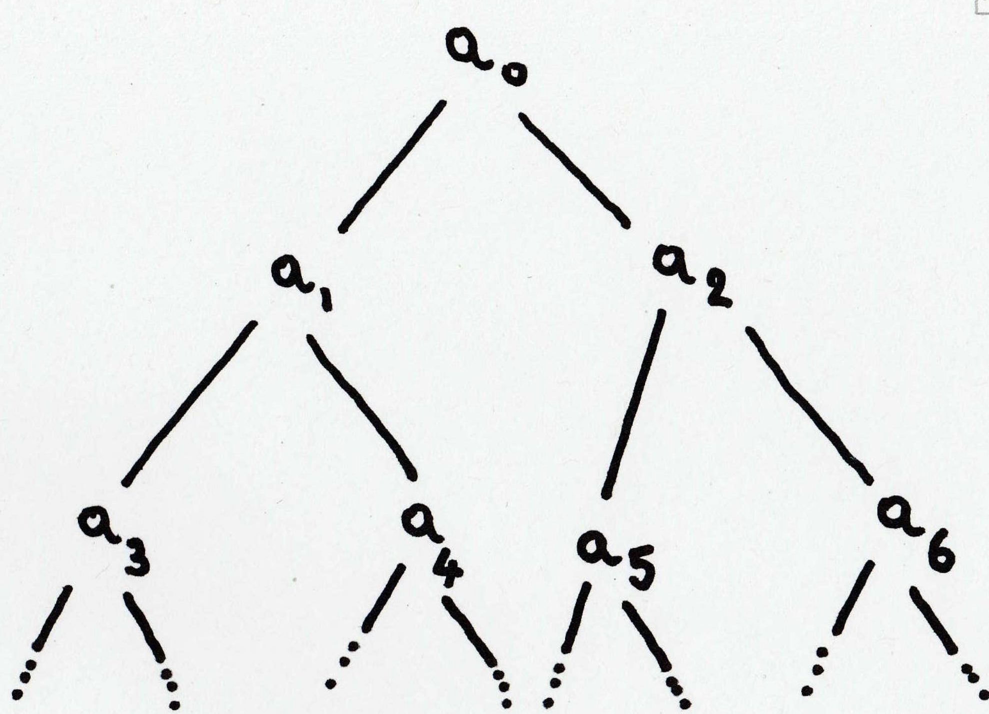
Coinductive Types

have elements with an infinite, non-well-founded structure

Example 1: Streams

Example 2: Infinite Trees

- Binary trees with no leaves
- Data on the nodes (every node is labelled with an element of type A)
- Branches go down forever



As for streams, it's not possible to give the whole structure directly:

node a_0 (node a_1 (node a_3 ⋯)
(node a_4 ⋯))

(node a_2 (node a_5 ⋯)
(node a_6 ⋯))

Instead, elements are generated by a process with dynamic states

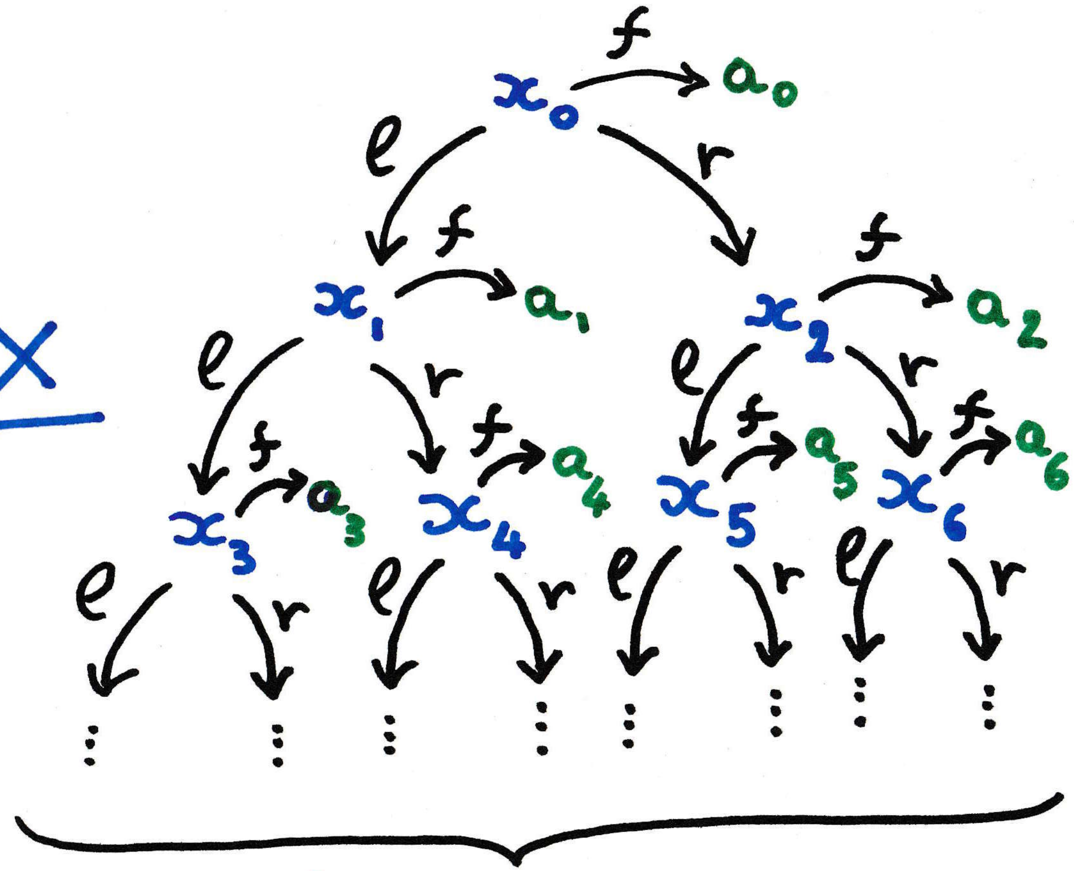
Introduction Rule:

For any type X

$$\frac{f : X \to A \quad \ell : X \to X \quad r : X \to X}{\mathrm{corec}\ f\ \ell\ r : X \to \mathrm{Tree}_A}$$

Idea: X is a set of states

- start with an initial state $x_0 : X$
- each state generates an element of A: $a_0 = f\,x_0$
- then splits into two distinct processes with states $x_1 = \ell\,x_0$ $\quad x_2 = r\,x_0$



corec $f\ \ell\ r\ x_0$ is the tree in the previous slide

Elimination Rules:

Just extract the components

$$\frac{t : \mathrm{Tree}_A}{\mathrm{label}\ t : A}$$

$$\frac{t : \mathrm{Tree}_A}{\mathrm{left}\ t : \mathrm{Tree}_A}$$

$$\frac{t : \mathrm{Tree}_A}{\mathrm{right}\ t : \mathrm{Tree}_A}$$

Reduction Rules:

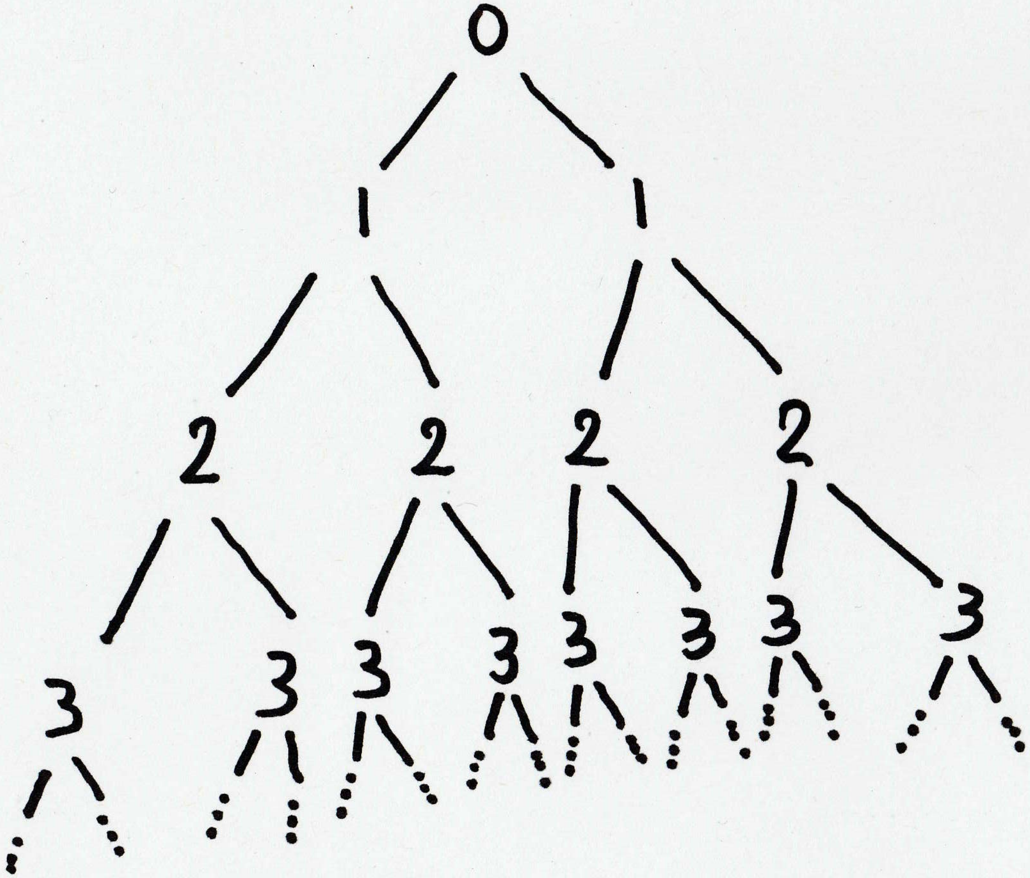
$$\mathrm{label}\ (\mathrm{corec}\ f\ \ell\ r\ x) \rightsquigarrow f\ x$$

$$\mathrm{left}\ (\mathrm{corec}\ f\ \ell\ r\ x) \rightsquigarrow \mathrm{corec}\ f\ \ell\ r\ (\ell\ x)$$

$$\mathrm{right}\ (\mathrm{corec}\ f\ \ell\ r\ x) \rightsquigarrow \mathrm{corec}\ f\ \ell\ r\ (r\ x)$$

Example:

Tree with nodes labelled by depth



We generalize it:
generator has a number/depth as state:

$$\mathrm{depthTree} : \mathrm{Nat} \longrightarrow \mathrm{Tree}_{\mathrm{Nat}}$$

$$\mathrm{depthTree} = \mathrm{corec}\ f\ \ell\ r$$

where

$f : \mathrm{Nat} \longrightarrow \mathrm{Nat}$

$f\,n = n$

$\ell : \mathrm{Nat} \rightarrow \mathrm{Nat}$

$\ell\,n = n+1$

$r : \mathrm{Nat} \rightarrow \mathrm{Nat}$

$r\,n = n+1$

Then the tree in the figure is

$$\mathrm{depthTree}\ 0$$

General Rules for Coinductive Types

F strictly positive functor

νF is the coinductive type associated with F

Examples:

- Stream_A is νF with the functor

$$FX = A \times X$$

- Tree_A is νF with

$$FX = A \times X \times X$$

Introduction:

For any type X

f is called a coalgebra of F

$$\frac{f : X \to FX}{\mathrm{ana}\ f : X \to \nu F}$$

this function is called the anamorphism of f

Elimination:

$$\frac{u : \nu F}{\mathrm{out}\ u : F\,\nu F}$$

Reduction:

$$\mathrm{out}\ (\mathrm{ana}\ f\ x)$$

$$\rightsquigarrow F(\mathrm{ana}\ f)\ (f\ x)$$