Dynamical Multi-Agent Systems in Dynamic Epistemic Logic

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Abstract. The paper analyzes dynamic epistemic logic from a topological perspective. The main contribution is a framework in which dynamic epistemic logic satisfies the requirements of being a topological dynamical system, thus interfacing discrete dynamic logics with continuous dynamical systems. Our setting is based on a notion of logical convergence, shown to be equivalent with convergence in the Stone topology. A flexible, parametrized family of metrics inducing the latter is presented and used as analytical aid. We show maps induced by action model transformations continuous with respect to the Stone topology and present results on the recurrent behavior of said maps. We present results from the unpublished [19] as-well as the just accepted [18].

Keywords: dynamical multi-agent systems, dynamic epistemic logic, limit behavior, convergence, recurrence, dynamical systems, metric spaces, general topology, modal logic

Dynamic epistemic logic is a framework for modeling information dynamics. In it, systematic change of Kripke models are punctiliously investigated through model transformers mapping Kripke models to Kripke models. The iterated application of such a map may constitute a model of an information dynamics, or be investigated purely for its mathematical properties [5, 7, 9–13, 25–28].

Dynamical systems theory is a mathematical field studying the long-term behavior of spaces under the action of a continuous function. In case of discrete time, this amounts to investigating the space under the iterations of a continuous map. The field is rich in concepts, methodologies and known results developed with the purpose of understanding general dynamics.

The two fields thus find common ground in the iterated application of maps. With dynamic epistemic logic investigating very specific map types, it may be hoped that general results from dynamical systems theory may be illuminating. There is, however, a chasm between the two: Dynamical systems theory concerns spaces imbued with metrical or topological structure with respect to which maps are continuous, but no such structure is found in dynamic epistemic logic.

This chasm has not gone unappreciated: In his 2011 *Logical Dynamics of Information and Interaction* [9], van Benthem writes

> **From discrete dynamic logics to continuous dynamical systems**
> "We conclude with what we see as a major challenge. Van Benthem [6, 7] pointed out how update evolution suggests a long-term perspective that is like the evolutionary dynamics found in dynamical systems. [...] Interfacing current dynamic and temporal logics with the continuous realm is a major issue, also for logic in general." [9, Sec. 4.8. Emph. is org. heading]

In this paper, we attempt to face this challenge, to bridge this chasm.

We proceed as follows. First, we present what we consider natural space when working with modal logic, namely sets of pointed Kripke models *modulo* logical equivalence. We refer to such as *modal spaces*. We moreover introduce a natural notion of "logical convergence" on modal spaces. Subsequently, we seek a topology on modal spaces for which topological convergence coincides with logical convergence. We consider and prove insufficient a metric topology based on n-bisimulation, but show an adapted Stone topology satisfactory. Saddened by the loss of the the useful aid the metric inducing the n-bisimulation topology provided, we introduce a family of metrics that induce the Stone topology. We thus obtain compact, metric spaces build from pointed Kripke models.

We then consider maps on modal spaces based on multi-pointed action models applied using product update [1–3]. We impose restrictions to ensure totality, and prove the resulting *clean maps* continuous with respect to the Stone topology. With that, we present our *main theorem*: A modal space under the action of a clean map satisfies the common requirements for being a *topological dynamical system*. Using Kripke models in their epistemic interpretation, we thus achieve (one version of) a *dynamical multi-agent system*.

We then turn to results. First, apply now-suited terminology from dynamical systems theory, and present some initial results on the recurrent behavior of clean maps on modal spaces. A first result strengthens that of [28] concerning the long-term behavior of non-epistemic actions in multi-agent epistemic logic: We show that the iterated application of such actions will always converge. Turning to more advanced dynamics, involving epistemic actions and epistemic actions with postconditions, we show that neither of these categories are ensured to have simple dynamics: Clean maps build from either may exhibit *nontrivial recurrence*, i.e., have non-periodic orbits with recurrent points.

With our main theorem, we have interfaced the discrete semantics of dynamic epistemic logic with dynamical systems, and have thus situated the former in the mathematical field of the latter. This allows for the application of results from dynamical systems theory and related fields to the information dynamics of dynamic epistemic logic.

The term *nontrivial recurrence* is adopted from Hasselblatt and Katok, [17]. They remark that "[nontrivial recurrence] is the first indication of complicated asymptotic behavior." Our results thus indicate that the dynamics of action models and product update may not be an easy landscape to map. Hasselblatt and Katok continue: "In certain low-dimensional situations [...] it is possible to give a comprehensive description of the nontrivial recurrence that can appear." [17, p. 24]. As the Stone topology is zero-dimensional, this fuels the hope that general topology and dynamical systems theory yet has perspectives to offer on dynamic epistemic logic. One possible direction is seeking a finer parametrization of clean maps combined with results specific to zero-dimensional spaces, as found, e.g., in the field of symbolic dynamics [23]. Also other venues are possible: The introduction of [17] is inspirational.

The approach presented also applies to model transformations beyond multi-pointed action models and product update. Given the equivalence shown in [20] between single-pointed action model product update and *general arrow updates*, we see no reason to suspect that "clean maps" based on the latter should not be continuous on modal spaces. We further conjecture that the *action-priority update* of [4] on plausibility models[3] yields "clean maps" continuous w.r.t. the suited Stone topology, and that this may be shown using variant of our proof of the continuity of clean maps. A more difficult case is the *PDL-transformations* of *General Dynamic Dynamic Logic* [15] given the signature change the operation involves.

We point out a possible clinch between the suggested approach and epistemic logic with common knowledge. The state space of a dynamical system is compact. The Stone topology for languages including a common knowledge operator is non-compact. Hence, it cannot constitute the space of a dynamical system—but it's *one-point compactification* can. We are currently working on this clinch, the consequences of compactification, and relations to the problem of attaining common knowledge, cf. [16].

Questions also arise concerning the *dynamic logic* of dynamic epistemic logic. Our results indicate that there is more to the semantic dynamics of dynamic epistemic logic than is representable by finite compositional dynamic modalities—even when including a Kleene star. An open question thus stands concerning how to reason about limit behavior. One interesting venue stems from van Benthem [9], where he notes[4] that the reduction axioms of dynamic epistemic logic could possibly be viewed on par with differential equations of quantitative dynamical systems. As modal spaces are zero-dimensional, they are imbeddable in $\mathbb{R}$ cf. [24, Thm 50.5], turning clean maps into functions from $\mathbb{R}$ to $\mathbb{R}$, possibly representable as discrete-time difference equations.

An alternative approach is possible through our main theorem. With it, a connection arises between dynamic epistemic logic and *dynamic topological logic* (see e.g. [**?**,14,21,22]): Each system $(\boldsymbol{X}_d, \boldsymbol{f})$ may be considered a dynamic topological model with atom set $\boldsymbol{\mathcal{L}}_\Lambda$ and the 'next' operator's semantic given by an application of $\boldsymbol{f}$, and thus equivalent to a $\langle \boldsymbol{f} \rangle$ dynamic modality of DEL. The

[3] Hence also the multi-agent belief revision policies *lexicographic upgrade* and *elite change*, also known as *radical* and *conservative upgrade*, introduced in [8], cf. [4].

[4] In the omitted part of the quotation from the introduction.

topological 'interior' operator has yet no DEL parallel. A 'henceforth' operator allows for a limited characterization of recurrence [22]. We wonder at the connections between a limit set operator with semantics $\boldsymbol{x} \models [\omega_{\boldsymbol{f}}]\varphi$ iff $\boldsymbol{y} \models \varphi$ for all $\boldsymbol{y} \in \omega_{\boldsymbol{f}}(\boldsymbol{x})$, dynamic topological logic and the study of oscillations suggested by van Benthem [10].

With the focus on pointed Kripke models and action model transformations, we have only regarded a special case of logical dynamics. It is our firm belief that much of the methodology here suggested generalizes: With structures described logically using a countable language, the notion of logical convergence will coincide with topological convergence in the Stone topology on the quotient space *modulo* logical equivalence, and the metrics introduced will, *mutatis mutandis*, be applicable to said space [19]. Continuity of maps and compactness of course depends on the specifics of the chosen model transformations and the compactness of the logic.

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