Mathematics for Computer Scientists 2 (G52MC2)

L11 : The ω-hotel, diagonalisation

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The ω-Hotel (1)

- The ω-hotel has infinitely many rooms, numbered $0, 1, 2, \ldots$.
- One night the hotel is completely full ...
- ... another guest arrives.
- Can we accomodate this guest?
- If we are prepared to move existing guests?

The ω-Hotel (1)

solution

Everybody has to move to the next room, i.e. the guest in room number n moves into room $n + 1$. Now 0 is free and the new guest can move in.

The ω-Hotel (2)

- Another night the hotel is completely full (again) ...
- ... a full ω-bus arrives
- An ω-bus has infinitely many seats, numbered $0, 1, 2, \ldots$
- Can we accomodate all these people?

The ω-Hotel (2)

solution

Every existing guest moves into the room with twice the number, i.e. the guest in room number n moves into room number $2n$.
As a consequence all the odd numbered rooms are free. The people in the bus are instructed to move into the odd rooms, i.e. if you have seat number m you move into room $2m + 1$.

The ω-Hotel (3)

- Another night the hotel is completely empty (relief) ...
- ... when all the ω-busses arrive at once!
- There are infinitely many ω-busses and they are numbered $0, 1, 2, 3, \ldots$.
- And they are all completely full.
- Can we accomodate all these people?

The ω-Hotel (3)

solution

We can assign rooms to the people in the busses using the following idea: if your bus number is m and your seat number is n your room number is given by the following table:

$m \backslash n$	0	1	2	3	4	...
0	0	2	5	9	14	...
1	1	4	8	13	...	
2	3	7	12	...		
3	6	11	...			
4	10	...				
$\vdots$						

However, this seems to require an infinite table. Can we give each guest a formula how to calculate their room number?

Isomorphisms with $\mathbb{N}$

The examples illustrate the following isomorphisms:

1. $1 + \mathbb{N} \simeq \mathbb{N}$
2. $\mathbb{N} + \mathbb{N} \simeq \mathbb{N}$
3. $\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$

It almost seems that all infinite sets are isomorphic. Is this correct?

Diagonalisation

Assume there are functions:

$$\begin{aligned} \phi &: \mathbb{N} \to (\mathbb{N} \to \mathbb{N}) \\ \psi &: (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \end{aligned}$$

$$\forall h : \mathbb{N} \to \mathbb{N}, \phi\,(\psi\, h) = h \qquad (1)$$

Then we can construct a function f by **diagonalisation**:

$$f \quad : \quad \mathbb{N} \to \mathbb{N}$$

$$f\, n \quad = \quad S\,(\phi\, n\, n) \qquad (2)$$

Now what is $f\,(\psi\, f) : \mathbb{N}$?

$$\begin{aligned} f\,(\psi\, f) &= S\,(\phi\,(\psi\, f)\,(\psi\, f)) && (2) \\ &= S\,(f\,(\psi f)) && (1) \end{aligned}$$

However, we know that there is no number n which is equal to its successor.

Diagonalisation . . .

Conclusion

We cannot embed $\mathbb{N} \to \mathbb{N}$ into $\mathbb{N}$, hence there cannot be an isomorphism between the two sets.

Cantor

Georg Cantor (1845 - 1918)

- Cantor used diagonalisation to show that the *continuum* (i.e. the real numbers: $\mathbb{R}$) has a larger *cardinality* then the natural numbers.
- He developed a theory of *cardinal numbers* to denote different infinities.
- $\aleph_0$ (aleph 0) is the cardinality of the natural numbers.
- $\aleph_1$ (aleph 1) is the cardinality of $\mathbb{R}$ and also $\mathbb{N} \to \mathbb{N}$.
- The *Continuums hypothesis* (CH) is the assumption that there are no cardinalities between $\aleph_0$ and $\aleph_1$.