Induction for Cycles

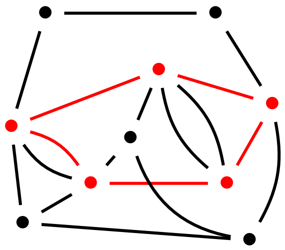
Nicolai Kraus

jww Jakob von Raumer

Types in Munich '20
(online substitution thereof)
11 March 2020

based on arxiv.org/abs/2001.07655

General Problem



Consider paths in a graph.

If we want to prove a property...

– *for all paths:* **Induction!**

– *for all closed paths:* **how???**

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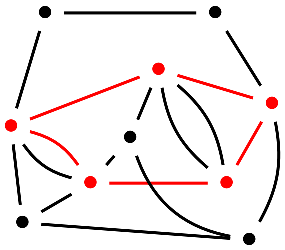

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If we want to prove a property...

– *for all paths:* **Induction!**

– *for all closed paths:* **how???**

Aim of this project:
approach for a special case
+ applications in HoTT.

Quotients in Type Theory (Hofmann)

Given: $A : \mathsf{Set}$
$\sim\, : A \to A \to \mathsf{Set}$

We get: $A/{\sim} : \mathsf{Set}$

Property: for $B : \mathsf{Set}$,

$$\simeq \quad \frac{\begin{array}{c} f : A \to B \\ h : (a_1 \sim a_2) \to f(a_1) = f(a_2) \end{array}}{g : (A/{\sim}) \to B}$$

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In *homotopy type theory:*

All of this is for *sets* (aka 0-truncated types, types satisfying UIP), *"set-quotients"*

What if B is only 1-truncated (e.g. the universe of *sets*)?

Set-Quotients in HoTT

Given: A : Type

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Property: for B : 1-Type,

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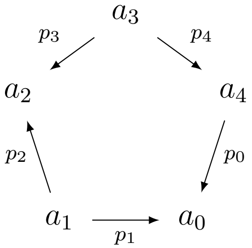
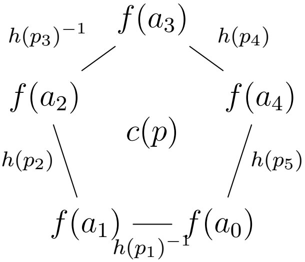


Its image in B:



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Pro

Instance of the general problem!!

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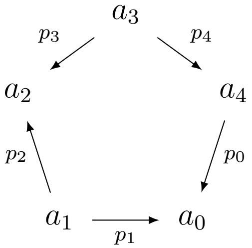

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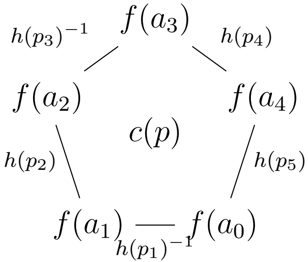

An Example in HoTT

Given: M : Set
Want: *Free Group* on M

In Sets (ordinary free group):

Set-quotient List$(M + M)/\sim$

$$[x_0, \dots, x_{k-1}, x_k, x_k^{-1}, x_{k+1}, \dots, x_n]$$

$$\sim$$

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Higher-categorical free group:

Ω(hcolim$(M \rightrightarrows 1)$)

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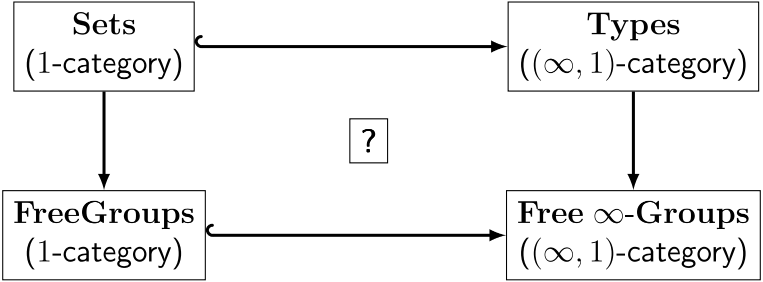
Higher-categorical free group:

$\Omega(\mathsf{hcolim}(M \rightrightarrows 1))$

$H :\equiv \mathsf{hcolim}(M \rightrightarrows 1)$
can be implemented as a higher inductive type:

inductive H
  base : H
  loops : $M \to \mathsf{base} = \mathsf{base}$

Free Groups



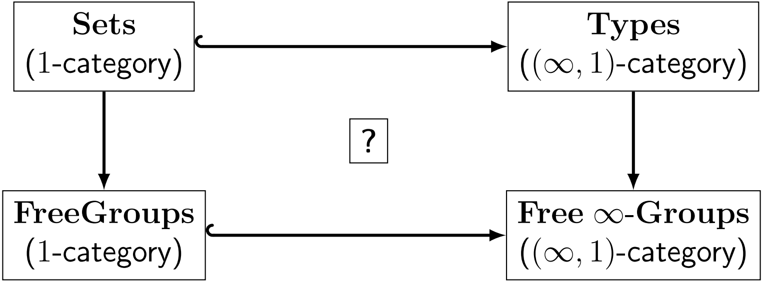
$$\mathsf{List}(M + M)/\sim \qquad \simeq \qquad \Omega(\mathsf{hcolim}(M \rightrightarrows 1))$$

?

Yes, with excluded middle.

Unknown (conjecture: independent) otherwise.

Free Groups



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Needed: map from set-quotient into (a priori) higher type!

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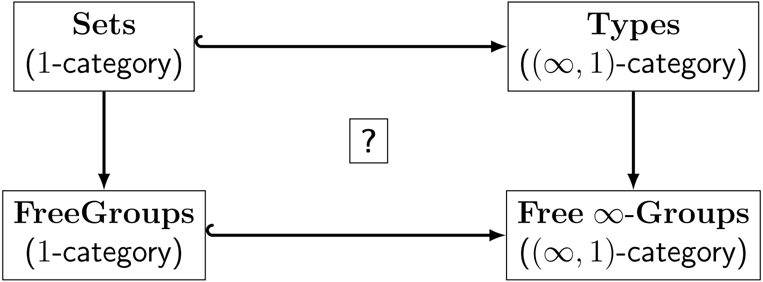

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Needed: map from set-quotient into (a priori) higher type!
First approximation: Does $\Omega(\text{hcolim}(M \rightrightarrows 1))$ have trivial fundamental groups? ($\rightsquigarrow \|\Omega(\text{hcolim}(M \rightrightarrows 1))\|_1$)

What would we need?

Recall: $\mathsf{List}(M + M)/\!\sim \quad \to \quad \|\Omega(\mathsf{hcolim}(M \rightrightarrows 1))\|_1$

is given by:

$f : \mathsf{List}(M + M) \to \|\Omega(\mathsf{hcolim}(M \rightrightarrows 1))\|_1$
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$c : h$ *maps every closed zig-zag to reflexivity*

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easy parts:
$f([+m_0, -m_1, +m_2]) :\equiv \mathsf{loops}(m_0) \cdot \mathsf{loops}(m_1)^{-1} \cdot \mathsf{loops}(m_2)$
h : *(use that inverses cancel, recall the def of* $\sim$:
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c : *(should be true, but how to prove it?)*

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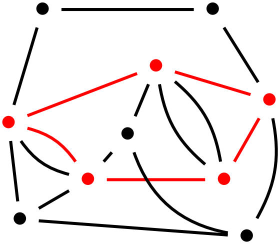

Problem: Prove a property *for every cycle* in a graph.

Assumption: The graph is given by the symmetric closure of a relation

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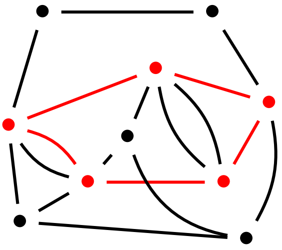

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- locally confluent
- Noetherian (co-wellfounded).

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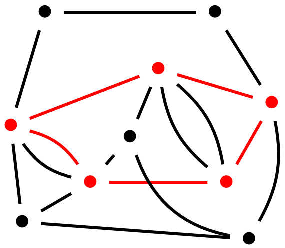

Problem: Prove a property *for every cycle* in a graph.

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Our proposed solution:

1. Given a relation $\rightsquigarrow$ on A, we define a new relation $\rightsquigarrow^{\circ}$ on cycles $a \rightsquigarrow^{s*} a$.
2. If $\rightsquigarrow$ is Noetherian, then so is $\rightsquigarrow^{\circ}$.
3. If $\rightsquigarrow$ further is locally confluent, then any cycle can be split into a $\rightsquigarrow^{\circ}$-smaller cycle and a confluence cycle

$\Rightarrow$ Induction is possible!

Step 1

Definition. Let $\rightsquigarrow$ be a relation on A.
Then, $\rightsquigarrow^L$ on $\mathsf{List}(A)$ is generated by

$$[\vec{a_1}, a, \vec{a_2}] \rightsquigarrow^L [\vec{a_1}, x_0, x_1, \ldots, x_k, \vec{a_2}]$$

for all x_i with $a \rightsquigarrow x_i$.

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3. If every a_i is $\rightsquigarrow$-accessible, then $[a_0, \ldots, a_n]$ is $\rightsquigarrow^L$-accessible. (Proof: first point.)

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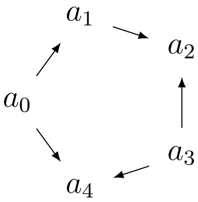
Lemma. ($\rightsquigarrow$ Noetherian) $\Rightarrow$
(any cycle is either empty or contains a span).



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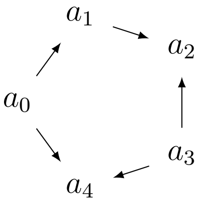

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Definition. For γ a cycle, write $\varphi(\gamma)$ for the *vertex sequence* of γ.
Write $\gamma \rightsquigarrow^{\circ} \delta$ if $\varphi(\gamma) \rightsquigarrow^{L} \varphi(\delta')$ for any rotation δ' of δ.

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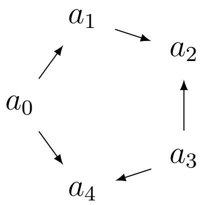

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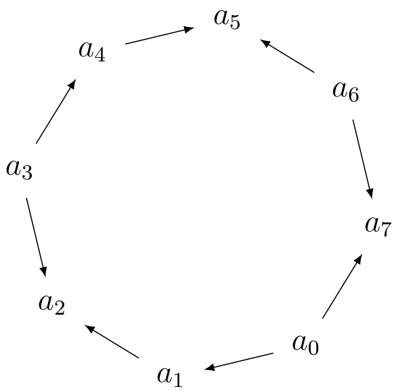

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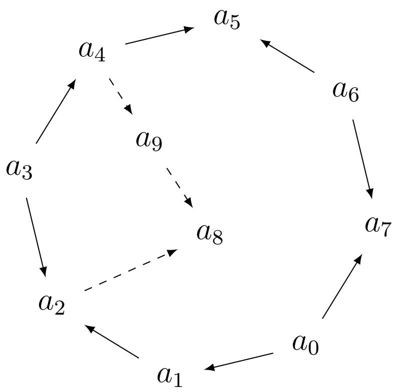

Back to type theory: consequence

Theorem (Noetherian Cycle Induction).

Given: A : Type
$(\rightsquigarrow) : A \to A \to$ Type
P : *cycles* $\to$ Type.

Assume further:

- relation $\rightsquigarrow$ Noetherian and locally confluent
- P stable under rotating of cycles:
  $P(\gamma) \to P($*some rotation of* $\gamma)$
- P stable under "merging" of cycles:
  $P(\alpha) \to P(\beta) \to P(\alpha + \gamma)$

Then: $P($*empty*$)$ and $P($*confluence cycle*$)$ $\Rightarrow$ $P($*any cycle*$)$.

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Then: $P($*empty*$)$ and $P($*confluence cycle*$) \;\Rightarrow\; P($*any cycle*$)$.

These conditions are easily checked in our HoTT-examples, where $P(\gamma) :\equiv$ *the cycle* γ *is mapped to a trivial equality*.

Conclusions

- Paper: NK and Jakob von Raumer, *Coherence via Wellfoundedness*, arxiv.org/abs/2001.07655.
- Can show approximations to other open questions in HoTT with this.
- Non-type theoretic applications? E.g. in graph rewriting, cf. Michael Löwe, *Van-Kampen pushouts for sets and graphs*, 2010.
- Formalised in Lean (great job by Jakob!).

Thanks!