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Normalisation by completeness

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The basic picture

- Soundness means:

$$\frac{\Gamma \vdash A}{\Gamma \models^{\mathcal{M}} A}$$

- Completeness via a universal model $\mathcal{U}$ means:

$$\frac{\Gamma \models^{\mathcal{U}} A}{\Gamma \vdash A}$$

- This can be refined to:

$$\frac{\Gamma \models^{\mathcal{U}} A}{\Gamma \vdash^{\mathrm{nf}} A}$$

- Putting this together with soundness we obtain:

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$$\frac{\Gamma \vdash A}{\Gamma \vdash^{\mathrm{nf}} A}$$

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What do we mean by normalisation (here) ?

- We always have (by definition)

$$\frac{\Gamma \vdash^{\mathrm{nf}} A}{\Gamma \vdash A}$$

- We define $\Gamma \vdash^{\mathrm{nf}} A$ s.t. we can show by induction:
  Consistency

$$\nvdash^{\mathrm{nf}} \bot$$

  Disjunction property

$$\frac{\vdash^{\mathrm{nf}} A \vee B}{(\vdash^{\mathrm{nf}} A) \vee (\vdash^{\mathrm{nf}} B)}$$

  Subformula principle A derivation of $\Gamma \vdash^{\mathrm{nf}} A$ contains only subformulas of Γ, A.

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- We do not talk about equality of proofs (or λ-terms) ,i.e.

$$\frac{\Gamma \vdash t = u : A}{\Gamma \vdash \mathrm{nf}(t) \equiv \mathrm{nf}(u)} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash t = \mathrm{nf}(t) : A}$$

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Overview

1. How to obtain normalisation from completeness of *Kripke models* for minimal (intuitionistic) propositional logic.
2. How to obtain normalisation from completeness of (a version of) *fallible Beth models* for full propositional logic.
3. Discussion: How does this relate to λ-calculus. Further work.

Note that most of this material has been formalized in LEGO and ALF.

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Minimal Logic

The basic ingredients are :

Atomic propositions

$$P, Q, R, \ldots$$

Propositions A, B, C

$$A \to B$$

Contexts Γ, Δ

$$\text{empty}$$

$$\Gamma.A$$

Sequents

$$\Gamma \vdash A$$

$$\Gamma \vdash \Delta$$

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Inductive definition of derivable sequents in normal form

$$\frac{}{\Gamma.A \vdash^{\mathsf{ne}} A}\,\text{ax} \qquad \frac{\Gamma \vdash^{\mathsf{ne}} A}{\Gamma.B \vdash^{\mathsf{ne}} A}\,\text{wk} \qquad \frac{\Gamma \vdash^{\mathsf{ne}} P}{\Gamma \vdash^{\mathsf{nf}} P}\,\text{ne} - \text{nf}$$

$$\frac{\Gamma.A \vdash^{\mathsf{nf}} B}{\Gamma \vdash^{\mathsf{nf}} A \to B}\,\to_{\mathsf{i}} \qquad \frac{\Gamma \vdash^{\mathsf{ne}} A \to B \qquad \Gamma \vdash^{\mathsf{nf}} A}{\Gamma \vdash^{\mathsf{ne}} B}\,\to_{\mathsf{e}}$$

$$\frac{}{\Gamma \vdash^{\alpha} \text{empty}}\,\text{empty} \qquad \frac{\Gamma \vdash^{\alpha} \Delta \qquad \Gamma \vdash^{\alpha} A}{\Gamma \vdash^{\alpha} \Delta.A}\,\text{ext} \qquad \alpha \in \{\mathsf{ne}, \mathsf{nf}\}$$

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Admissable sequents in normal form

$$\frac{\Gamma \vdash^{\alpha} \Delta}{\Gamma.A \vdash^{\alpha} \Delta}\,\mathrm{Wk} \qquad \alpha \in \{\mathrm{ne}, \mathrm{nf}\}$$

by induction over the structure of Δ.

$$\frac{}{\Gamma \vdash^{\mathrm{ne}} \Gamma}\,\mathrm{Id}$$

by induction over the structure of Γ using Wk.

$$\frac{\Gamma \vdash^{\alpha} A \qquad \Delta \vdash^{\mathrm{ne}} \Gamma}{\Delta \vdash^{\alpha} A}\,\mathrm{cut} \qquad \alpha \in \{\mathrm{ne}, \mathrm{nf}\}$$

by induction over the structure of A using Wk.

$$\frac{\Gamma \vdash^{\alpha} \Theta \qquad \Delta \vdash^{\mathrm{ne}} \Gamma}{\Delta \vdash^{\alpha} \Theta}\,\mathrm{Cut} \qquad \alpha \in \{\mathrm{ne}, \mathrm{nf}\}$$

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by induction over the structure of Θ using cut.

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Kripke models

A Kripke model $\mathcal{K} = (\mathcal{W}, \leq, \Vdash)$ is given by

- A preordered set $(W, \leq)$.
- A relation $\Vdash \subseteq W \times P$ (forcing) s.t.

$$\frac{w \Vdash P \qquad w' \leq w}{w' \Vdash P}\ \text{mon}$$

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Forcing

We extend $\Vdash$ to formulas and contexts:

$$\begin{aligned} w \Vdash A \to B \quad &\iff \quad \forall_{w' \leq w} w' \Vdash A \to w' \Vdash B \\ w \Vdash \text{empty} \quad & \\ w \Vdash \Gamma.A \quad &\iff \quad w \Vdash \Gamma \wedge w \vdash A \end{aligned}$$

Lemma: Monotonicity

$$\frac{w \Vdash A \qquad w' \leq w}{w' \Vdash A} \qquad \frac{w \Vdash \Gamma \qquad w' \leq w}{w' \Vdash \Gamma}$$

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Soundness of Kripke models

Definition: Validity
We say that the sequent $\Gamma \vdash A$ is valid

$$\Gamma \models A$$

iff

$$\forall_w w \Vdash \Gamma \rightarrow w \Vdash A$$

Theorem: Soundness

$$\frac{\Gamma \vdash A}{\Gamma \models A}$$

Proof by induction over the structure of $\Gamma \vdash A$.

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The universal model $\mathcal{U}$ with normal forms

$$\begin{array}{rcl} W & = & \mathit{Contexts} \\ \Gamma \leq \Delta & \Longleftrightarrow & \Gamma \vdash^{\mathsf{ne}} \Delta \\ \Gamma \Vdash P & \Longleftrightarrow & \Gamma \vdash^{\mathsf{ne}} P \end{array}$$

That $\vdash^{\mathsf{ne}}$ is a preorder follows from Refl and Cut.

Monotonicity follows from cut.

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Quote and unquote

Lemma: quote

$$\Gamma \Vdash A \to \Gamma \vdash^{\mathsf{nf}} A$$

Lemma: unquote

$$\Gamma \vdash^{\mathsf{ne}} A \to \Gamma \Vdash A$$

By mutual induction over the structure of A.

Note that the base case P follows from the definition.

Lemma: Unquote

$$\Gamma \vdash^{\mathsf{ne}} \Delta \to \Gamma \Vdash \Delta$$

By induction over the structure of Δ using unquote.

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Proving quote and unquote

quote We have to show $\Gamma \Vdash A \to B \to \Gamma \vdash^{\mathsf{nf}} A \to B$.

Assume $\Gamma \Vdash A \to B$ that is

$$\forall_{\Delta \vdash^{\mathsf{ne}} \Gamma} \Delta \Vdash A \to \Delta \Vdash B \qquad (*)$$

Set $\Delta = \Gamma.A$. Clearly $\Delta \vdash^{\mathsf{ne}} \Gamma$ (Id and Weak).

Using ax we know that $\Gamma.A \vdash^{\mathsf{ne}} A$ and by ind.hyp.(unquote) we have $\Gamma.A \Vdash A$.

Using $(*)$ we get $\Gamma.A \Vdash B$ and by ind.hyp. (quote) we get $\Gamma.A \vdash^{\mathsf{nf}} B$ and using $\to_i$:

$$\Gamma \vdash^{\mathsf{nf}} A \to B$$

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unquote We have to show $\Gamma \vdash^{\mathrm{ne}} A \to B \to \Gamma \Vdash A \to B$.

Assume $\Gamma \vdash^{\mathrm{ne}} A \to B$ (1)

To show $\Gamma \Vdash A \to B$, assume $\Delta \Vdash A$ (2). We have to show $\Delta \Vdash B$.

Using (2) and ind.hyp. (quote) we get $\Delta \vdash^{\mathrm{nf}} A$.

Using cut on (1) we get $\Delta \vdash^{\mathrm{ne}} A \to B$ and using $\to_e$ with the previous result we get $\Delta \vdash^{\mathrm{ne}} B$.

Apply ind.hyp.(unquote) to obtain $\Delta \Vdash B$.

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Completeness with normal forms

Theorem Completeness

$$\frac{\Gamma \models A}{\Gamma \vdash^{\mathrm{nf}} A}$$

Proof:

$\Gamma \models A$ means that for all Δ: $\Delta \Vdash \Gamma \to \Delta \Vdash A$ (1).

Set $\Delta = \Gamma$.

From Id we know $\Gamma \vdash^{\mathrm{ne}} \Gamma$ and with Unquote we get $\Gamma \Vdash \Gamma$.

Using (1) we obtain $\Gamma \Vdash A$.

By quote we get $\Gamma \vdash^{\mathrm{nf}} A$. □

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Normal derivations

We define:

Neutral derivations

$$\Gamma \vdash^{\text{ne}} A$$

Normal derivations

$$\Gamma \vdash^{\text{nf}} A$$

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Normalisation

Theorem:

$$\frac{\Gamma \vdash A}{\Gamma \vdash^{\text{nf}} A}$$

From soundness and completenesss with normal forms.

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Derivable sequents (disjunction) in normal form

$$\frac{\Gamma \vdash^{\mathrm{nf}} A_i \qquad i \in \{1, 2\}}{\Gamma \vdash^{\mathrm{nf}} A_1 \vee A_2} \vee_{\mathrm{i}}$$

$$\frac{\Gamma.A_1 \vdash^{\mathrm{nf}} C \qquad \Gamma.A_2 \vdash^{\mathrm{nf}} C \qquad \Gamma \vdash^{\mathrm{ne}} A_1 \vee A_2}{\Gamma \vdash^{\alpha} C \quad \begin{array}{ll} \alpha = \mathrm{nf} & \text{if } C = C_1 \vee C_2 \\ \alpha = \mathrm{ne} & \text{otherwise} \end{array}} \vee_{\mathrm{e}}$$

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Disjunction

Forcing

$$w \Vdash A \vee B \quad \iff \quad (w \Vdash A) \vee (w \Vdash B)$$

Soundness holds as before.

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Completeness ?

From quote and unquote we could derive:

$$\begin{aligned} \Gamma \vdash A \vee B &\rightarrow \Gamma \Vdash A \vee B \\ &\rightarrow (\Gamma \Vdash A) \vee (\Gamma \Vdash B) \\ &\rightarrow (\Gamma \vdash A) \vee (\Gamma \vdash B) \end{aligned}$$

However, this cannot be true, consider $\Gamma = P \vee Q$ and $A = P, B = Q$.

In fact a completeness proof would have to exploit decidability $\Gamma \vdash A \vee \Gamma \nvdash A$.

For predicate logic it is known that no intuitionistic completeness proof for Kripke models exists.

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Fallible Beth models (or proof-irrelevant sheaf models)

Definition: Beth model

A Beth model $(W, \leq, \Vdash, \lhd)$ is given by:

- A Kripke model $(W, \leq, \Vdash)$.
- A relation of *covering*: $\lhd \subseteq W \times \mathcal{P}(W)$ s.t.
  1. $w \lhd \{w' | w' \leq w\} = w^+$
  2. $\dfrac{w \lhd \mathcal{P} \qquad w' \leq w}{w' \lhd \{w'' \in \mathcal{P} \mid w'' \leq w'\}}$
  3. $\dfrac{w \lhd \mathcal{P} \qquad \mathcal{P} \lhd \mathcal{Q} = \forall_{w' \in \mathcal{P}} w' \lhd \mathcal{Q}}{w \lhd \mathcal{Q}}$

  and

$$\frac{w \lhd \mathcal{P} \qquad \forall_{w' \in \mathcal{P}} w' \Vdash P}{w \Vdash P}\,\text{paste}$$

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Soundness of Beth models

$$w \Vdash A \vee B \iff \exists_P w \lhd P \wedge \forall_{w' \in P} w' \Vdash A \vee w' \Vdash B$$

Lemma:

$$\frac{w \in \mathcal{P} \qquad \forall_{w' \in \mathcal{P}} w' \Vdash A}{w \lhd A}\text{Paste}$$

Theorem: Soundness

$$\frac{\Gamma \vdash A}{\Gamma \models A}$$

For $\vee_i$ take $\mathcal{P} = \Gamma^+$.

For $\vee_e$ use Paste.

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The universal Beth model with normal forms

$W, \leq, \Vdash$ are defined as for the universal Kripke model.

$\lhd$ is defined inductively:

$$\frac{}{\Gamma \lhd \Gamma^+ = \{\Delta \mid \Delta \vdash^{\mathrm{ne}} \Gamma\}} \qquad \frac{\Gamma \vdash^{\mathrm{ne}} A \vee B \qquad \Gamma.A \lhd \mathcal{P} \qquad \Gamma.B \lhd \mathcal{Q}}{\Gamma \lhd \mathcal{P} \cup \mathcal{Q}}$$

Lemma:

$$\frac{\Gamma \lhd \mathcal{P} \qquad \forall_{\Delta \in \mathcal{P}} \Delta \vdash^{\mathrm{nf}} A}{\Gamma \vdash^{\alpha} A \quad \begin{array}{ll} \alpha = \mathrm{nf} & \text{if } A = A_1 \vee A_2 \\ \alpha = \mathrm{ne} & \text{otherwise} \end{array}}\text{Paste}_{\mathcal{U}}$$

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Extending quote and unquote

quote We have to show $\Gamma \Vdash A \vee B \rightarrow \Gamma \vdash^{\mathsf{nf}} A \vee B$

Assume $\Gamma \Vdash A \vee B$ i.e. there is a $\Gamma \lhd \mathcal{P}$ s.t.

$$\forall_{\Delta \in \mathcal{P}} (\Delta \Vdash A) \vee (\Delta \Vdash B)$$

Using the induction hypothesis (quote) for A and B we get

$$\forall_{\Delta \in \mathcal{P}} (\Delta \vdash^{\mathsf{nf}} A) \vee (\Delta \vdash^{\mathsf{nf}} B)$$

Using $\vee_i$ we get

$$\forall_{\Delta \in \mathcal{P}} (\Delta \vdash^{\mathsf{nf}} A \vee B)$$

And by paste we get:

$$\Gamma \vdash^{\mathsf{nf}} A \vee B$$

unquote We have to show $\Gamma \vdash^{\mathsf{ne}} A \vee B \rightarrow \Gamma \Vdash A \vee B$.

Assume $\Gamma \vdash^{\mathsf{ne}} A \vee B$. it is easy to see that $\mathcal{P} = \Gamma.A^{+} \cup \Gamma.B^{+}$ is

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a covering $\Gamma \lhd \mathcal{P}$.

And we have that

$$\forall_{\Delta \in \mathcal{P}} (\Delta \vdash^{\mathsf{ne}} A) \vee (\Delta \vdash^{\mathsf{ne}} B)$$

Using the induction hypothesis (unquote) we get that

$$\forall_{\Delta \in \mathcal{P}} (\Delta \Vdash A) \vee (\Gamma \Vdash B)$$

And hence (by definition)

$$\Gamma \Vdash A \vee B$$

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Completeness for Beth models with normal forms

Theorem Completeness

$$\frac{\Gamma \models A}{\Gamma \vdash^{\mathbf{nf}} A}$$

The proof is the same as for Kripke models.

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Normalisation for Beth models

Theorem:

$$\frac{\Gamma \vdash A}{\Gamma \vdash^{\mathbf{nf}} A}$$

From soundness and completenesss with normal forms.

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Extension to λ-calculi

proof-irrelevant	proof-relevant
preorder	category
Kripke model	presheaves
Beth model	sheaves

The proof-relevant version of normalisation for Kripke models has been explored in

- Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. Categorical reconstruction of a reduction free normalization proof. CTCS 95, February 1995.
- Peter Dybjer, Djordje Cubric, and Philip Scott. Normalization and the Yoneda embedding. Mathematical Structures in Computer Science, 1997.