# Multi-agent timed Modal Epistemic Logic with Common Knowledge*

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**Abstract**

The timed Modal Epistemic Logic, tMEL is an epistemic logical framework extended from the traditional Modal Epistemic logic MEL, with the ability to represent the reasoning time taken by the agent. It has formulas of the form $K^i\phi$, instead of just $K\phi$, with the intended meaning that $\phi$ is known by the agent at the time $i$. In this paper we will discuss several multi-agent versions of tMEL, including the case in which a common-knowledge-like operator is introduced.

## 1 Introduction

*The Nell & Dudley Problem: Dudley wishes to save Nell, who is tied to the railroad tracks as a train bears down. He must come up with a plan and carry it out before it is too late. Clearly he must not waste time searching through all possible plans for a theoretical best one. That is, he must take into account the fact that every second he spends planning is one more second gone by and hence one second less in which to carry out a plan — and also one second less before the train reaches Nell.*

This is an old story distributed in the circle of Artificial Intelligence for years [2].It indicates two things: First, knowledge and reasoning ability are critical for an agent to carry out a task — to list possible plans and come up with the best one; and second, in a case like this, the passage of reasoning time is a decisive factor to determine if a plan can be successfully and meaningfully carried out by the agent, or not. We have seen the usefulness of the modal approach of epistemic logical formalism in the study of the science of computing and artificial intelligence. However, it is obvious that this traditional modal formalism, with its modest expressivity, can't help to deal with the problems raised in these scenarios. With the formalism, all we can get is what is known by the agent, but we also need the information about how long it would be taken by the agent to derive the knowledge, the information which can be utilized by us, the modeler, to judge if the modeled agent will pass the "*deadline constraints*."

*This is an ongoing project.

Recently, a proposed epistemic logical framework *timed Modal Epistemic Logic*, tMEL seems fit the need [3]. It is extended from Modal Epistemic Logic, MEL with the mechanism that can representing the reasoning time of the agent. Formulas of the form $K^i\phi$, instead of just $K\phi$, are introduced, with the intended meaning that $\phi$ is known by the agent at the time $i$, and the magnitude of $i$ will reflect the agent's reasoning ability and the difficulty of deducing the formula $\phi$. It is then a natural step to consider multi-agent versions of tMEL, especially examine the case in which common knowledge is involved. But there is a difficulty to directly subsume common knowledge into multi-agent consideration of tMEL.

The genuine concept of common knowledge is a static one, which is different from the ideology set in the formalism of tMEL which is dynamic in nature. Common knowledge is a kind of statements of which the information is shared by a group of anticipated agents. But rather in tMEL, we consider the knowledge that is actively generated by the agents with their reasoning abilities. Fortunately, when we trace the development of formulating common knowledge in the modal formalism, we find our solution of it. We find the first trial in [1], in which a modal operator "fool $O$" is introduced. The idea of "fool $O$" is that if the fool knows, then everybody knows ($O\phi{\rightarrow}K_a\phi$)). So the fool is just like another agent in this formulation, and hence we can generalize this idea to consider the knowledge that the fool takes time to deduce, which will be the our substitute of common knowledge, and it will be accompanied with the time that everybody needs to consume in order to obtain the knowledge.

## 2 timed Modal Epistemic Logic

In this section we sketch the framework of tMEL. The languages of tMEL is adapted from tMEL such that the modal formula is formulated by the formation rule: if $\phi$ is a formula, then $K^i\phi$, not $K\phi$ is also a formula, where $i$ a natural number. One thing distinctive of tMEL logics is that they are *base-parameterized*. By a base, formally we mean a tuple $\mathcal{A}{=}\langle \mathbf{A}, \mathbf{f}\rangle$, with $\mathbf{A}$ a set of tMEL formulas, and $\mathbf{f}\colon \mathbf{A} \mapsto \mathbb{N}$ (natural numbers). We will explain the significance of bases later. Now given a base $\mathcal{A}$, an $\mathcal{A}$-*awareness function* $\alpha$ is defined as a partial function mapping tMEL formulas to natural numbers and satisfying the following conditions ($\alpha(\phi)\downarrow$ denotes $\alpha(\phi)$ is defined):

**0**. If $A \in \mathbf{A}$, then $\alpha(A) \leq \mathbf{f}(A)$.

(*Initial Condition*)

**1**. If $\alpha(\phi \rightarrow \psi)\downarrow$ and $\alpha(\phi)\downarrow$, then $\alpha(\psi) \leq \max(\alpha(\phi \rightarrow \psi), \alpha(\phi)) + 1$.

(*Deduction by Modus Ponens*)

**2**. If $A \in \mathbf{A}$ and $f(A) \leq i$, then $\alpha(K^i A) \leq i + 1$.

(*Deduction by $\mathcal{A}$-Epistemization*)

**3**. If $\alpha(\phi)\downarrow$ and $\alpha(\phi) \leq i$, then $\alpha(K^i\phi) \leq i + 1$.

(*Inner Positive Introspection*)

The goal of these awareness functions is nothing but chronologically recording the deductive reasoning that the modeled agent practices, and these conditions reflect the logical rules that the depicted agent is assumed to perform, with each application of the rules taking one unit of time.

Fix a base $\mathcal{A}$. By a $\mathsf{tS4}(\mathcal{A})$-*structure* we mean a tuple $M = \langle W, R, \mathfrak{A}, \mathcal{V}\rangle$, where $\langle W, R, \mathcal{V}\rangle$ is an $\mathsf{S4}$ structure and $\mathfrak{A}$ is a collection of $\mathcal{A}$-awareness functions $\alpha_w$, one for each world $w \in W$, satisfying the *monotonicity condition*: given a $\mathsf{tMEL}$ formula $\phi$, if $\alpha_w(\phi)\downarrow$, then $\alpha_{w'}(\phi) \leq \alpha_w(\phi)$. So the basic idea behind this semantics is that at each world, there is a static world view, like before, which includes the truth-values of those basic events, represented by propositional letters, and the reasoning process taken by the agent at the world.

Now we can define the truth-value for $\mathsf{tMEL}$ formulas. For the propositional cases, they are treated in the same way as those of $\mathsf{MEL}$ formulas, and for a labelled modal formulas, its truth-value is given as follows:

$$(M, w) \vDash K^i\phi \quad \text{iff} \quad (M, w') \vDash \phi \text{ for all } w' \in W \text{ with } wRw', \text{ and } \alpha_w(\phi) \leq i.$$

According to this analysis, an agent knows at the time $i$ that $\phi$ is the case if and only if it is true at all accessible possible worlds and the agent is able to, by activating her/his deductive reasoning faculty on the information the agent has at the world, be aware of the truth of the formula before that time.

Given bases $\mathcal{A} = \langle \mathbf{A}, f\rangle$ and $\mathcal{B} = \langle \mathbf{B}, g\rangle$, $\mathcal{B} \subseteq \mathcal{A}$ means $\mathbf{B} \subseteq \mathbf{A}$ and $f(B) \leq g(B)$ for all $B \in \mathbf{B}$. A collection of bases $\{\mathcal{A}_i (= \langle \mathbf{A}_i, f_i\rangle)\}_{i\in\mathbb{N}}$ is an *ascending chain* if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$, and a base $\mathcal{A}$ is the *limit* of the chain if $\mathcal{A} = \bigcup \mathcal{A}_i$, i.e., $\mathbf{A} = \bigcup \mathbf{A}_i$ and $f(A) = \min\{f_i(A) : f_i(A)\downarrow\}$.

**Definition 2.1.** *A base $\mathcal{A}$ is* $\mathsf{tS4}$ logical *if one of the following is true:*
*(1) $\mathcal{A}$ is empty,*
*(2) in $\mathcal{A} = \langle \mathbf{A}, f\rangle$, $\mathbf{A}$ consists of* $\mathsf{tS4}$*($\mathcal{B}$) valid formulas with $\mathcal{B}$ a* $\mathsf{tS4}$ *logical base,*
*(3) $\mathcal{A}$ is the limit of an ascending chain of* $\mathsf{tS4}$ *logical bases $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$, with $\mathcal{A}_{i+1}$ consisting of* $\mathsf{tS4}(\mathcal{A}_i)$ *valid formulas.*

**Lemma 2.2.** *If $\mathcal{A} = \langle \mathbf{A}, \mathbf{f}\rangle$ is a tS4 logical base, every formula in $\mathbf{A}$ is tS4($\mathcal{A}$) valid.*

Let $\mathcal{A}$ be a $\mathsf{tS4}$ logical base. The axiom system of tS4($\mathcal{A}$) is as follows:

Axioms:
A0 Classical propositional axiom schemes,
A1 $K^i(\phi \to \psi) \to (K^j\phi \to K^k\psi) \quad i, j < k$,
A1' $K^iA \to K^j(K^iA) \quad i < j$ if $A \in \mathbf{A}$ and $f(A) \leq i$,
A2 $K^i\phi \to K^j(K^i\phi) \quad i < j$,
A3 $K^i\phi \to \phi$,
A4 $K^i\phi \to K^j\phi \quad i < j$, (*Monotonicity Axiom*)

Inference Rules
R1 if $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$,
R2 if $A \in \mathbf{A}$ and $f(A) \leq i$, then $\vdash K^iA$, ($\mathcal{A}$-*Epistemization*)

for all $i, j, k \in \mathbb{N}$.

**Theorem 2.3.** *Given a* $\mathsf{tS4}$ *logical base $\mathcal{A}$, a* $\mathsf{tMEL}$ *formula is* $\mathsf{tS4}$*($\mathcal{A}$) valid if and only if it is provable in the axiom system of* $\mathsf{tS4}$*($\mathcal{A}$).*

So corresponding to an MEL logic, such as S4, there is in fact a family of tMEL logics, tS4($\mathcal{A}$), introduced, with each tS4 logical base $\mathcal{A}$ to indicate the basic logical truths to be used by the agent. We have seen the empty base. We can also have a *comprehensive logical base* which include all valid formulas. One logical base especially interests us. We call a tMEL logic, also the logics introduced later on, has the *internalization property* if for any valid formula $\phi$, there is an $i \in \mathbb{N}$ such that $K^i\phi$ is valid. Then we call a logical base $\mathcal{A}$ *full*, if, using the base, the tMEL logic, as well as the logics defined later, have the internalization property. Certainly a comprehensive base is full, but so is a base containing all the axiom instances of the schemes listed in the above system. Given the completeness result, this can be proved by induction on a proof of the valid formula, with only axioms A1 and A1' applied. We can understand agents with full or stronger logical bases as having enough basic logical truths from which they can have *complete* epistemic logical knowledge about themselves. Logics with full or stronger logical bases normally have nice properties, which include the following one concerning the formal relation between MEL logics and their tMEL counterparts. Here's the version for S4 and tS4:

**Theorem 2.4.** *Given a full* tS4 *logical base $\mathcal{A}$, an* MEL *formula $\phi$ is a* S4 *valid formula if and only if there is a suitable label for each of the epistemic modalities $K$ in $\phi$ such that turning all the epistemic modalities in $\phi$ into modalities with their suitable labels $K^i$, we have an* tS4($\mathcal{A}$) *valid formula.*

# References


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