Eliminating out of Truncations

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Nicolai Kraus

If we want to perform a construction or show a result which does not hold for types with non-trivial higher equality structure, we often choose to only work with n-types, for some suitable number $n \geq -1$. To give examples: for algebraic structures such as groups, we may require the type of elements to be a 0-type, and for categories, the type of objects has to be a 1-type, while one might want to do some form of "traditional logic" with (-1)-types. This way, we can avoid coherence problems that could potentially occur on higher levels that we may not even be interested in. The truncation operator $\|-\|_n$, which transforms any type A into an n-type $\|A\|_n$, can be viewed and implemented as a higher inductive type, but is certainly somewhat special. It is a modality (an idempotent monad in some appropriate sense), and it allows us to work completely in the "subuniverse" of n-types. This becomes difficult if, at some point, we need to leave this "subuniverse". The universal property of $\|-\|_n$ says that functions $(\|A\|_n \to B)$ correspond to functions $(A \to B)$, but only if B happens to be an n-type.

It may therefore be interesting to derive a more powerful "universal property" for $\|-\|_n$ which is not restricted to n-types B, but works for any m-type B. Here, m is a fixed number that may be anything greater than n, including ∞, in which case we do not put any restriction on B. Intuitively, what we need to do is to require the functions $(A \to B)$ to satisfy certain coherences if we want them to correspond to functions $(\|A\|_n \to B)$.

I will present an outline of my solution for the propositional truncation [2], i.e. $n \equiv -1$, where (in the currently considered type theory) m is any number, but has to be fixed externally. This needs some specific "semi-simplicial type". I use the construction to illustrate that we might want a type theory that allows the construction of "Reedy-fibrant diagrams" and its limits (sometimes called "infinitary type theory"). Joint work with Paolo Capriotti and Andrea Vezzosi has further yielded a solution for the case $m \equiv n+1$ (i.e. n is no longer required to be -1) [1]. I will try to explain why the remaining cases (general $n > -1$, arbitrary m greater than n) seem to be harder than the solved ones. Intuitively, this is because they combine two different kinds of coherence problems.

References

[1] Paolo Capriotti, Nicolai Kraus, and Andrea Vezzosi. Functions out of higher truncations. *Computer Science Logic (CSL) 2015*, volume 41 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 359–373, 2015.

[2] Nicolai Kraus. The general universal property of the propositional truncation. *20th International Conference on Types for Proofs and Programs (TYPES 2014)*, volume 39 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 111–145, 2015.