Non-Normalizability of Cauchy Sequences

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Abstract

We show that any definable function from the type of Cauchy sequences into a discrete type is constant, provided that the function respects the usual equivalence relation on Cauchy sequences. The core reason is that the real numbers, viewed as a space in standard mathematics, are connected.

In particular, there is no definable normalization function on the set of Cauchy sequences in any extension of basic MLTT which admits the standard property that definable functions are continuous. In the language of [AAL], this means that the Reals are not definable.

For homotopy type theory, a consequence is that any definable function from the real numbers into a discrete type is constant. This also implies that it is not possible to calculate an approximation for a real number.

For technical reasons, we use $\mathbb{N}^+$, the set of positive natural numbers, instead of $\mathbb{N}$. We assume familiarity with the standard topological structure that can be given to the type of sequences over a discrete type; for example, the *distance* of two functions $f, g : \mathbb{N}^+ \to \mathbb{Q}$ can be defined as

$$d(f, g) :\equiv 2^{-\mathsf{inf}\{k \in \mathbb{N}^+ \mid f(k) \neq g(k)\}}, \tag{1}$$

making $\mathbb{N}^+ \to \mathbb{Q}$ a metric space. It is folklore that, with this notation, definable functions are continuous: for example, for a definable endofunction on $\mathbb{N}^+ \to \mathbb{Q}$, a finite part of the output can always be calculated using only a finite part of the input.

Definition 1. We call a function $f : \mathbb{N}^+ \to \mathbb{Q}$ a *Cauchy Sequence* if it satisfies

$$\mathsf{isCauchy}(f) :\equiv \forall (n : \mathbb{N}^+).\, \forall (m : \mathbb{N}^+).\, m > n \to |f(n) - f(m)| < \frac{1}{n}. \tag{2}$$

The type of Cauchy Sequences is thus

$$\mathsf{R}_0 :\equiv \Sigma_{f : \mathbb{N}^+ \to \mathbb{Q}} \mathsf{isCauchy}(f).$$

Remark 1. Alternative definitions of the property of being a Cauchy Sequence (which lead to an "essentially equivalent" type R_0) include

$$\forall (n : \mathbb{N}^+).\, \forall (m_1\, m_2 : \mathbb{N}^+).\, m_1, m_2 > n \to |f(m_1) - f(m_2)| < \frac{1}{n} \tag{3}$$

as well as

$$\forall (\epsilon : \mathbb{Q}).\, \epsilon > 0 \to \Sigma_{n : \mathbb{N}^+} \forall (m_1\, m_2 : \mathbb{N}^+).\, m_1, m_2 > n \to |f(m_1) - f(m_2)| < \epsilon \tag{4}$$

and

$$\forall(\epsilon : \mathbb{Q}).\, \epsilon > 0 \to \Sigma_{n:\mathbb{N}^+} \forall(m : \mathbb{N}^+).\, m > n \to |f(n) - f(m)| < \epsilon. \tag{5}$$

In particular, 3, 2 have the property that the proof of f being a Cauchy sequence is propositional, which might be desirable for various reasons. What we prove in this note is valid for each of these possible definitions as well.

Remark 2. For another possible definition, see [UF]. Note that what they call *Cauchy approximation* is essentially our Cauchy sequence.

Remark 3. In [AAL], R_0 is defined using 5 with an existential quantifier, that is, starting with Σ_n, everything is truncated. We would argue that this is not a good idea as we want to be able to approximate the real number represented by a Cauchy sequence.[1] In [UF, Chapter 11], an argument in the same direction is made. Besides, it requires truncation to be part of the theory and we try to be minimalistic in this note. Truncation would simplify the proof of this note's main result significantly.

Definition 2. If $(f, p), (g, q)$ are two Cauchy sequences, we say $(f, p) \sim (g, q)$ if

$$\forall(n : \mathbb{N}^+).\, |f(n) - g(n)| \leq \frac{1}{n}. \tag{6}$$

Remark 4. Note that this property is propositional again. For 5, the definition would have to be something like

$$\Pi_{\epsilon:\mathbb{Q},\epsilon>0} \Sigma_{n:\mathbb{N}^+} \Pi_{m:\mathbb{N}^+} m > n \to |f(m) - g(m)| < \epsilon. \tag{7}$$

Let us introduce the following auxiliary definition:

Definition 3. For a sequence $f : \mathbb{N}^+ \to \mathbb{Q}$, we say that f is *Cauchy with factor* k, written $\mathsf{isCauchy}_k$, for some $k \in \mathbb{Q}$, $k > 0$, if

$$\mathsf{isCauchy}_k(f) \;:\equiv\; \forall(n\, m : \mathbb{N}^+).\, m > n \to |f(n) - f(m)| < \frac{1}{k \cdot n}. \tag{8}$$

The usual Cauchy condition $\mathsf{isCauchy}$ is therefore "Cauchy with factor 1".

Proposition 4. $\mathsf{R}_0/\sim$ *is connected. In type theory, this means: Assume*

$$f : \mathsf{R}_0 \to \mathbf{2} \tag{9}$$

is a continuous[2] *function.*[3] *Assume f respects $\sim$, that is we have a proof*

$$p : \Pi_{c_1,c_2:\mathsf{R}_0} c_1 \sim c_2 \to f(c_1) = f(c_2). \tag{10}$$

Under these assumptions, f is constant in the sense that it is impossible to find $c_1, c_2 : \mathsf{R}_0$ such that $f(c_1) \neq f(c_2)$.[4]

[1] This is actually partially possible; for example, it will be possible to *prove* that a given Cauchy sequence represents a positive number iff this is the case. However, it will not be possible to *calculate* an approximation directly: a priori there will be no way for us to find out whether a given natural number is approximately −17 or 149.

[2] The justification is, of course: Definable functions are continuous.

[3] The metric of R_0 comes from the first component. Technically, if R_0 is defined by 5, this would not make it a metric space (as the distance between non-equal elements could be 0); however, this would not matter, and for our definition, there is no problem anyway.

[4] This is a meta-theoretic statement. I do not expect much of the argument to be internalizable, but it would be interesting to explore this. (With the "truncated" definition of $\mathsf{isCauchy}$ 5, the proof becomes much simpler.)

Proof. Assume f, p are given.

Consider the "naive" set model (with "classical standard mathematics" as meta-theory). This clearly works if we are in a minimalisitic type theory with Π, Σ, W, $=$, $\mathbb{N}$; however, if we restrict ourselves to the types in the lowest universe of homotopy type theory (which is enough), it also works for HoTT. We use $[\![\cdot]\!]$ as an interpreation function; for example, we write $\mathbb{R}$ for the field of real numbers which can be defined as $[\![\mathsf{R}_0]\!]/[\![\sim]\!]$. By abuse of notation, we write $[\![\mathsf{R}_0]\!]$ for the set of Cauchy sequences in the model that fulfill the Cauchy condition, without the actual proof thereof. This is justified as this property is propositional.

For readability, we use the symbol $=$ for equality in the theory as well in the model, and we do not use the semantic brackets for natural numbers such as 2 or 4. In the model, we use $\bar{\cdot} : [\![\mathsf{R}_0]\!] \to \mathbb{R}$ as the function that gives us the limit of a Cauchy sequence. Thus, for $r : \mathsf{R}_0$, we write $\overline{[\![r]\!]} \in \mathbb{R}$ for the real number it represents.

We prove that $[\![f]\!] : [\![\mathsf{R}_0]\!] \to [\![\mathbf{2}]\!]$ is constant in the model, which implies the statement of Proposition 4. Thus, assume there are $c_1, c_2 : [\![\mathsf{R}_0]\!]$ with $[\![f]\!](c_1) \neq [\![f]\!](c_2)$.

Define

$$m_1 :\equiv \mathsf{sup}\{\overline{d} \in \mathbb{R} \mid d \in [\![\mathsf{R}_0]\!], \overline{d} \leq \mathsf{max}(\overline{c_1}, \overline{c_2}), [\![f]\!](d) = [\![1_{\mathbf{2}}]\!]\} \tag{11}$$

$$m_2 :\equiv \mathsf{sup}\{\overline{d} \in \mathbb{R} \mid d \in [\![\mathsf{R}_0]\!], \overline{d} \leq \mathsf{max}(\overline{c_1}, \overline{c_2}), [\![f]\!](d) = [\![0_{\mathbf{2}}]\!]\} \tag{12}$$

(note that one of these two necessarily has to be $\overline{c_1}$ or $\overline{c_2}$, whichever is bigger). Set $m :\equiv \mathsf{min}(m_1, m_2)$.

Let $c \in [\![\mathsf{R}_0]\!]$ be a Cauchy sequence such that $\bar{c}$ is equal to m. We may assume that c satisfies the condition $[\![\mathsf{isCauchy}_5]\!]$.

As f (and thereby $[\![f]\!]$) is continuous, there is $n_0 \in [\![\mathbb{N}^+]\!]$ such that for any Cauchy sequence $c' \in [\![\mathsf{R}_0]\!]$, if the first n_0 sequence elements of c' coincide with those of c, then $[\![f]\!](c') = [\![f]\!](c)$. Write $U \subset [\![\mathsf{R}_0]\!]$ for the set of Cauchy sequences which fulfill this property, and $\overline{U} :\equiv \{\overline{d} \mid d \in U\}$ for the set of reals that U corresponds to.

We claim that U is a neighborhood of m. More precisely, we prove: The interval $(m - \frac{1}{2n_0}, m + \frac{1}{2n_0})$ is contained in $\overline{U}$. Let $x \in \mathbb{R}$ be in that interval. There is a sequence $t : [\![\mathbb{N}^+ \to \mathbb{Q}]\!]$ such that $[\![\mathsf{isCauchy}_{5n_0}]\!](t)$ and $\bar{t} = x$. Let us now "concatenate" the first n_0 elements of the sequence c with t, that is, define

$$g : [\![\mathbb{N}^+ \to \mathbb{Q}]\!] \tag{13}$$

$$g(n) = \begin{cases} c(n) & \text{if } n \leq n_0 \\ t(n - n_0) & \text{else.} \end{cases} \tag{14}$$

Observe that g is a Cauchy sequence, i.e. $[\![\mathsf{isCauchy}]\!](g)$: The only thing that needs to be checked is whether the two "parts" of g work well together. Let $0 < n \leq n_0$ and $m > n_0$ be two natural numbers. We need to show that

$$|g(n) - g(m)| < \frac{1}{n}. \tag{15}$$

Calculate

$$
\begin{aligned}
& |g(n) - g(m)| && (16)\\
= \; & |c(n) - t(m - n_0)| && (17)\\
= \; & |c(n) - \overline{c} + \overline{c} - \overline{t} + \overline{t} - t(m - n_0)| && (18)\\
\leq \; & |c(n) - \overline{c}| + |\overline{c} - \overline{t}| + |\overline{t} - t(m - n_0)| && (19)\\
\leq \; & \frac{1}{5n} + \frac{1}{2n_0} + \frac{1}{5n_0 \cdot (m - n_0)} && (20)\\
\leq \; & \frac{1}{5n} + \frac{1}{2n} + \frac{1}{5n_0} && (21)\\
< \; & \frac{1}{n}. && (22)
\end{aligned}
$$

Clearly, $\overline{g} = \overline{t} = x \in (m - \frac{1}{2n_0}, m + \frac{1}{2n_0})$. By the condition that f (and thereby $[\![f]\!]$) repects $\sim$, *all* $s \in [\![\mathsf{R}_0]\!]$ with $|\overline{s} - m| < \frac{1}{2n_0}$ satisfy $[\![f]\!](s) = [\![f]\!](c)$.

However, the definition of m implies that in *every* neighborhood of m, and thus in particular in $(m - \frac{1}{2n_0}, m + \frac{1}{2n_0})$, there is an x such that $x = \overline{e}$ (for some e) with $[\![f]\!](e) \neq [\![f]\!](c)$, in contradiction to the just established statement. □

Theorem 5. *Any continuous endofunction f on R_0 that respects $\sim$ in the sense of*

$$p : \Pi_{c_1, c_2 : \mathsf{R}_0} c_1 \sim c_2 \to m(c_1) = m(c_2). \tag{23}$$

is constant (in the sense of proposition 4).

Proof. We only need to show that $\pi_1 \circ f$ (the actual sequence) is constant as the proof of being a Cauchy sequence is propositional.[5] Again, by slight abuse of notation, we write $[\![f]\!] : [\![\mathsf{R}_0]\!] \to [\![\mathsf{R}_0]\!]$, omitting the proof part of f.

Given $c : [\![\mathsf{R}_0]\!]$, we want to show $[\![f]\!](c) = [\![f]\!](\mathsf{0})$, where $\mathsf{0}$ is the sequence that is constantly 0. To do so, it is enough to show that, for a given $k : [\![\mathbb{N}^+]\!]$, we have $[\![f]\!](c)(k) = [\![f]\!](\mathsf{0})(k)$. If this was not true, we would have a function $[\![\mathsf{R}_0]\!] \to [\![\mathbf{2}]\!]$, defined by

$$\lambda c.\mathsf{isEquval}\,([\![f]\!](c)(k))\,([\![f]\!](\mathsf{0})(k))$$

that is not constant, contradicting Proposition 4. □

Corollary 6. *There is no definable normalization function on R_0, that is, there is no function*

$$m : \mathsf{R}_0 \to \mathsf{R}_0 \tag{24}$$

with the property

$$p : \Pi_{c_1, c_2 : \mathsf{R}_0} c_1 \sim c_2 \to m(c_1) = m(c_2). \tag{25}$$

such that

$$\Pi_{c : \mathsf{R}_0} c \sim m(c). \tag{26}$$

Corollary 7. $\mathsf{R}_0 / \sim$ *is not definable in the sense of [AAL].*

[5] Even if $\mathsf{isCauchy}$ is not a propositional predicate (as in 5), it will still be true that m is constant. This is simply because $\sim$ is defined only in terms of the actual sequence part.

Corollary 8 (Corollary of Proposition 4)**.** *In Homotopy Type Theory, every definable function from the real numbers* $\mathbb{R}$ *(as defined in [UF, Chapter 11]) into a discrete type is constant. In particular, it is not possible to approximate a real number: We can* not *define a function* $f : \mathbb{R} \to \mathbb{Q}$ *such that, for all* $r : \mathbb{R}$*, we have* $f(r) - 1 < r < f(r) + 1$.[6]

Conjecture 9. If T is a definable type in the minimalisitic type theory with Σ, Π, W, $=$, $\mathbb{N}$, and T does have two distinguishable elements, then T is not connected (note that this is much weaker than the original wrong conjecture of [AAL]). This in particular implies that the reals are not a definable type.

References

[AAL] Thorsten Altenkirch, Thomas Anberrée, and Nuo Li. "Definable Quotients in Type Theory". 2011.

[UF] The Univalent Foundations Program UF. *Homotopy type theory: Univalent foundations of mathematics.* first. Available online at homotopytypetheory.org/book. 2013.

[6]For any [reasonable] definition of $<$ and as long as HoTT admits the principle that definable functions are continuous, which is certainly true, but of which I do not know a proof.