An Induction Principle for Cycles

Nicolai Kraus and Jakob von Raumer

Consider a graph. Imagine we want to prove a certain property for every path within the graph, where a path is a sequence $(v_0, e_0, v_1, e_1, \ldots, v_k)$ of edges e_i from the vertex v_i to the vertex v_{i+1}. The obvious first approach is induction: We show that the property holds for the empty path, and that adding an edge at the end of a path preserves the property. We now make the situation more challenging by changing it slightly:

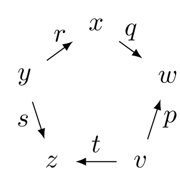
Problem 1. Imagine we want to prove a certain property for every *cycle*, i.e. every *closed* path within the graph (a path satisfying the condition $v_0 = v_k$). How can this be approached?

An example for a property could be the statement that every vertex in a cycle has even degree, something which is not true for paths in general. Straightforward induction does not work in the situation of Problem 1 anymore, since a cycle can in general not be created from a smaller cycle by adding an edge (cycles are simply not inductively generated).

This situation occurs in the context of coherence conditions in homotopy type theory.[1] Given a *set* (a 0-truncated type) A and a binary proof-relevant relation on A, i.e. a type family $(\rightsquigarrow) : A \to A \to \mathsf{Type}$, recall that the property of the *set-quotient* $A/\rightsquigarrow$ is that a function $g : A/\rightsquigarrow \to B$ is uniquely given by a function $f : A \to B$ which respects $\rightsquigarrow$, i.e. comes together with $h : \Pi\{x, y : A\}.(x \rightsquigarrow y) \to f(x) = f(y)$. This property is only guaranteed as long as B is a *set*. Sometimes, this condition not satisfied. We show in the paper [3] the following result:

Lemma 2. *If B is a 1-type and A is a set, then a function $g : A/\rightsquigarrow \to B$ is uniquely given by a triple (f, h, c) with f, h as above, and c witnessing that any* closed zig-zag *(or simply* cycle*) of $\rightsquigarrow$ is sent by h to a trivial equality* refl *in B.*

The coherence condition c says, for example, that if we are given $v, w, x, y, z : A$ with $p : v \rightsquigarrow w$, $q : x \rightsquigarrow w$, $r : y \rightsquigarrow x$, $s : y \rightsquigarrow z$, $t : v \rightsquigarrow z$ as in the diagram on the right, then the equality $h(p) \cdot h(q)^{-1} \cdot h(r)^{-1} \cdot h(s) \cdot h(t)^{-1} : f(v) = f(v)$ is equal to $\mathsf{refl}_{f(v)}$.



An example for a set-quotient in homotopy type theory is the explicit construction of the *free group* as in [6, Thm 6.11.17]. For a given set Y, we set $A :\equiv \mathsf{List}(Y + Y)$. We think of the left copy of Y as positive and the right as negative, and write $\_^{-1} : Y + Y \to Y + Y$ for the "swap" operation. The relation of interest is then generated by

$$[x_0, \ldots, x_{i-1}, x_i, x_i^{-1}, x_{i+1}, \ldots, x_n] \rightsquigarrow [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n], \tag{1}$$

and $\mathsf{List}(Y + Y)/\rightsquigarrow$ has the correct universal property of the free group on Y. It is an open question (a variation of the long-standing unsolved problem recorded in [6, Ex 8.2]) whether $\mathsf{List}(Y + Y)/\rightsquigarrow$ is equivalent to the *free higher group*, namely the loop space of the wedge of Y-many circles; details can be found in [3]. Constructing functions from the set-quotient $\mathsf{List}(Y + Y)/\rightsquigarrow$ into a 1-type is the key to showing an approximation to the mentioned question. To do this, we would like to apply Lemma 2. However, on its own, this lemma is not useful

[1] The situation described in the first paragraph is of course reminiscent of a central observation of (homotopy) type theory, namely the fact that the equality eliminator J (a.k.a. *path induction*) does not imply Streicher's K (which one could call *loop induction*): with J, we can only replace an equality by refl if it is a general equality between two different points. This is not the connection that we study in this work.

precisely because of Problem 1: the condition that any cycle is sent to a trivial equality is hard to verify. Overcoming this difficulty is the motivation for this talk. The title of the current talk description alludes to an approach to Problem 1 that we present in [3].

If we look at (1) (and the relevant relations in similar examples), we can observe that the relation $\rightsquigarrow$ has two properties (one of them proof-relevant) that are familiar from the theory of reduction systems in general (for example, see [1]). First, (1) is *co-wellfounded* (a.k.a. *Noetherian*): this is clear, since we can only reduce a finite number of times until no redexes of the form $[x, x^{-1}]$ are left. The general definition is:

Definition 3 (accessibility [6, Chp. 10.3])**.** The family $\mathsf{acc}^{\rightsquigarrow} : A \to \mathsf{Type}$ is generated inductively by a single constructor $\mathsf{step} : \Pi(a : A).(\Pi(x : A).(x \rightsquigarrow a) \to \mathsf{acc}^{\rightsquigarrow}(x)) \to \mathsf{acc}^{\rightsquigarrow}(a)$. We say that a is *accessible* if we have $\mathsf{acc}^{\rightsquigarrow}(a)$, and if all a are accessible, then $\rightsquigarrow$ is *wellfounded*. We say that $\rightsquigarrow$ is *Noetherian* if $\rightsquigarrow^{\mathsf{op}}$ is wellfounded.

Second, the relation (1) is *locally confluent*: If we have two redexes and reduce one, we can still reduce the other (or both reductions arrive at the same result). In general:

Definition 4 (local confluence)**.** We say that $\rightsquigarrow$ is *locally confluent* if, for any span, there is a matching extended cospan. This means that, given $x, y, z : A$ with $x \rightsquigarrow y$ and $x \rightsquigarrow z$, we have $w : A$ such that $y \rightsquigarrow^* w$ and $z \rightsquigarrow^* w$.

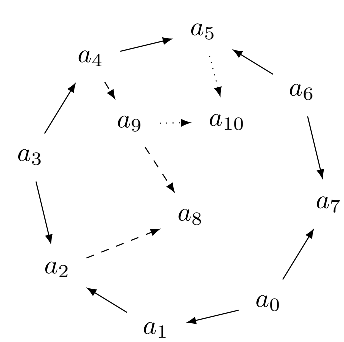
The key construction in our paper [3] is the following. Starting from a relation $\rightsquigarrow$ on A, we construct a new relation $\rightsquigarrow^\circ$ on the type of cycles of $\rightsquigarrow$. This new relation is given by a variation of the *multiset extension*, which is known to preserve wellfoundedness (a proof essentially building on Definition 3 has been given by Nipkow [5]). In a nutshell, for cycles α and β, we have $\alpha \rightsquigarrow^\circ \beta$ if α can be transformed into β by removing one vertex (one element $a_0 : A$) and replacing it by a finite number $b_1, \dots, b_k$ of vertices such that we have $a_0 \rightsquigarrow b_i$ for all i.

Lemma 5. *If $\rightsquigarrow$ is Noetherian on A, then so is $\rightsquigarrow^\circ$ on the type of cycles. If $\rightsquigarrow$ is in addition locally confluent, then any cycle α can be split (see illustration below) into a cycle β with $\alpha \rightsquigarrow^\circ \beta$, and a cycle which is given by a single span and local confluence.*

Let now a type family Q indexed over cycles be given (e.g. the type witnessing that any cycle gets mapped to the trivial equality, as in Lemma 2). Under some natural assumption (Q respects "merging" and "rotating" of cycles), the above lemmas give us the following principle:

Theorem 6 (Noetherian cycle induction [3])**.** *Assume $\rightsquigarrow$ is Noetherian and locally confluent. Assume further that Q is inhabited at every empty cycle and at every cycle that comes from a span and the local confluence property. Then, Q is inhabited at every cycle.*

On the right is an illustration of *Noetherian cycle induction*. Assume we want to show a property Q for the big octagon $a_0 - a_7$. We first "remove" the confluence cycle spanned by $a_2 \leftsquigarrow a_3 \rightsquigarrow a_4$ to get the nonagon consisting of $a_0 - a_9$ without a_3 but with the dashed edges: we now have more vertices, but the nonagon is smaller than the octagon in the order $\rightsquigarrow^\circ$. In the next step, we "remove" the confluence cycle spanned by $a_9 \leftsquigarrow a_4 \rightsquigarrow a_5$, and so on, until the empty cycle is reached.



This induction principle allows us to approach questions involving set-quotients such as the problem of the free higher group mentioned above; in particular, we can re-prove the result of [2]. We also believe that there are use-cases outside of type theory, e.g. for graph rewriting as in [4].

References

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