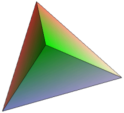
On the Role of
Semisimplicial Types

An open problem in
homotopy type theoy



Nicolai Kraus
Braga, 18 June'18

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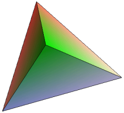

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What are semisimplicial types? What's the open problem?

Why is this important?

Semisimplicial types

A *semisimplicial type* restricted to level 2 is a triple (A_0, A_1, A_2) of types:

$$A_0 : \mathbf{Type}$$

$$A_1 : A_0 \to A_0 \to \mathbf{Type}$$

$$A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(y, z) \to A_1(x, z) \to \mathbf{Type}$$

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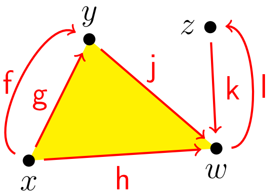

Example:

$A_0 \equiv \{x, y, z, w\}$

$A_1(x, y) \equiv \{f, g\}$

$A_1(x, w) \equiv \{h\}, \ldots$

$A_2(x, y, w, g, j, h) \equiv$ *yellow* Δ

Can we define semisimplicial types?

Can we define $\mathbf{F} : \mathbb{N} \to \mathbf{Type_1}$ such that $\mathbf{F(n)}$
encodes the type of tuples $(\mathbf{A_0}, \ldots, \mathbf{A_n})$?

Can we define the type of "infinite tuples" $(\mathbf{A_0}, \mathbf{A_1}, \ldots)$?

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Remark: We actually want a type of diagrams $\Delta_+^{\mathsf{op}} \to \mathbf{Type}$.

Δ_+^{op} is the category $[0] \leftleftarrows [1] \overset{\leftarrow}{\leftleftarrows} [2] \leftleftarrows\!\!\leftleftarrows \cdots$

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Semisimplicial types are an encoding avoiding equalities!

(A_0, A_1, A_2) encodes:

$$
\begin{aligned}
[0] &\mapsto A_0 \\
[1] &\mapsto \Sigma(x, y : A_0), A_1(x, y) \\
[2] &\mapsto \Sigma x, y, z, f, g, h, A_2(x, y, z, f, g, h)
\end{aligned}
$$

Can we define semisimplicial types?

While semisimplicial types are unsolved in "book HoTT", they work in other settings:

- Voevodsky's HTS (*homotopy type system*)
- many models (c.f. Shulman's work)
- 2LTT (*2-level type theory*) by Capriotti et al. is flexible:
  - plain 2LTT $\approx$ book HoTT (Capr.)
  - 2LTT + axiom "external $\mathbb{N}$ is $\mathbb{N}$" $\approx$ HTS (Hofmann)
  - 2LTT + axiom "towers of fibrations have limits" $\approx$ Shulman's condition
- Boulier-Tabareau's version of HTS/2LTT
- Part-Luo's logic-enriched HoTT
- some "cubical 2LTT's" (Angiuli, Favonia, Harper)
- ...?

Why do we **want** to define semisimplicial types?

(1) Because it looks like it should be possible...

(2) For higher categories

(3) For "HoTT eating HoTT" (conjectured)

(4) For certain "internal elementary" open problems (conjectured).

Let's talk about (2), (4).

Higher univalent categories

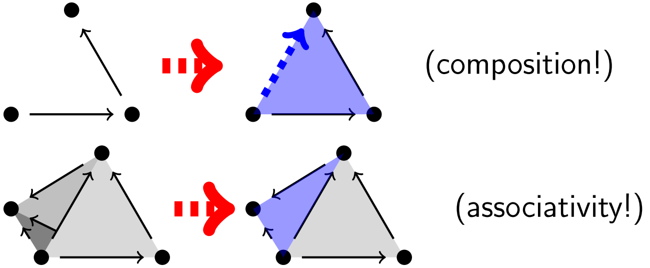
Capriotti-K., POPL'18:

- A *univalent (n,1)-category* is a semisimplicial type $(A_0, \ldots, A_{n+2})$ with two (three) propositional properties
- This coincides with the "manual" definitions on levels ≤ 2

Higher univalent categories

Capriotti-K., POPL'18:

- A *univalent (n,1)-category* is a semisimplicial type $(A_0, \ldots, A_{n+2})$ with two (three) propositional properties
- This coincides with the "manual" definitions on levels ≤ 2
- one of the properties is the *Segal condition* (horn filling):



- identities come from the *completeness* property

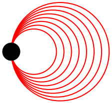
Internal elementary Problems

Wedge of $\mathbf{A}$-many circles

HIT *WA* where

$b : WA$

$l : A \to b = b$



$$A \rightrightarrows \mathbf{Unit} \dashrightarrow WA$$

Question:

A is set $\overset{?}{\Rightarrow}$ *WA is 1-type*

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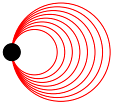

$$A \rightrightarrows \mathbf{Unit} \dashrightarrow WA$$

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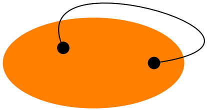
Adding a path from x to $y : \mathbf{B}$

HIT $\tilde{B}$ where

$\eta : B \to \tilde{B}$

$p : \eta(x) = \eta(y)$



$$\begin{array}{ccc} \mathbf{Bool} & \longrightarrow & \mathbf{Unit} \\ \downarrow & & \downarrow \\ B & \dashrightarrow & \tilde{B} \end{array}$$

Question:
B *is 1-type* $\overset{?}{\Rightarrow}$ $\tilde{B}$ *is 1-type*

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Problem: To describe the path spaces of *WA* and $\tilde{B}$, we need infinite towers of coherences — cf. K.-Altenkirch LiCS'18.

Conjecture: In a type theory with semisimplicial types, we can do this and answer these questions positively.

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