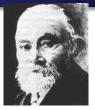
Mathematics for Computer Scientists 2 (G52MC2) L06 : More predicate logic

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Friedrich Ludwig Gottlob Frege (1848-1925)

- Frege developed predicate logic in the 1870ies.
- He also introduced a graphic notation system nobody is using anymore (⊢ for provable is a relict).
- Frege wrote a book called *Begriffsschrift* with the goal to make the logical foundations of Mathematics precise.
- Unluckily, there was a serious bug, which was discovered by Betrand Russell. Frege's theory was inconsistent.
- Later predicate logic was used for: Arithmetic The theory of the natural numbers. ZF Set Theory The classical theory of sets.

Sets or types.

(Not exactly the same sets as in ZF set theory).

Terms or *expressions*. Made from constants, variables and functions.

Propositions Propositional logic + quantifiers (\forall, \exists) .

Predicates and relations Functions into **Prop**. If they have more than ane argument they are called relations. Equality (=) is a special relation.

| connective | Introduction | Elimination |
|-------------------|-------------------|--------------------|
| $P \rightarrow Q$ | intro(s) | apply Hyp |
| $P \wedge Q$ | split | elim <i>Hyp</i> |
| True | split | |
| $P \lor Q$ | left,right | case Hyp |
| False | | case Hyp |
| forall $x : A, P$ | intro(s) | apply Hyp |
| exists $x : A, P$ | exists wit | elim <i>Hyp</i> |
| a = b | reflexivity | rewrite Hyp |

Examples of tautologies

- $(\forall x : D.P x \land Q x) \leftrightarrow (\forall y : D, P y) \land (\forall y : D, Q y)$
- $(\exists x : D.P x \lor Q x) \leftrightarrow (\exists y : D, P y) \lor (\exists y : D, Q y)$
- $(\forall x : D.P x \to R) \leftrightarrow (\exists x : D.P x) \to R$
- $(\forall x : D. \neg Px) \leftrightarrow \neg \exists x : D, Px$
- $(\exists x : D. \neg Px) \rightarrow \neg \forall x : D, Px$

The following require the principle of the excluded middle (Import Classical):

•
$$(\neg \forall x : D, Px) \rightarrow \exists x : D. \neg Px$$

•
$$\forall x : D, Px \lor \neg Px$$

We say a logical system is **decidable**, if there is a computer program which can determine wether a proposition is true (or provable) or not.

Decidable:

- Propositional logic
- Predicate logic over Bool

Undecidable:

• General predicate logic (with predicate and set variables)

However, there are some heuristics which work in many cases (In Coq: firstorder). Not for the homework!