\mathsf{Sub}\,\Gamma\,\bullet$
$\bullet\eta$	:	$\forall(\sigma : \mathsf{Sub}\,\Gamma\,\bullet).\ \sigma = \epsilon$

terminal object

Ty	:	Con $\to$ Type
$\_[\_]^{\mathsf{T}}$	:	$\mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma$
$[\mathsf{id}]^{\mathsf{T}}$	:	$A[\mathsf{id}]^{\mathsf{T}} = A$
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p	:	$\mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma$	
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presheaf

Tm	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type}$	
$\_[\_]^{\mathsf{t}}$	:	$\mathsf{Tm}\,\Delta\,A \to (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}})$	
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category

$\bullet : \mathsf{Con}$

$\epsilon : \mathsf{Sub}\,\Gamma\,\bullet$

$\bullet\eta : \forall(\sigma : \mathsf{Sub}\,\Gamma\,\bullet).\ \sigma = \epsilon$

terminal object

$\mathsf{Ty} : \mathsf{Con} \to \mathsf{Type}$

$\_[\_]^{\mathsf{T}} : \mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma$

$[\mathsf{id}]^{\mathsf{T}} : A[\mathsf{id}]^{\mathsf{T}} = A$

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$\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type}$

$\_[\_]^{\mathsf{t}} : \mathsf{Tm}\,\Delta\,A \to (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}})$

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another functor

$\_\triangleright\_ : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}$

$\mathsf{p} : \mathsf{Sub}\,(\Gamma \triangleright A)\,\Gamma$

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context extension

$\_\triangleright\_$	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}$	
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(Good definition in a
type theory with K/UIP)

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$[\diamond]^{\mathsf{t}}$	:	$t[\sigma \diamond \delta]^{\mathsf{t}} = t[\sigma]^{\mathsf{t}}[\delta]^{\mathsf{t}} t$	over $[\diamond]^{\mathsf{T}}$
		$\Gamma\,(\Delta \triangleright A)$	
$\triangleright\beta_2$	:	$\mathsf{q}[\sigma, t]^{\mathsf{t}} = tt$	over $[\diamond]^{\mathsf{T}}$ and $\triangleright\beta_1$
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See e.g.:

- **Altenkirch and Kaposi, *Type Theory in Type Theory using Quotient Inductive Types*, 2016**
- **Kaposi, Huber, and Sattler, *Gluing for Type Theory*, 2019**

(Good definition in a
type theory with K/UIP)

First example: the **syntax** / (intended) **initial CwF**

Possible implementation:

(I) via **raw syntax**

- possibly ill-typed expressions plus wellformedness predicates

$\Rightarrow$ Initial by the **Initiality Theorem** (Brunerie, de Boer, Lumsdaine, Mörtberg 2019–20).

(II) via a **a quotient inductive-inductive type** (Altenkirch-Kaposi 2016)

- mutually defined inductive families Con, Sub, Ty, Tm
- a constructor for every component of the previous

$\Rightarrow$ Initial by construction.

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Second example of a CwF: **"Standard Model"**, a.k.a.
the universe with the obvious structure

- Con is the universe $\mathcal{U}$
- $\mathsf{Sub}\,\Gamma\,\Delta$ is the function type $(\Gamma \to \Delta)$
- $\mathsf{Ty}\,\Gamma$ is given as $(\Gamma \to \mathcal{U})$
- $\mathsf{Tm}\,\Gamma\,A$ is given as $\Pi(x:\Gamma).(A\,x)$
- all operations are canonical
- all equations hold judgmentally (assuming enough η-laws)

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e.g. in Agda

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Recall: **UIP** (*uniqueness of identity proofs*) a.k.a. **Axiom K** says:

$$\mathbf{\Pi(x\,y : A).\Pi(p\,q : x = y).(p = q)}$$

The above definition of a CwF works assuming this axiom!

What if UIP is not assumed (or even inconsistent, e.g. in homotopy type theory)?

Two obvious approaches:

(I) Ignore it: Do everything as before.

or

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But then:

$$\mathsf{idl}_\sigma : \mathsf{id} \diamond \sigma = \sigma$$
$$\mathsf{idr}_\sigma : \sigma \diamond \mathsf{id} = \sigma$$

Initial model (w/ base types) does **not** satisfy $\mathsf{idl}_{\mathsf{id}} = \mathsf{idr}_{\mathsf{id}}$.

$\Rightarrow$ Initial model is **not** based on h-sets & does **not** have decidable equality.

$\Rightarrow$ The "syntax" (first example) is not initial.

(II) Bake UIP into the definition of CWF: Require Con etc. to be h-sets.

Typical "HoTT solution".

But: The universe is not an h-set.

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Why we really want both examples (syntax and standard model)

Shulman 2014:
Is the n^{th} universe a model of HoTT with (n-1) universes?

I.e.: Can we define the syntax and *interpret* it in $\mathcal{U}_n$?

Work by: Escardó-Xu, K., Bucholtz, Lumsdaine, Kaposi-Kovaćs, Altenkirch, ...

However: Even the simplest[1] version of this is still open!

[1] (where the core problem occurs)

The two examples would give us:

Syntax (raw) ⟷ (initiality theorem) Syntax (as QIIT) ⟶ (by initiality) universe $\mathcal{U}$ (standard model)

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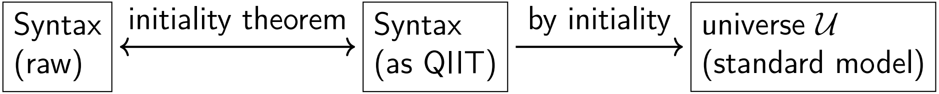

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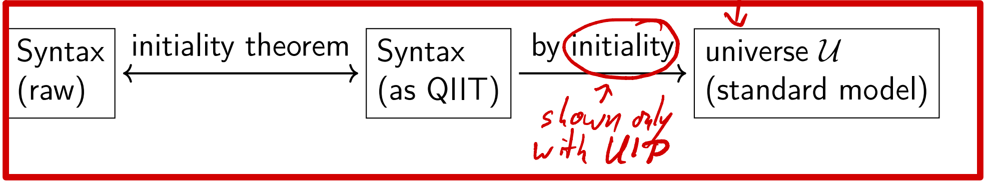

Back to the definition from slide 4:

Con	:	Type
Sub	:	$\mathsf{Con} \to \mathsf{Con} \to \mathsf{Type}$
$\_ \diamond \_$	:	$\mathsf{Sub}\,\Theta\,\Delta \to \mathsf{Sub}\,\Gamma\,\Theta \to \mathsf{Sub}\,\Gamma\,\Delta$
assoc	:	$(\sigma \diamond \delta) \diamond \nu = \sigma \diamond (\delta \diamond \nu)$
id	:	$\mathsf{Sub}\,\Gamma\,\Gamma$
idl_σ	:	$\mathsf{id} \diamond \sigma = \sigma$
idr_σ	:	$\sigma \diamond \mathsf{id} = \sigma$
$\bullet$	:	Con
ϵ	:	$\mathsf{Sub}\,\Gamma\,\bullet$
$\bullet\eta$	:	$\forall(\sigma : \mathsf{Sub}\,\Gamma\,\bullet).\ \sigma = \epsilon$
Ty	:	$\mathsf{Con} \to \mathsf{Type}$
$\_[\_]^{\mathsf{T}}$	:	$\mathsf{Ty}\,\Delta \to \mathsf{Sub}\,\Gamma\,\Delta \to \mathsf{Ty}\,\Gamma$
$[\mathsf{id}]^{\mathsf{T}}$	:	$A[\mathsf{id}]^{\mathsf{T}} = A$
$[\diamond]^{\mathsf{T}}$	:	$A[\sigma \diamond \delta]^{\mathsf{T}} = A[\sigma]^{\mathsf{T}}[\delta]^{\mathsf{T}}$

$\mathsf{idl}_{\mathsf{id}} \overset{?}{=} \mathsf{idr}_{\mathsf{id}}$

Tm	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type}$	
$\_[\_]^{\mathsf{t}}$	:	$\mathsf{Tm}\,\Delta\,A \to (\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}})$	
$[\mathsf{id}]^{\mathsf{t}}$	:	$t[\mathsf{id}]^{\mathsf{t}} = t$	over $[\mathsf{id}]^{\mathsf{T}}$
$[\diamond]^{\mathsf{t}}$	:	$t[\sigma \diamond \delta]^{\mathsf{t}} = t[\sigma]^{\mathsf{t}}[\delta]^{\mathsf{t}} t$	over $[\diamond]^{\mathsf{T}}$
$\_ \rhd \_$	:	$(\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}$	
p	:	$\mathsf{Sub}\,(\Gamma \rhd A)\,\Gamma$	
q	:	$\mathsf{Tm}\,(\Gamma \rhd A)\,(A[\mathsf{p}]^{\mathsf{T}})$	
$\_ , \_$	:	$(\sigma : \mathsf{Sub}\,\Gamma\,\Delta) \to \mathsf{Tm}\,\Gamma\,(A[\sigma]^{\mathsf{T}}) \to \mathsf{Sub}\,\Gamma\,(\Delta \rhd A)$	
$\rhd\beta_1$	:	$\mathsf{p} \diamond (\sigma, t) = \sigma$	
$\rhd\beta_2$	:	$\mathsf{q}[\sigma, t]^{\mathsf{t}} = tt$	over $[\diamond]^{\mathsf{T}}$ and $\rhd\beta_1$
$\rhd\eta$	:	$(\mathsf{p}, \mathsf{q}) = \mathsf{id}$	
$,\diamond$	:	$(\sigma, t) \diamond \nu = (\sigma \diamond \nu, t[\nu]^{\mathsf{t}}) t$	over $[\diamond]^{\mathsf{T}}$

Goal: Make this coherent! E.g. we really need $\mathsf{idl}_{\mathsf{id}} = \mathsf{idr}_{\mathsf{id}}$.

Brutal method: Require h-sets everywhere (too restrictive).

Proposed method: Use higher categories $\Longrightarrow$ $(\infty, 1)$-CwF's.

How?

As discussed above: A 1-CwF consists of

- a category $\mathcal{C}$ of contexts and substitutions
- a presheaf of types
- another functor for terms
- a context extension operation.

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What is an ∞-category? Model used: Rezk's Segal spaces.

Strategy:

(1) Start with a *semisimplicial type* ("basic data")

(2) Add Segal condition ($\Rightarrow$ ∞-semicategory)

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propositional properties!

(1) Recall: semisimplicial type up to dimension 2 is tuple (A_0, A_1, A_2) where

$$A_0 : \mathsf{Type}$$

$$A_1 : A_0 \to A_0 \to \mathsf{Type}$$

$$A_2 : \Pi\{x\,y\,z : A_0\}.\,(A_1\,x\,y) \to (A_1\,y\,z) \to (A_1\,x\,z) \to \mathsf{Type}$$

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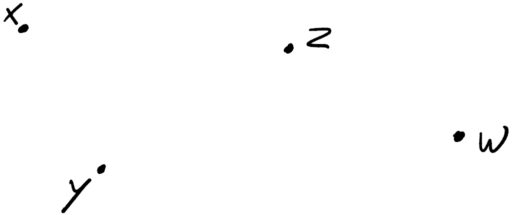

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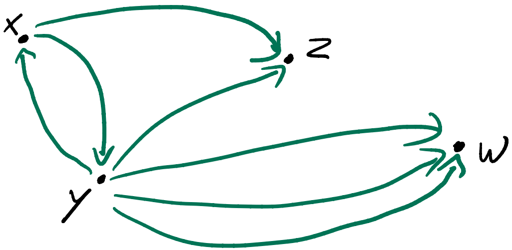

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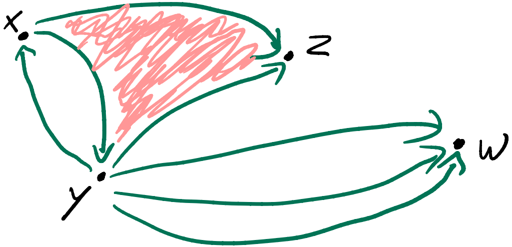

(2) Adding the Segal condition

Semicategory (beginning)

Ob : Type

Hom : Ob $\to$ Ob $\to$ Type

_ $\circ$ _ : $\{x\, y\, z : \mathsf{Ob}\} \to (\mathsf{Hom}\, y\, z) \to (\mathsf{Hom}\, x\, y) \to (\mathsf{Hom}\, x\, z)$

Semisimplicial type (beginning)

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In previous work: *Completeness* (Lurie/Harpaz/Capriotti) corresponding to univalent identities (cf. Capriotti-Kraus 2018).

Here: We don't want built-in univalence. Instead:

Def: A line $f : A_1\, x\, x$ is a *good identity* if it is an *idempotent equivalence*.

Def: f is *idempotent* if $A_2\, f\, f\, f$.

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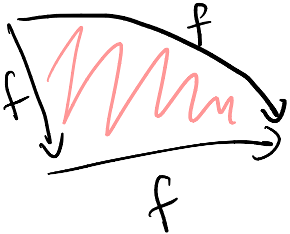

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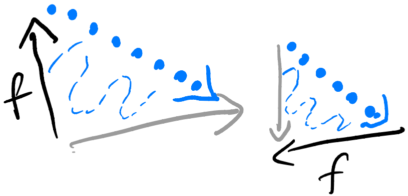

Definition: A semicategory (higher semicategory, semi-Segal type) has a *good identity structure* if every object (point) is equipped with an *idempotent equivalence*.

Theorem: "Having a good identity structure":
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