Dependent Containers

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based on joint work with Neil Ghani, Conor McBride and Peter Hancock
University of Nottingham

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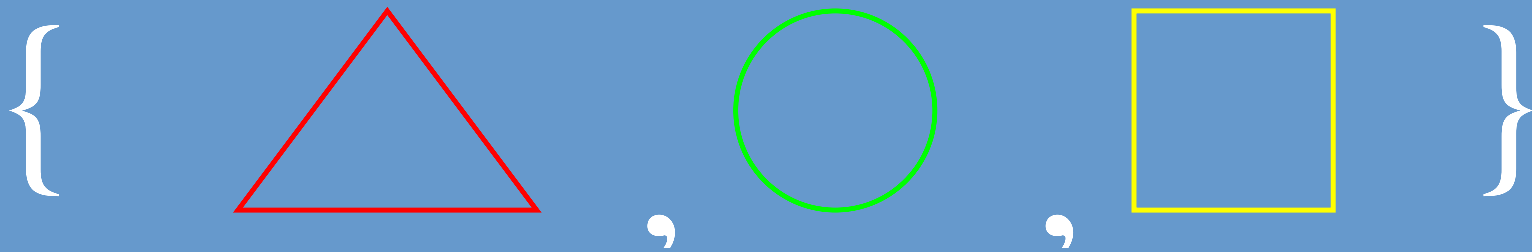
- A set S of shapes, e.g.



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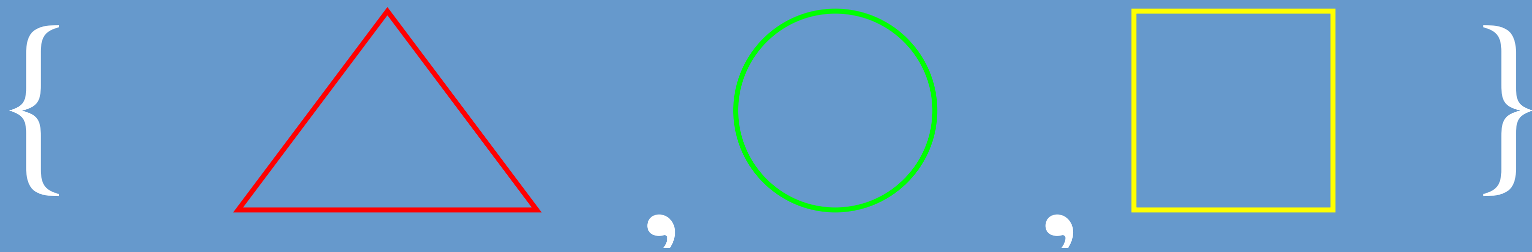
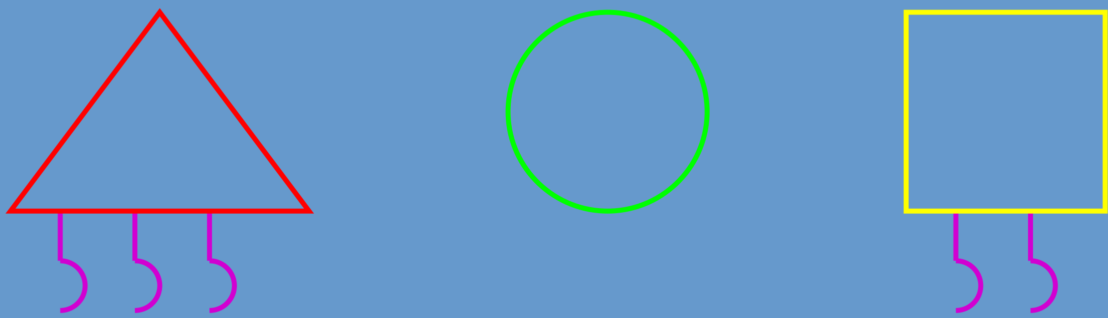


- For any shape $s \in S$ a set of positions $P(s)$, e.g.



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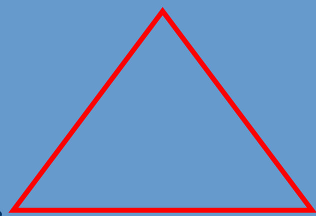
- Choosing a shape, e.g. 

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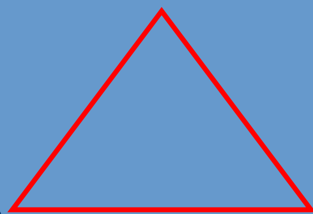
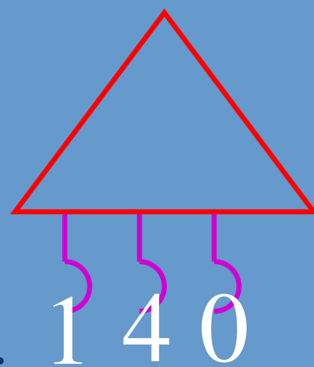
- Choosing a shape, e.g. 

- Filling the positions with payload, e.g. e.g. 

Extension of a container type

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The extension $[\![S \triangleright P]\!]$ of a container is given by an endofunctor $\mathbf{Set} \to \mathbf{Set}$:

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where

$$\Sigma a \in A.B(a) = \{(a, b) \mid a \in A \wedge b \in B(a)\}$$

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$$\underline{\text{data}}\ \frac{X : \star}{\mathsf{List}\ X : \star}\ \underline{\text{where}}\ \frac{}{\mathsf{nil} : \mathsf{List}\ X}\quad \frac{x : X \quad xs : \mathsf{List}\ X}{x :: xs : \mathsf{List}\ X}$$

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Its extension is an endofunctor $\mathbf{Set}^n \to \mathbf{Set}$ is:

$$[\![S \;\triangleright\; \vec{P}]\!](X) = \Sigma s \in S.\Pi i < n.P\, i\, s \to X\, i$$

Morphisms of containers

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Given containers

$$\begin{aligned} F(X) &= \Sigma s \in S.P(s) \to X \\ G(X) &= \Sigma t \in T.Q(t) \to X \end{aligned}$$

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Theorem : Every natural transformation (i.e. polymorphic function) between containers can be represented as a container morphisms.
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Example: any natural transformation $g \in \Pi X.\text{List}\,X \to \text{List}\,X$ is given by:

$$\begin{aligned} f &: \mathsf{Nat} \to \mathsf{Nat} \\ u &: \Pi n : \mathsf{Nat}.(f\,n) \to n \end{aligned}$$

Strictly positive types

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- A **Martin-Löf category** is an extensive, locally cartesian closed category with W-types.
- **Theorem:** All strictly positive types are representable as containers in any Martin-Löf category.
- **Corollary :** All closed strictly positive types are representable in any Martin-Löf category.

Observations

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- Framework to define and reason about datatype generic programming, e.g. see our papers on derivatives of containers.

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- Martin-Löf categories have representations of all strictly positive **non-dependent** inductive and coinductive types.
- We have developed a theory of **non-dependent** datatypes in a **dependently** typed framework.

Dependently typed programming

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$$\underline{\text{data}}\ \frac{n : \mathsf{Nat} \quad X : \star}{\mathsf{Vec}\ n\ X : \star}\ \underline{\text{where}}\ \frac{}{\overline{\mathsf{nil}} : \mathsf{Vec}\ 0\ X}\quad \frac{x : X \quad xs : \mathsf{Vec}\ n\ X}{x \mathbin{\bar{::}} xs : \mathsf{Vec}\ (1{+}n)\ X}$$

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$$\begin{array}{lllll}
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\end{array}$$

Dependent datatypes

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Given $I : \mathbf{Set}$ we define the slice category $\mathbf{Set}/I$ as:

Objects $F : I \to \mathbf{Set}$

Morphisms $\mathbf{Set}/I(F, G) = \Pi i : I.(F\, i) \to (G\, i)$

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Dependent (inductive) datatypes arise as initial algebras of endofunctors on slice categories.
E.g. $\mathsf{Fin} = \mu F : \mathsf{Nat} \to \mathbf{Set}. T_{\mathsf{Fin}}\, F$, where

$$
\begin{aligned}
T_{\mathsf{Fin}} &: \mathbf{Set}/\mathsf{Nat} \to \mathbf{Set}/\mathsf{Nat} \\
T_{\mathsf{Fin}}\, F\, n &= \Sigma m : \mathsf{Nat}. n = 1 + m \\
&\quad + \Sigma m : \mathsf{Nat}.(n = 1 + m) \times (F\, m)
\end{aligned}
$$

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Given I, J : **Set** a dependent container $S \triangleright P$: **Con** $I\ J$ is given by

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The extension of a dependent container is a functor on slices, that is $[\![S \triangleright P]\!] : \mathbf{Set}/I \to \mathbf{Set}/J$, on objects

$$[\![S \triangleright P]\!]\, F\, j = \Sigma s : S\, j.\Pi i : I.(P\, j\, s\, i) \to (F\, i).$$

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Given two dependent containers $S \triangleright P, T \triangleright Q \ : \ \mathbf{Con}(I,\ J)$ a morphism $f \triangleright u$ is given by

- $f \ : \ \Pi j \ : \ J.(S\,j) \to T\,j$
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The extension of a container morphism is a natural transformation which is given by the following family of maps (for $F \,:\, J \to \mathbf{Set}$):

$$\begin{aligned} [\![f \;\triangleright\; u]\!]\, F &: \Pi j \,:\, J.[\![S \;\triangleright\; P]\!]\, F\, j \to [\![T \;\triangleright\; Q]\!]\, F\, j \\ [\![f \;\triangleright\; u]\!]\, F\, j\, (s, h) &= (f\, j\, s, \lambda i.(h\, i) \circ (u\, i)) \end{aligned}$$

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Theorem : Every natural transformation (i.e. polymorphic function) between dependent containers can be represented as a dependent container morphisms.
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- What is a dependent strictly positive type?
- **Inductive Schemes**, as in Luo's UTT or COQ's Type Theoy give rise to dependent containers.
- Better: define a collection of combinators to generate **strictly positive dependent types**.

Application to schema checking

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- Schema checking is complex, incomplete and potentially unsound.
- Using dependent containers we can implement extensible schemes which produce evidence by translating the scheme into core Type Theory with W-types.
- This requires a Type Theory with an extensional propositional equality (under development).

Dependent Signatures?

Our current approach doesn't capture inductive definitions like the definition of the syntax of Type Theory which simultanbously introduces:

$$
\begin{aligned}
\mathrm{Con} &: \mathbf{Set} \\
\mathrm{Ty} &: \mathrm{Con} \to \mathrm{Set} \\
\mathrm{Tm} &: \Pi\Gamma : \mathrm{Con}.(\mathrm{Ty}\,\Gamma) \to \mathbf{Set}
\end{aligned}
$$

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- Initial algebras of unary dependent containers correspond to the Petersson and Synek's tree types.
- The categoy of dependent containers is equivalent to the category of Interaction Structures investigated by Hancock,Hyvernat and Setzer.