Epistemic Quantification and Quantified Announcements

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Logics of quantified announcements – APAL, GAL, and CAL – have been studied extensively in recent years. In this paper, we consider them through the formalism based on the one from [8], which explicitly contains quantifiers in its language. Such a logic is an extension of Public Announcement Logic with added epistemic quantification. The logic allows us to express operators of APAL, GAL, and CAL, and provides an alternative viewpoint on them. Moreover, we show that the logic is sound and complete.

1 Introduction

We start with an example of an epistemic model in Figure 1.

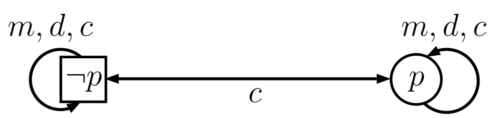

Figure 1: An example

There are three agents: m (Mum), d (Dad), and c (Child). Let p denote the proposition that 'Santa Claus exists.' There are two states in the model: a square state with $\neg p$, and a circle state with p. Let the square state be the actual one. Arrows connect states which an agent cannot distinguish. In the actual state, a child (agent c) does not know whether Santa Claus exists. Mum and Dad, on the contrary, are quite sure of his non-existence. Now, suppose someone, e.g. a very reliable person on the Internet, announces that Santa Claus does not exist. This truthful public announcement 'deletes' all the states incompatible with it and corresponding relations; in this example the circle state is 'deleted.' After such an announcement Child knows $\neg p$, or, formally, $[\neg p]K_c\neg p$. In this paper we are primarily interested in quantification over announcements. So, in the square state, since Mum and Dad know $\neg p$, no announcement can force them to 'unknow' it, or, formally, $\Box(K_m p \wedge K_d p)$. Apart from such arbitrary public announcements, we will consider also announcements by groups and coalitions of agents. For instance, if a group consists only of Child, the following holds: $[c](\neg K_c\neg p \wedge \neg K_c p)$, i.e. whatever agent c announces, she does not know whether p. Also, Mum and Dad can remain silent (or announce $\top$) and preclude Child from knowing that Santa Claus does not exist. In other words, there is announcement by their group such that agent c does not know the value of p: $\langle m \cup d\rangle(\neg K_c\neg p \wedge \neg K_c p)$. Moreover, this holds whatever Child announces at the same time: $\langle\!\![m \cup d]\!\!\rangle(\neg K_c\neg p \wedge \neg K_c p)$. In this competitive case we say that Mum and Dad form a coalition. On the other hand, a coalition consisting of Mum and Child does not have such a power, since Dad can always announce that Santa Claus does not exist and put an end to Child's doubts: $\neg\langle\!\![m \cup c]\!\!\rangle(\neg K_c\neg p \wedge \neg K_c p)$, or, equally, $[\!\langle m \cup c\rangle\!](K_c\neg p \vee K_c p)$.

The formalism which allows reasoning about public announcements is Public Announcement Logic (**PAL**) [9, 6]. Its extension with arbitrary public announcement operators $\Box\varphi$ is called Arbitrary Public

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Announcement Logic (**APAL**) [5, 4]. Logic for reasoning about announcements by groups of agents – Group Announcement Logic (**GAL**) – was developed in [1]. Finally, Coalition Announcement Logic (**CAL**) was introduced in [2] to reason about competitive announcements by agents. However, there is no complete axiomatisation of **CAL** [2, 3]. Moreover, there are no finitary axiomatisations of the logics of quantified announcements. In this paper, we propose a complete logic where operators of **APAL**, **GAL**, and **CAL** are expressible. Such a formalism provides an additional tool for studying the aforementioned logics. In particular, we interpret quantified announcements from the standpoint of epistemic quantification. For that, we use a fragment of the logic for reasoning about knowledge of unawareness [8] (we call this fragment **HR**), which allows quantification over epistemic formulas, and enrich the fragment with public announcements. In the resulting logic, $\mathbf{PAL}^\forall$, we define operators for quantified announcements, and then show that the logic is complete.

2 Syntax and Semantics

Throughout the paper, let A be a finite set of agents with $G \subseteq A$, and let P and X be disjoint countable sets of propositional variables.

Definition 2.1. The *language* $\mathscr{L}_{PAL^\forall}$ of $\mathbf{PAL}^\forall$ is described by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \forall x\varphi,$$

where $x \in X$, $p \in P$, and $a \in A$. The language $\mathscr{L}_{PAL}$ is the language without quantifiers and variables from X, $\mathscr{L}_{HR}$ – without public announcements, and the language $\mathscr{L}_{EL}$ of epistemic logic is the language without quantifiers, variables form X, and public announcements. As usual, $\varphi \to \psi$ is a shorthand for $\neg(\varphi \wedge \neg\psi)$, and $\forall x_1,\ldots,x_n\varphi$ stands for $\forall x_1 \ldots \forall x_n\varphi$, where $x_1,\ldots,x_n \in X$.

Definition 2.2. Let φ and ψ be formulas, then $\varphi[x\backslash\psi]$ is the formula where all free occurrences of x in φ are replaced by ψ. Formula ψ is called *substitutable* for x in φ if for all propositional variables $y \in X$ their free occurrence in φ remains free in $\varphi[x\backslash\psi]$.

Formulas of all logics mentioned in the paper are interpreted in epistemic models (Figure 1 is an example of such a model).

Definition 2.3. An *epistemic model* is a triple $M = (W, \sim, V)$, where

- W is a non-empty set of states;
- $\sim: A \to \mathscr{P}(W \times W)$ assigns an equivalence relation to each agent; we will denote relation assigned to agent $a \in A$ by $\sim_a$;
- $V : P \to \mathscr{P}(W)$ assigns a set of states to each propositional variable.

A pair $(W,\sim)$ is called an *epistemic frame*, and a pair (M,w) with $w \in W$ is called a *pointed model*. An announcement in a pointed model (M,w) results in an *updated pointed model* (M^φ,w). Here $M^\varphi = (W^\varphi, \sim^\varphi, V^\varphi)$, and $W^\varphi = [\![\varphi]\!]_M$, $\sim_a^\varphi = \sim_a \cap ([\![\varphi]\!]_M \times [\![\varphi]\!]_M)$, and $V^\varphi(p) = V(p) \cap [\![\varphi]\!]_M$. Generally speaking, an updated pointed model (M^φ,w) is a restriction of the original one to the states where φ holds.

Quantification in $\mathbf{PAL}^\forall$, following [8], is non-standard. As opposed to the classic quantification over propositional variables [7], it allows for quantification over epistemic formulas. This enables us to reason about 'whatever agents announce,' 'there is an epistemic announcement,' etc. The formal treatment of such quantification is provided in the following definition.

Definition 2.4. Let a pointed model (M,w) with $M = (W, \sim, V)$, $a \in A$, and φ, $\psi \in \mathscr{L}_{PAL^\forall}$ be given.

$$
\begin{array}{lll}
(M,w) \models p & \text{iff} & w \in V(p) \\
(M,w) \models \neg\varphi & \text{iff} & (M,w) \not\models \varphi \\
(M,w) \models \varphi \wedge \psi & \text{iff} & (M,w) \models \varphi \text{ and } (M,w) \models \psi \\
(M,w) \models K_a\varphi & \text{iff} & \forall v \in W : w \sim_a v \text{ implies } (M,v) \models \varphi \\
(M,w) \models [\varphi]\psi & \text{iff} & (M,w) \models \varphi \text{ implies } (M^{\varphi},w) \models \psi \\
(M,w) \models \forall x\varphi & \text{iff} & (M,w) \models \varphi[x\backslash\psi] \text{ for all } \psi \in \mathscr{L}_{EL}
\end{array}
$$

Formula φ is called *valid* if for any pointed model (M,w) it holds that $(M,w) \models \varphi$.

Although operators of quantified announcements are not the part of our language, we provide their semantics. First, arbitrary announcements.

$$(M,w) \models \Box\varphi \quad \text{iff} \quad \forall\psi{\in}\mathscr{L}_{EL} : (M,w) \models [\psi]\varphi,$$

which corresponds to 'after any epistemic announcement, φ holds.'

Now, let $\mathscr{L}^G_{EL}$ denote the set of formulas of the type $\bigwedge_{i\in G} K_i\varphi_i$, where for every $i \in G$ it holds that $\varphi_i \in \mathscr{L}_{EL}$. Group announcements are as follows:

$$(M,w) \models [G]\varphi \quad \text{iff} \quad \forall\psi{\in}\mathscr{L}^G_{EL} : (M,w) \models [\psi]\varphi,$$

which reads as 'whatever agents from G announce, they cannot avoid φ.'

Finally, coalition announcements:

$$(M,w) \models [\langle G\rangle]\varphi \quad \text{iff} \quad \forall\psi{\in}\mathscr{L}^G_{EL}\ \exists\chi{\in}\mathscr{L}^{A\backslash G}_{EL} : (M,w) \models \psi \rightarrow \langle\psi \wedge \chi\rangle\varphi,$$

which corresponds to 'whatever agents from G announce, agents from $A \setminus G$ have a simultaneous announcements, such that φ holds.'

3 Axiomatisation and Quantified Announcements

In this section we present an axiomatisation of $\mathbf{PAL}^{\forall}$ and show its soundness. Also, we define arbitrary, group, and coalition announcements within our language.

Definition 3.1. *Axiomatisation* of $\mathbf{PAL}^{\forall}$ is a union of axiomatisation of **HR** (which is the fragment of a logic for reasoning about knowledge of unawareness [8]), **PAL** [6], and additional axiom for interaction between public announcements and quantifiers.

$(A0)$	all instantiations of propositional tautologies,
$(A1)$	$K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$,
$(A2)$	$K_a\varphi \rightarrow \varphi$,
$(A3)$	$K_a\varphi \rightarrow K_aK_a\varphi$,
$(A4)$	$\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$,
$(A5)$	$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$,
$(A6)$	$[\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$,
$(A7)$	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$,
$(A8)$	$[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi)$,
$(A9)$	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$,
$(A10)$	$[\psi]\forall x\varphi \leftrightarrow (\psi \rightarrow \forall x[\psi]\varphi)$, where x is not free in ψ,
$(A11)$	$\forall x\varphi \rightarrow \varphi[x\backslash\psi]$,
$(A12)$	$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$,
$(A13)$	$\varphi \rightarrow \forall x\varphi$, if x is not free in φ,
$(A14)$	$\forall x K_a\varphi \rightarrow K_a\forall x\varphi$,
$(R0)$	$\vdash \varphi, \varphi \rightarrow \psi \Rightarrow \vdash \psi$,
$(R1)$	$\vdash \varphi \Rightarrow \vdash K_a\varphi$,
$(R2)$	$\vdash \varphi \Rightarrow \vdash [\psi]\varphi$,
$(R3)$	$\vdash \varphi \Rightarrow \forall x\varphi$.

Note, that in $A11$, ψ is a quantifier-free formula substitutable for x in φ.

Definition 3.2. (Soundness and completeness) An axiomatisation is *sound*, if for any formula φ of the language, it holds that $\vdash \varphi$ implies $\models \varphi$ is valid. And vice versa for *completeness*.

Soundness of **PAL** is due to [6, 9], and soundness of epistemic logic with epistemic quantification is due to soundness of a richer logic in [8]. The remaining case is to show validity of $A10$.

Proposition 3.3. $\models [\psi]\forall x\varphi \leftrightarrow (\psi \rightarrow \forall x[\psi]\varphi)$*, where x is not free in ψ, is valid.*

Proof. Let an arbitrary pointed model (M,w) be given.

From left to right. Suppose $(M,w) \models [\psi]\forall x\varphi$, which means that $(M,w) \models \psi$ implies $(M^\psi,w) \models \forall x\varphi$ by the semantics. The consequent is $(M^\psi,w) \models \varphi[x\backslash\tau]$ for all $\tau \in \mathscr{L}_{EL}$. The latter means that $(M,w) \models \psi$ implies that for all $\tau \in \mathscr{L}_{EL}$ it holds that $(M,w) \models \psi$ implies $(M^\psi,w) \models \varphi[x\backslash\tau]$. By the semantics, and by the fact that ψ does not contain x free, we have $(M,w) \models (\psi \to \forall x[\psi]\varphi)$.

From right to left. Suppose $(M,w) \models \psi \to \forall x[\psi]\varphi$, which means $(M,w) \models \psi$ implies that for all $\tau \in \mathscr{L}_{EL}$ we have $(M,w) \models [\psi]\varphi[x\backslash\tau]$. The latter is $(M,w) \models \psi$ implies $(M^\psi,w) \models \varphi[x\backslash\tau]$. Since x is not free in ψ, we can move the universal quantifier inside the implication to the consequent: $(M,w) \models \psi$ implies that $(M,w) \models \psi$ implies that for all $\tau \in \mathscr{L}_{EL}$: $(M^\psi,w) \models \varphi[x\backslash\tau]$. And this is $(M,w) \models \psi \to [\psi]\forall x\varphi$ by the semantics. □

Now, quantification over epistemic formulas allows us to define operators of **APAL**, **GAL**, and **CAL** within $\mathbf{PAL}^\forall$. Let us start with $\Box\varphi$:

$$\Box\varphi \leftrightarrow \forall x[x]\varphi.$$

Then, group announcements $[G]\varphi$:

$$[G]\varphi \leftrightarrow \forall x_{g_1},\dots,x_{g_n}[K_{g_1}x_{g_1} \wedge \dots \wedge K_{g_n}x_{g_n}]\varphi,$$

where $G = \{g_1,\dots,g_n\}$.

And, finally, coalition announcements $[\langle G\rangle]\varphi$:

$$[\langle G\rangle]\varphi \leftrightarrow \forall x_{g_1},\dots,x_{g_n},\exists y_{h_1},\dots,y_{h_k} : K_{g_1}x_{g_1} \wedge \dots \wedge K_{g_n}x_{g_n} \to \langle K_{g_1}x_{g_1} \wedge \dots \wedge K_{g_n}x_{g_n} \wedge K_{h_1}y_{h_1} \wedge \dots \wedge K_{h_k}y_{h_k}\rangle\varphi,$$

where $G = \{g_1,\dots,g_n\}$ and $A \setminus G = \{h_1,\dots,h_k\}$.

4 Completeness

In order to show completeness of $\mathbf{PAL}^\forall$ we translate sentences of this logic into sentences of **HR**. We use the following translation to get rid of public announcements.

Definition 4.1. The translation $t : \mathscr{L}_{PAL^\forall} \to \mathscr{L}_{HR}$ is defined as follows:

$$\begin{array}{lll lll}
t(p) &=& p, & t([\psi](\varphi\wedge\chi)) &=& t([\psi]\varphi \wedge [\psi]\chi),\\
t(\neg\varphi) &=& \neg t(\varphi), & t([\psi]K_a\varphi) &=& t(\psi \to K_a[\psi]\varphi),\\
t(\varphi\wedge\psi) &=& t(\varphi)\wedge t(\psi), & t([\psi][\chi]\varphi) &=& t([\psi\wedge[\psi]\chi]\varphi),\\
t(K_a\varphi) &=& K_a t(\varphi), & t([\psi]x) &=& t(\psi)\to x,\\
t([\psi]p) &=& t(\psi\to p), & t(\forall x\varphi) &=& \forall x t(\varphi),\\
t([\psi]\neg\varphi) &=& t(\psi\to\neg[\psi]\varphi), & t([\psi]\forall x\varphi) &=& t(\psi\to\forall x[\psi]\varphi).
\end{array}$$

Definition 4.2. The complexity $d : \mathscr{L}_{PAL^\forall} \to \mathbb{N}$ is defined as follows:

$$\begin{array}{ll}
d(p) = d(x) = 1, & d(\varphi\wedge\psi) = 1+\max(d(\varphi),d(\psi)),\\
d(\neg\varphi) = d(K_a\varphi) = d(\forall x\varphi) = 1+d(\varphi), & d([\psi]\varphi) = (4+d(\psi))\cdot d(\varphi).
\end{array}$$

Now, we are ready to show that every formula of $\mathbf{PAL}^\forall$ is provably equivalent to its translation.

Proposition 4.3. *For all formulas $\varphi \in \mathscr{L}_{PAL^\forall}$ it is the case that $\vdash \varphi \leftrightarrow t(\varphi)$.*

Proof. We show only the case of universal quantifier and announcement.

Base case: $\vdash p \leftrightarrow p$.

Induction Hypothesis: for all φ such that $c(\varphi) \leqslant n$ it holds that $\vdash \varphi \leftrightarrow t(\varphi)$.

Universal quantifier and announcement $[\psi]\forall x\varphi$: this is equivalent to $\psi \to \forall x[\psi]\varphi$ by $A10$. $d(\psi \to \forall x[\psi]\varphi) = 2+\max(d(\psi),d(\neg\forall x[\psi]\varphi)) = 2+\max(d(\psi),2+d([\psi]\varphi)) = 2+\max(d(\psi),2+(4+d(\psi))\cdot d(\varphi)) = 2+2+4\cdot d(\varphi)+d(\psi)\cdot d(\varphi) = 4+4\cdot d(\varphi)+d(\psi)\cdot d(\varphi)$. Now, $d([\psi]\forall x\varphi) = (4+d(\psi))\cdot d(\forall x\varphi) = (4+d(\psi))\cdot(1+d(\varphi)) = 4+4\cdot d(\varphi)+d(\psi)+d(\psi)\cdot d(\varphi)$. So, $d([\psi]\forall x\varphi) > d(\psi\to\forall x[\psi]\varphi)$, and by induction hypothesis, we conclude that $\vdash [\psi]\forall x\varphi \leftrightarrow t([\psi]\forall x\varphi)$. □

Induction in the previous proof is based on complexity of $\mathbf{PAL}^\forall$ formulas. Since we are interested in decreasing complexity of a subformula within the scope of an announcement, the requirement of x to be bound does not impede reduction. We can always assume that in some $[\psi]\varphi$, ψ does not contain free any variables bound in φ. This holds, since in the opposite case we can rename bound variables φ and have an equivalent formula.

Finally, we show completeness of $\mathbf{PAL}^\forall$.

Proposition 4.4. *For every $\varphi \in \mathscr{L}_{PAL^\forall}$ it holds that $\models \varphi$ implies $\vdash \varphi$.*

Proof. Suppose that $\models \varphi$. Then, by Proposition 4.3 and soundness of $\mathbf{PAL}^\forall$, we have $\models t(\varphi)$. Next, due to completeness of **HR**, we conclude that $\vdash_{\mathbf{HR}} t(\varphi)$, and $\vdash t(\varphi)$ since **HR** is a fragment of $\mathbf{PAL}^\forall$. Finally, by Proposition 4.3, it holds that $\vdash \varphi$. □

5 Conclusion

In this paper we presented an extension of **HR** with public announcements. The resulting logic, $\mathbf{PAL}^\forall$, allows us to define operators of **APAL**, **GAL**, and **CAL** within its language. We showed soundness and completeness of $\mathbf{PAL}^\forall$. Although the original modal logic with propositional quantification [7] (for a single agent) is decidable, $\mathbf{PAL}^\forall$ is not [8]. Note, that **APAL**, **GAL**, and **CAL** are also undecidable [3]. Hence, an interesting direction of further research is finding decidable fragments of the given logics.

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