Yoneda Groupoids

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Abstract

These notes are an attempt to structure the author's thoughts and conjectures related to higher relations and their quotients.

We define the notion of a *Yoneda Groupoid* in HoTT, the name of which is inspired by the relation to the Yoneda lemma, and show how a *weak ω groupoid* structure can be extracted.

1 Introduction to the General Problem

Many open problems of Homotopy Type Theory are related to the question how an (infinite) tower of coherence conditions can be stated. Assume that $\sim : A \times A \to \mathcal{U}$ is any family of types, indexed twice over A, and to be read as a binary relation. A priori, $\sim$ needs not to have the properties of an equivalence relation. We would like to state that it does satisfy the usual corresponding properties. On the other hand, we explicitly do not want $\sim$ to be a propositional relation. We explore a special case in which we get the requires structure basically for free.

Let us start with the following question:

Question 1 (Altenkirch and possibly others)**.** Given the terms

- $\mathsf{refl}^{\sim} : \forall a.\, a \sim a$,
- $\mathsf{sym}^{\sim} : \forall ab.\, a \sim b \to b \sim a$,
- $\mathsf{trans}^{\sim} : \forall abc.\, a \sim b \to b \sim c \to a \sim c$,

how can we formalise the statement that they give A the structure of a weak ω groupoid?

A straightforward idea of approaching this question is stating all the coherence conditions. For example,

$$\lambda : \forall p.\, (\mathsf{trans}^{\sim}\ \mathsf{refl}^{\sim} p) = p$$
$$\rho : \forall p.\, p = (\mathsf{trans}^{\sim}\ \mathsf{refl}^{\sim} p)$$

(where we hide arguments that can be inferred for readability) are necessary coherence conditions. But now, we get a new coherence condition,

$$\lambda\, \mathsf{refl}^{\sim} = \rho\, \mathsf{refl}^{\sim},$$

and in general, every new condition gives rise to even more new conditions. Nevertheless, a similar approach was taken by Altenkirch & Rypacek [1].

2 Yoneda Groupoids

Definition 2 (Yoneda Groupoid)**.** A relation $\sim$ is a *Yoneda Groupoid* if there is a function mapping every $a : A$ to a pair (n, X), where $n : \mathbb{N}$ is the "label" and X represents the structure of the corresponding equivalence class (we discuss the latter point later in detail).

$$\mathsf{isYonedaGrp}(\sim) :\equiv \Sigma_{F:A\to\mathbb{N}\times\mathbf{U}}\, \forall(ab : A).\, (a \sim b) = (F\, a = F\, b).$$

$\mathbf{U}$ could be any available universe or type. However, if $\mathbf{U}$ is just some type $Q : \mathcal{U}$, then Q would already have to be a "supertype" of the required quotiend. Therefore, we consider this case rather uninteresting. Our focus shall lie on the possibility that $\mathbf{U}$ is a universe, as univalence provides then additional equality proofs. For our discussion, we find it convenient to choose $\mathbf{U} :\equiv \mathcal{U}$, so let us assume that we are using the smallest universe.

If the cardinality of $\mathbb{N}$ is not sufficient, any other proper set could serve for the labelling. In fact, we could even make the indexing set part of the definition in the form of

$$\mathsf{isYonedaGrp}(\sim) :\equiv \Sigma_{I:\mathcal{U},\mathsf{isSet}\, I}\Sigma_{F:A\to I\times\mathbf{U}}\forall(ab : A).\, (a \sim b) = (F\, a = F\, b).$$

This definition is inspired by two different formalisations of equivalence relations in the proof-irrelevant case and can actually be understood as a combination of those. The first is, for equivalence relations $\sim : A \to A \to \mathbf{Prop}$, the "Yoneda"-characterisation

$$\forall ab.\, a \sim b \leftrightarrow \Pi_{x:A} a \sim x \leftrightarrow b \sim x$$

Unfortunately, it is not possible to generalise this in the straightforward way to higher relation as there is an unwanted "shift" of the level by 1 included. If we try to use the type

$$\forall ab.\, (a \sim b) \equiv \Pi_{x:A}(a \sim x) \equiv (b \sim x),$$

we quickly realise that the right-hand side goes up "one equality level too much". For example, if we have a type with only one term a, and $a \sim a \equiv \mathbf{5}$, then the left-hand side is $\mathbf{5}$, while the right-hand side is $\mathbf{120}$ (there are 5! automorphisms on the set $\mathbf{5}$). We could try to fix this by stating

$$\forall ab.\, a \sim b \leftrightarrow \Pi_{x:A} a \sim x \equiv b \sim x,$$

but clearly, logical equivalence is not enough for a valid characterisation.

The second source of inspiration has been the definition of an equivalence class by Voevodsky. For $P : A \to \mathbf{Prop}$, we may define the statement that P *is an equivalence class* by saying that it is non-empty and, for any two elements of A,

$$\mathsf{isCl}(P : A \to \mathbf{Prop}) :\equiv \|\Sigma_{a:A} P\, a\| \times \Pi_{a:A} P\, a \to \Pi_{b:A} \|P\, b \leftrightarrow a \sim b\|$$

Here, it is already assumed that $\sim$ is an equivalence relation and the idea is that the quotient is just the collection of equivalence classes. Originally, our definition used equivalence classes, making it very similar to the one of Voevodsky. After realising that it is not necessary to have a whole type of equivalence classes indexed over A (which works, but it involved), we were able to simplify it by just using a single $F : A \to \mathcal{U}$ which also adds indexes from a proper set (such as the natural numbers) to distinguish classes that are isomorphic, but distinct.

We want to sketch the fairly simple proof that a Yoneda Groupoid is indeed a weak ω groupoid.

Lemma 3. *Given* $p : \mathsf{isYonedaGrp}(\sim)$*, the higher relation* $\sim$ *carries the structure of a weak* ω *groupoid and this structure can be extracted purely syntactically from the proof* p*.*

Remark 4. Of course, the structure is not unique in general, as we have no way to distinguish between terms of $a \sim b$ (without looking at p). But, and this is more important, even up to isomorphism, there are fundamentally different structures. For example, for $A :\equiv \mathbf{1}$ and $\_ \sim \_ :\equiv \mathbf{6}$, $\sim$ could be either the equivalent of the group $\mathbb{Z}/(6)$ or the equivalent of the permutation group S_3. It really is the proof of $\mathsf{isYonedaGrp}$ that makes the choice.

Proof. The main ingredient of our construction is the groupoid property of equality itself. In particular, equality provides the usual terms $\mathsf{refl} : \forall a.\, a = a$ and $\mathsf{sym} : \forall ab.\, a = b \to b = a$ as well as $\mathsf{trans} : \forall abc.\, b = c \to a = b \to a = c$.

The proof $p : \mathsf{isYonedaGrp}(\sim)$ is necessarily a pair (F, i) with

$$\begin{aligned} &F : A \to \mathbb{N} \times \mathbf{U} \\ &i : \forall (ab : A).\, (a \sim b) = (F\, a = F\, b). \end{aligned}$$

We define $\mathsf{refl}^\sim$, $\mathsf{sym}^\sim$ and $\mathsf{trans}^\sim$ in terms of p. For readability, we first omit arguments in the definitions that can easily be inferred. We also implicitly use the usual function that transforms an equality between types into a function between types.

$$\begin{aligned} \mathsf{refl}^\sim &:\equiv \mathsf{sym}\, i\, \mathsf{refl} \\ \mathsf{sym}^\sim\, s &:\equiv \mathsf{sym}\, i\, (\mathsf{sym}\, (i\, s)) \\ \mathsf{trans}^\sim\, t\, s &:\equiv \mathsf{sym}\, i\, (\mathsf{trans}\, (i\, t)\, (i\, s)) \end{aligned}$$

The strategy is the same in each case: We use the isomorphism (or equality) i to translate the problem to the case where $\sim$ is replaced by $=$. Now, in the case of equality, we know exactly how the required operation can be done, and we can just transport the result back using the inverse of i.

Every single coherence condition just holds because it holds for equality. For example,

$$\mathsf{sym}^{\sim} \circ \mathsf{sym}^{\sim}(s) = \mathsf{sym}\, i\, (\mathsf{sym}\, (i\, (\mathsf{sym}\, i\, (\mathsf{sym}\, (i\, s)))))$$

is propositionally equal to s. For a proof, we just need to use that $i \circ (\mathsf{sym}\, i)$ is the identity, then the same for $\mathsf{sym} \circ \mathsf{sym}$, and finally, that $(\mathsf{sym}\, i) \circ i$ is the identity as well. It becomes even clearer if we write $\cdot^{-1}$ for sym and $f \circ g(a)$ instead of $f(g(a))$:

$$\mathsf{sym}^{\sim} \circ \mathsf{sym}^{\sim}(s) = i^{-1}\, ((i \circ i^{-1}\, (i\, s)^{-1})^{-1}$$

□

3 Quotienting by a Yoneda Groupoid

With the above developments and *propositional truncation* we can form a quotient (in a reasonable way). However, the quotient that we define will not live inside the same universe. Let $\sim$ together with (F, i) : $\mathsf{isYonedaGrp}(\sim)$ be a Yoneda Groupoid.

Define the "carrier" of the quotient

$$Q :\equiv \Sigma_{x:\mathbb{N}\times\mathcal{U}} \|\Sigma_{a:A} F(a) = x\|_{-1}$$

and the projection into the carrier to be

$$\begin{aligned} &q : A \to Q \\ &q(a) :\equiv (F(a), |a, \mathsf{refl}|). \end{aligned}$$

We then trivially have

$$\forall ab.\, (a \sim b) = (q(a) = q(b))$$

which corresponds to some form of *soundness and exactness* property.

4 Examples

Some examples, where we omit the natural number index of the equivalence classes (thus, we only give $F : A \to \mathcal{U}$ instead of $F : A \to \mathbb{N} \times \mathcal{U}$:

- $A \equiv \mathbf{1}$, $\_ \sim \_ \equiv \mathbf{6}$ is a Yoneda Groupoid, proved by $(\lambda\_ \to \mathbf{3}, \textit{someproof})$. The quotient is the symmetric group on $\mathbf{3}$, which is not inside the universe $\mathcal{U}$ anymore. This is exactly how it should have been expected, as it is consistent to assume that the equality of types in the lowest univalent universe is propositional. One universe above $\mathcal{U}$ does not allow this assumption anymore, and indeed, we have constructed the group $S_3 : \mathcal{U}_1$. $(A, \sim)$ has another possible quotient which is the group $\mathbb{Z}/(6)$, but unfortunately, we cannot get it with our construction.

- $A \equiv \mathbf{1}$, $\_ \sim \_ \equiv S_3$ (where we already need $\sim$ to be of the type $A \times A \to \mathcal{U}_1$) is a Yoneda Groupoid as $(\lambda\_ \to S_3, \mathit{someotherproof}) : \mathsf{isYonedaGrp}(\sim)$ (the symmetric group over S_3 is S_3 again). The quotient gives us a type of truncation level 2, let us call it $1_{S_3} : \mathcal{U}_2$. Obviously, we could carry on this example to get types with higher and higher structure, making more and more universes necessary.

- In the same way, we can construct the quotients for $A \equiv \mathbf{1}$, $\_ \sim \_ \equiv \mathbf{n}!$ for any natural number n. It is always a Yoneda Groupoid by $(\lambda\_ \to \mathbf{n}, \mathit{yetanotherproof})$ and the quotient will be the symmetric group S_n. However, our construction does not provide us with any other group structure on $\mathbf{n}!$. If we carry on as in the example before, the only thing we have to care about is that the automorphism groups of S_2 and S_6 is not, as in every other case, S_2 and S_6 again, thereby making these two cases special.

- We can now freely combine the groups on different levels constructed above, for example, we get $S_3 \times 1_{S_7} + S_5 + 3 : \mathcal{U}_2$, which is a groupoid with 5 distinguishable cells on level zero, 11 on level one, and 29 on level two. There are also 29 n-cells for every $n > 2$.

5 The Root of Equality

Our definitions immediately give rise to the question: when does this function F exist? Put differently: Given a type C, in which cases is $\Sigma_{B:\mathcal{U}} C = (B = B)$ inhabited?

In the example $C \equiv \mathbf{6}$, a solution exists, namely $B \equiv \mathbf{3}$, leading to the symmetric group S_3 as discussed before. However, we cannot construct the group $\mathbb{Z}/(6)$. In the case of $C \equiv \mathbf{1}$, we get two solutions, namely S_0 and S_1.

Can we find an appropriate structure if C is not a discrete set where the number of terms equals a factorial? First, does this structure exist in the ω groupoid model (resp. the simplicial set model)? Such a group does exist indeed for $C \equiv \mathbf{3}$ which can be generalised to other non-factorial sets (Christian Sattler). We conjecture that for any n-groupoid there always is an $n{+}1$-groupoid with the required property.

We do not know whether there are solutions or partial solutions to the corresponding problem of ω-groupoids (or ω-categories, (ω, n)-categories, ...).

References

[1] Thorsten Altenkirch and Ondrej Rypacek. *A syntactical approach to weak omega groupoids.* 2012.