Homotopy Type Theory

Nicolai Kraus

Functional Programming Laboratory Away Day

8th July 2011

What is it all about?

A connection...

Type Theory | **Topology**

What is it all about?

A connection...

Type Theory	**Topology**

related to logic

What is it all about?

A connection...

Type Theory	**Topology**

related to logic

related to

- Algebra
- Analysis
- Geometry
- ...

So, what is Topology actually?

So, what is Topology actually?

A major area of mathematics that examines continuity!

So, what is Topology actually?

A major area of mathematics that examines continuity!

Sets often have a natural notion of *open* subsets, e.g. in $\mathbb{R}$:

$(1, 2) := \{x \mid 1 < x < 2\}$ is open, but $[1, 2] := \{x \mid 1 \leq x \leq 2\}$ is not.

So, what is Topology actually?

A major area of mathematics that examines continuity!

Sets often have a natural notion of *open* subsets, e.g. in $\mathbb{R}$:

$(1, 2) := \{x \mid 1 < x < 2\}$ is open, but $[1, 2] := \{x \mid 1 \leq x \leq 2\}$ is not.

Definition (Continuity of $f : X \to Y$):

f is continuous iff inverse images of open sets are open, i.e. if $V \subset Y$ is open, so is $f^{-1}(V)$.

Remark

The word "Continuity" is used a lot:

Remark

The word "Continuity" is used a lot:

- functions $\mathbb{R} \to \mathbb{R}$. "Definition":
  $\forall x, \epsilon > 0. \exists \delta > 0. |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Remark

The word "Continuity" is used a lot:

- functions $\mathbb{R} \to \mathbb{R}$. "Definition":
  $\forall x, \epsilon > 0. \exists \delta > 0. |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$
- complete partial orders: $f(sup\ D) = sup\ f(D)$ for every directed D

Remark

The word "Continuity" is used a lot:

- functions $\mathbb{R} \to \mathbb{R}$. "Definition": $\forall x, \epsilon > 0. \exists \delta > 0. |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$
- complete partial orders: $f(sup\ D) = sup\ f(D)$ for every directed D
- "Computable functions are continuous."
- ...

Remark

The word "Continuity" is used a lot:

- functions $\mathbb{R} \to \mathbb{R}$. "Definition":
  $\forall x, \epsilon > 0. \exists \delta > 0. |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$
- complete partial orders: $f(sup\ D) = sup\ f(D)$ for every directed D
- "Computable functions are continuous."
- ...

all notions can be broken down to:

Inverse images of open sets are open.

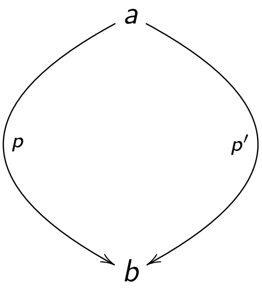
Identity Types without UIP - a Reminder

$$\frac{a, b : A}{a \equiv b \quad \text{Type}}$$

$$\frac{}{\mathsf{refl}_a : a \equiv a}$$

$$\begin{aligned} p &: a \equiv b \\ p^{-1} &: b \equiv a \\ q &: b \equiv c \\ q \circ p &: a \equiv c \end{aligned}$$

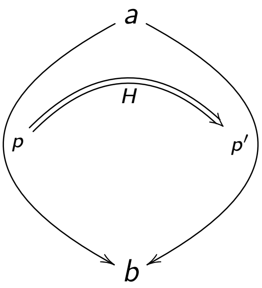
Identity Types without UIP - a Reminder



for example:

- $a := b := x$
- $p := p' := \mathrm{refl}_x$

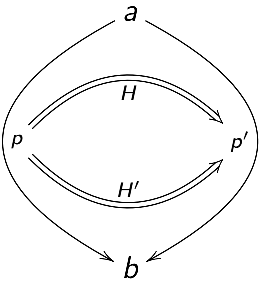
Identity Types without UIP - a Reminder



for example:

- $a := b := x$
- $p := p' := \mathrm{refl}_x$

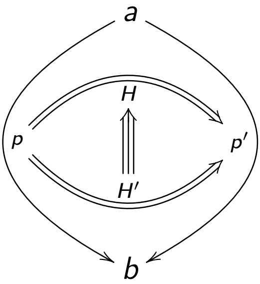
Identity Types without UIP - a Reminder



for example:

- $a := b := x$
- $p := p' := \mathsf{refl}_x$
- $H := H' := \mathsf{refl}_{\mathsf{refl}_x}$

Identity Types without UIP - a Reminder



for example:

- $a := b := x$
- $p := p' := \mathsf{refl}_x$
- $H := H' := \mathsf{refl}_{\mathsf{refl}_x}$
- $\mathsf{refl}_{\mathsf{refl}_{\mathsf{refl}_x}}$
- $\ldots$

Back to Topology

Structures:

Topological spaces

Back to Topology

Structures:

Topological spaces

$\rightarrow$ Special case: Hausdorff spaces (or T_2)

Back to Topology

Structures:

Topological spaces

$\rightarrow$ Special case: Hausdorff spaces (or T_2)

$\rightarrow\rightarrow$ Special case of this special case: Metric spaces

Back to Topology

Structures:

Topological spaces

$\rightarrow$ Special case: Hausdorff spaces (or T_2)

$\rightarrow\rightarrow$ Special case of this special case: Metric spaces

$\rightarrow\rightarrow\rightarrow$ Even much more special: Normed vector spaces

Back to Topology

Structures:

Topological spaces

$\rightarrow$ Special case: Hausdorff spaces (or T_2)

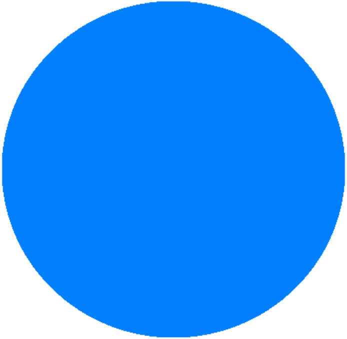
$\rightarrow\rightarrow$ Special case of this special case: Metric spaces

$\rightarrow\rightarrow\rightarrow$ Even much more special: Normed vector spaces

$\rightarrow\rightarrow\rightarrow\rightarrow$ ...and finally: $\mathbb{R}^n$, or just subsets of it!

A disc

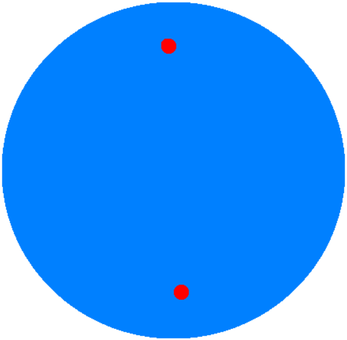
a type - we call it X



a topological space - we call it X

A disc

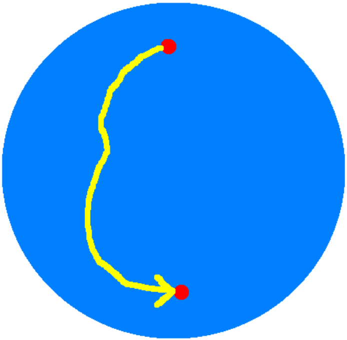
two terms



two points

A disc

?

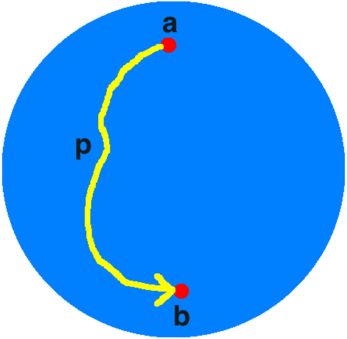


a path

A disc

$a, b : X$

$p : a \equiv b$



$a, b \in X$

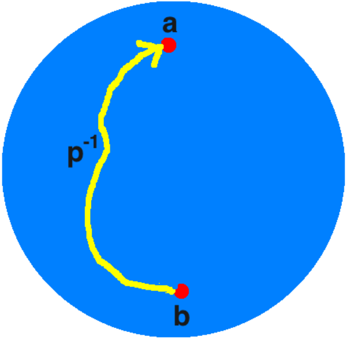
$p : [0, 1] \to X$

$p(0) = a$

$p(1) = b$

A disc

$p^{-1} : b \equiv a$



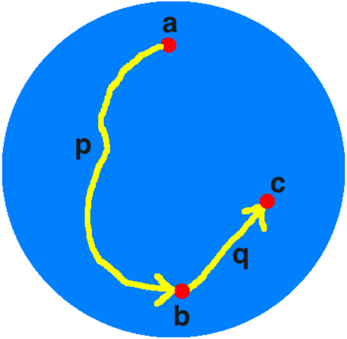
$p^{-1} : [0,1] \to X$

$p^{-1}(t) = p(1-t)$

A disc

$p : a \equiv b$

$q : b \equiv c$



$a, b \in X$

$p : [0,1] \to X$

$p(0) = a$

$p(1) = b$

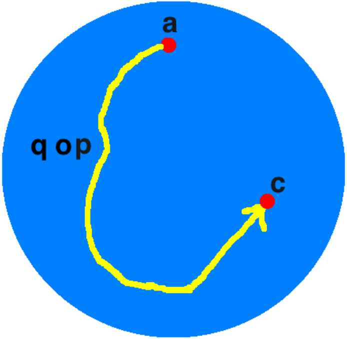
$q : [0,1] \to X$

$q(0) = b$

$q(1) = c$

A disc

$q \circ p \; : \; a \equiv c$



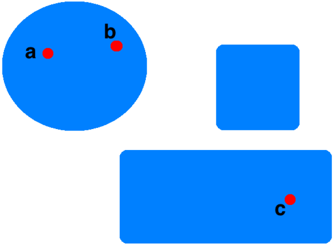
$q \circ p$:

$[0,1] \to X$

$x \mapsto$
$$\begin{cases} p(2x), x < 0.5 \\ q(2x-1), \text{else} \end{cases}$$

Another set

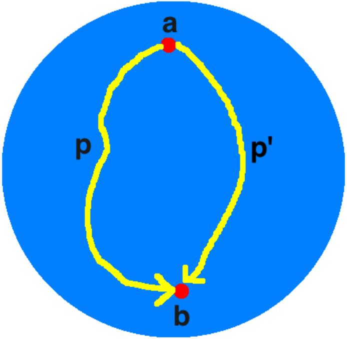
$a \equiv c$ not inhabited



not path-connected

A disc

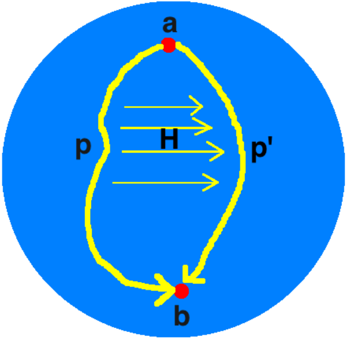
$p, p' \; : \; a \equiv b$



$p, p' \; : \; [0,1] \to X$

A disc

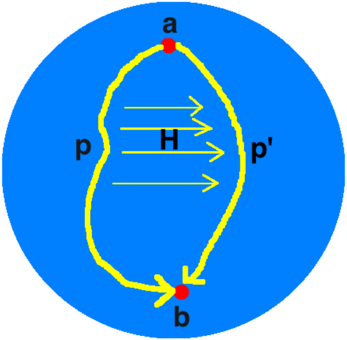
$H : p \equiv p'$



$$H : [0,1]^2 \to X$$
$$H(0,\cdot) = p$$
$$H(1,\cdot) = p'$$
$$H(t,0) = a$$
$$H(t,1) = b$$

A disc

$H : p \equiv p'$



$H : [0,1]^2 \to X$

$H(0,\cdot) = p$

$H(1,\cdot) = p'$

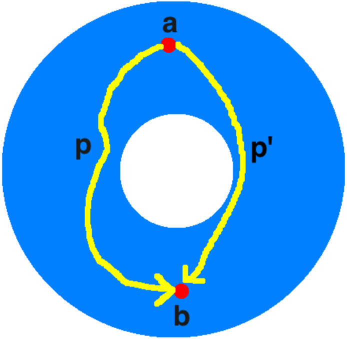
$H(t,0) = a$

$H(t,1) = b$

$p : [0,1]^1 \to X$

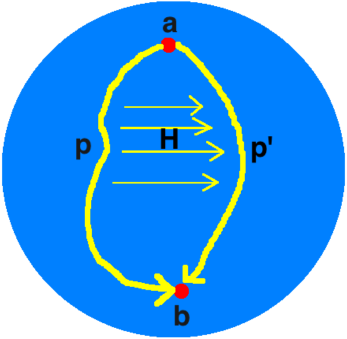
$a : [0,1]^0 \to X$

A ring



A disc

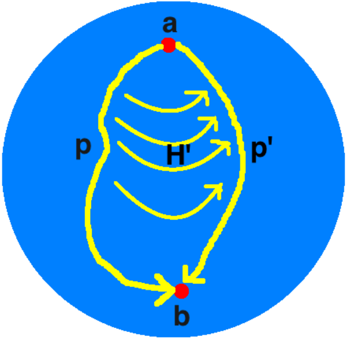
$H : p \equiv p'$



$$H : [0,1]^2 \to X$$
$$H(0,\cdot) = p$$
$$H(1,\cdot) = p'$$
$$H(t,0) = a$$
$$H(t,1) = b$$

A disc

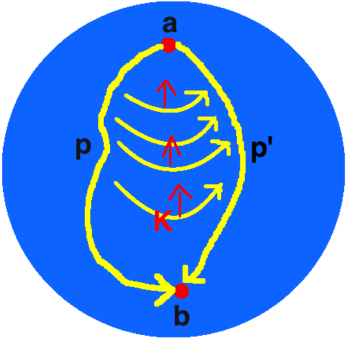
$H' \ : \ p \equiv p'$



$$H' \ : \ [0,1]^2 \to X$$
$$H'(0,\cdot) = p$$
$$H'(1,\cdot) = p'$$
$$H'(t,0) = a$$
$$H'(t,1) = b$$

A disc

$K : H' \equiv H$



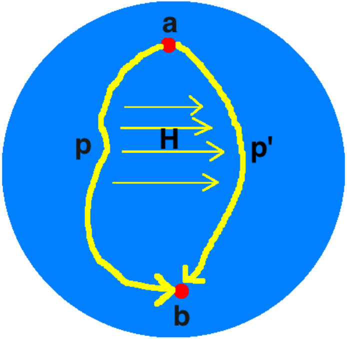
$K : [0,1]^3 \to X$

$K(0,\cdot,\cdot) = H'$

$\ldots$

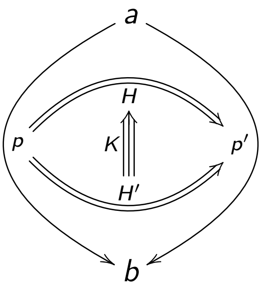
A disc

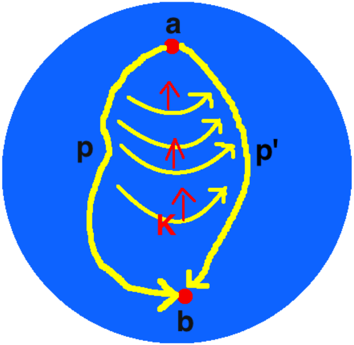
$H : p \equiv p'$



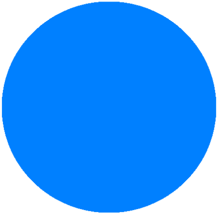
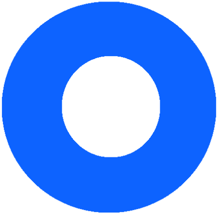
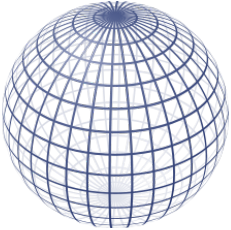
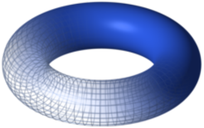
$$H : [0,1]^2 \to X$$
$$H(0,\cdot) = p$$
$$H(1,\cdot) = p'$$
$$H(t,0) = a$$
$$H(t,1) = b$$

Putting it together





So, which types can we get?

any CW complex?

Where is it going?

All has been done in abstract homotopy theory.

Where is it going?

All has been done in abstract homotopy theory.

What I (at the moment) hope:

- Creating a simple model
- that is complete
- and easy to understand and to use

Where is it going?

e.g. for this problem (Thorsten):

- subst-refl P (subst P (refl x) p)

  and
- cong (subst P (refl x)) (subst-refl P p)

  both prove
- subst P (refl x) (subst P (refl x) p) $\equiv$ subst P (refl x) p

  But are they equal?