# Extending Finite Memory Determinacy to Multiplayer Games (Extended Abstract)

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We show that under some general conditions the finite memory determinacy of a class of two-player win/lose games played on finite graphs implies the existence of a Nash equilibrium built from finite memory strategies for the corresponding class of multi-player multi-outcome games. This generalizes a previous result by Brihaye, De Pril and Schewe. We provide a number of example that separate the various criteria we explore.

Our proofs are generally constructive, that is, provide upper bounds for the memory required, as well as algorithms to compute the relevant Nash equilibria.

## 1 Introduction

The usual model employed for synthesis are sequential two-player win/lose games played on finite graphs. The vertices of the graph correspond to states of a system, and the two players jointly generate an infinite path through the graph (the *run*). One player, the protagonist, models the aspects of the system under the control of the designer. In particular, the protagonist will win the game iff the run satisfies the intended specification. The other player is assumed to be fully antagonistic, thus wins iff the protagonist loses. One then would like to find winning strategies of the protagonist, that is, a strategy for her to play the game in such a way that she will win regardless of the antagonist's moves. Particularly desirable winning strategies are those which can be executed by a finite automaton.

Our goal is to dispose of two restrictions of this setting: First, we would like to consider any number of players; and second allow them to have far more complicated preferences than just preferring winning over losing. The former generalization is crucial in a distributed setting (also e.g. [1, 2]): If different designers control different parts of the system, they may have different specifications they would like to enforce, which may be partially but not entirely overlapping. The latter seems desirable in a broad range of contexts. Indeed, rarely is the intention for the behaviour of a system formulated entirely in black and white: We prefer a programm just crashing to accidently erasing our hard-drive; we prefer a programm to complete its task in 1 minute to it taking 5 minutes, etc. We point to [5] for a recent survey on such notions of quality in synthesis.

Rather than achieving this goal by revisiting each individual type of game and proving the desired results directly (e.g. by generalizing the original proofs of the existence of winning strategies), we shall provide two transfer theorems: In both Theorem 5 and Theorem 7, we will show that (under some conditions), if the two-player win/lose version of a game is finite-memory

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determined, the corresponding multi-player multi-outcome games all have finite-memory Nash equilibria. The difference is that Theorem 5 refers to all games played on finite graphs using certain preferences, whereas Theorem 7 refers to one fixed graph only.

Theorem 7 is more general than a similar one obtained by Brihaye, De Pril and Schewe [1],[11, Theorem 4.4.14]. A particular class of games covered by our result but not the previous one are (a variant of) energy parity games as introduced by Chatterjee and Doyen [3]. The high-level proof idea follows earlier work by the authors on equilibria in infinite sequential games, using Borel determinacy as a blackbox [7] – unlike the constructions there (cf. [8]), the present ones however are constructive and thus give rise to algorithms computing the equilibria in the multi-player multi-outcome games given suitable winning strategies in the two-player win/lose versions.

The general theme of transferring determinacy results from antagonistic two-player games to the existence of Nash equilibria in multiplayer games is already present in [12] by Thuijsman and Raghavan, as well as [4] by Grädel and Ummels.

Echoing De Pril in [11], we would like to stress that our conditions apply to the preferences of each player individually. For example, some players could pursue energy parity conditions, whereas others have preferences based on Muller conditions: Our results apply just as they would do if all players had preferences of the same type.

This work extends and supersedes the earlier [10] which appeared in the proceedings of Strategic Reasoning 2016. A full version is available on the arXiv as [9].

## 2 Definitions and notations

**Threshold games and future games:** Our results, including transfer from the two-player win/lose case to the general case, rely on the idea that each general game induces a collection of two-player win/lose games, namely the threshold games of the future games, as below.

**Definition 1** (Future game and one-vs-all threshold game)**.**

Let $g = \langle (V,E), v_0, A, \{V_a\}_{a\in A}, (\prec_a)_{a\in A}\rangle$ be a game played on a finite graph.

- For $a_0 \in A$ and $\rho \in [\mathcal{H}]$, the one-vs-all threshold game $g_{a_0,\rho}$ for $a_0$ and $\rho$ is the win-lose two-player game played on $(V,E)$, starting at $v_0$, with vertex subsets $V_{a_0}$ and $\bigcup_{a\in A\setminus\{a_0\}} V_a$, and with winning set $\{\rho' \in [\mathcal{H}] \mid \rho \prec_{a_0} \rho'\}$ for Player 1.
- Let $v \in V$. For paths $hv$ and $vh'$ in $(V,E)$ let $hv\hat{}vh' := hvh'$.
- For all $h \in \mathcal{H}$ with last vertex $v$ let $g^h := \langle (V,E), v, A, \{V_a\}_{a\in A}, (\prec^h_a)_{a\in A}\rangle$ be called the future game of $g$ after $h$, where for all $\rho,\rho' \in [\mathcal{H}_{g^h}]$ we set $\rho \prec^h_a \rho'$ iff $h\hat{}\rho \prec_a h\hat{}\rho'$. If $s$ is a strategy in $g$, let $s^h$ be the strategy in $g^h$ such that $s^h(h') := s(h\hat{}h')$ for all $h' \in \mathcal{H}_{g^h}$.

Our (transfer) results rely on players having winning strategies that are implementable with *uniformly* finite memory, so that for every game they may be picked from a finite set of strategies. The following (shortenable) shorthands will be useful. Let $g$ be a game, let $a$ be a player.

- Let $m \in \mathbb{N}$ be such that in all threshold games for $a$ in $g$, if player $a$ has a winning strategy, she has one that is implementable using $m$ bits of memory. Then we say that player $a$ wins her winnable threshold games in $g$ using uniformly finite memory $m$.
- Let $m \in \mathbb{N}$ be such that all (future) threshold games for $a$ in $g$ have finite-memory winning strategies that are implementable using $m$ bits of memory. Then we say that the (future) threshold games for $a$ in $g$ are uniformly-finite-memory determined using $m$ bits.

Note that speaking about future games above is the more general statement, as we prefix some finite history, and the sufficient memory depends on neither the threshold nor the history.

**Guarantees** Definition 2 below rephrases Definitions 2.3 and 2.5 from [6]: Given a strategy of a player $a$, the guarantee consists of the compatible runs plus the runs that are, according to $\prec_a$, not worse than all the compatible runs. The guarantee is thus upper-closed w.r.t. $\prec_a$. The best guarantee is the intersection of all the guarantees, and is thus also upper-closed.

**Definition 2** (Player (best) future guarantee)**.** Let $g$ be the game $\langle (V,E), v_0, A, \{V_a\}_{a\in A}, (\prec_a)_{a\in A}\rangle$ and let $a \in A$. For all $h \in \mathcal{H}$ and strategies $s_a$ for $a$ in $g^h$ let $\gamma_a(h, s_a) := \{\rho \in [\mathcal{H}_{g^h}] \mid \exists \rho' \in [\mathcal{H}_{g^h}(s_a)], \neg(\rho \prec_a^h \rho')\}$ be the player future guarantee by $s_a$ in $g^h$. Let $\Gamma_a(h) := \bigcap_{s_a} \gamma_a(h, s_a)$ be the best future guarantee of $a$ in $g^h$. We write $\gamma_a(s_a)$ and $\Gamma_a$ when $h$ is the trivial history.

Note that in general the best guarantee may be empty, but our assumptions will (indirectly) rule this out: they will imply that each player has indeed a strategy realizing her best guarantee, i.e. a strategy $s_a$ with $\Gamma_a = \gamma_a(s_a)$.

**Optimality** Strategies may be optimal either at the beginning of the game or at all histories. This useful game-theoretic concept is rephrased below in terms of best guarantee.

**Definition 3.** Let $s_a$ be a strategy for some player $a$ in some game $g$.

- $s_a$ is *optimal*, if $\gamma_a(s_a) = \Gamma_a$.
- $s_a$ is *optimal at history* $h \in \mathcal{H}$, if $s_a^h$ is optimal in $g^h$.
- $s_a$ is *subgame-optimal*, if it is optimal at all $h \in \mathcal{H}$.
- $s_a$ is *consistent-optimal*, if it is optimal at all $h \in \mathcal{H}(s_a)$.

Note that the notion of optimality is orthogonal to that of determinacy. Especially, the players having optimal strategies does not imply determinacy of the derived threshold games (in an undetermined win/lose game, the guarantee of both players is the set of all runs – hence, any strategy is optimal).

A key ingredient in our proof essentially consists in a quantifier inversion. Assuming that for all histories there is a finite-memory optimal strategy, we will use them to construct a finite-memory strategy that is subgame-optimal. To know when to use which strategy, we will use the assumption that optimality of a given strategy at an arbitrary history can be decided regularly.

**Definition 4.** A player $a$ in a game $g$ has the *optimality is regular* (OIR) property, if for all finite-memory strategies $s_a$ for $a$ in $g$, there exists a finite automaton that decides on input $h \in \mathcal{H}$ whether or not $s_a$ is optimal at $h$.

A game $g$ has the OIR property, if all the players have it in $g$.

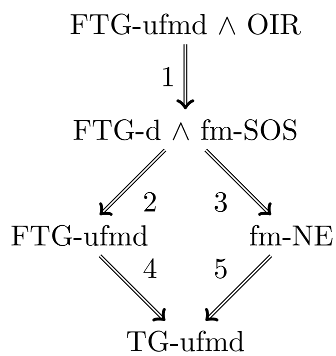
A preference has the optimality is regular property, if players with this preference have the OIR in all games.

## 3 The results

In this extended abstract, the games are always played on finite graphs, and they always involve colors in $C$, players in $A$, and strict weak order preferences $(\prec_a)_{a\in A}$. Theorem 5 presents implications, and absence of stated implication is discussed afterwards. For several implications we use the assumption that the set of preferences is closed under antagonism, *i.e.* for all $\prec_a$ there exists $b \in A$ such that $\prec_b = \prec_a^{-1}$.

**Theorem 5.** Let $(\prec_a)_{a\in A}$ be closed under antagonism. The statements below refer to all the games built with $C$, $A$, and $(\prec_a)_{a\in A}$, and the diagram displays implications between the statements.

- **OIR:** Optimality is regular.
- **fm-SOS:** There are finite-memory subgame-optimal strategies.
- **fm-NE:** There are finite-memory Nash equilibrium.
- **FTG-d:** The future threshold games are determined.
- **TG-ufmd** In every game, the threshold games are determined using uniformly finite memory.
- **FTG-ufmd:** In every game, the future threshold games are determined using uniformly finite memory.




We can show that neither Implication (1) nor Implication (2) are reversible. For the remaining implications, we do not have answers regarding their reversibility. Regarding Implication (4), note that if we were to fix a specific graph, it would be trivial to separate the two notions. The difficulty lies in the requirement to deal with all finite graphs simultaneously.

Theorem 5 relies on many games having the assumed property, *i.e* the statements are universal quantifications over games. However, a given game might enjoy some property, *e.g.* existence of NE, not only due to the preferences on infinite sequences of colors, but also due to the specific structure of the underlying graph. Theorem 7 below captures this idea. It is a generalization of our main result in [10]. (The new Assumption 2 is indeed weaker.)

**Definition 6.** Given a game $g$, a preference $\prec$ is Mont (in $g$) if for every regular run $h_0\hat{}\rho \in [\mathcal{H}]$ and for every family $(h_n)_{n\in\mathbb{N}}$ of paths in $(V,E)$ such that $h_0\hat{}\dots\hat{}h_n\hat{}\rho \in [\mathcal{H}]$ for all $n$, if $h_0\hat{}\dots\hat{}h_n\hat{}\rho \prec h_0\hat{}\dots\hat{}h_{n+1}\hat{}\rho$ for all $n$ then $h_0\hat{}\rho \prec h_0\hat{}h_1\hat{}h_2\hat{}h_3\dots$.

**Theorem 7.** Let $g$ be a game such that

1. For all $a \in A$ the future threshold games for $a$ in $g$ are determined, and each of Player 1 and 2 can win their winnable games via $k$ fixed strategies using $T$ bits each.
2. Optimality for such a strategy is regular using $D$ bits.
3. The preferences in $g$ are Mont.

Then $g$ has a Nash equilibrium made of strategies using $|A|(kD + kT + \log k) + 1$ bits each.

The difference between Theorem 7 and Theorem 5 is similar to the difference between previous works by the authors for games on infinite trees (without memory concerns): Namely, between [6] and [7]. The first result characterizes the preferences that guarantee existence of NE for all games; the second result relies on the specific structure of a given game to guarantee existence of NE for the given game.

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