2nd Note for *A Taste of Proof Theory*

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This is just an update on the 1st note extending the systems to first order predicate logic. We assume as given a finite set of relation symbols P with a given arity, a finite set of function symbols F with a given arity and a countably infinite set of variable names X. We define the syntax of predicate logic by extending the syntax of propositional logic (atomic formulas correspond to relation symbols with arity 0).

$$\begin{array}{lcl} A & :: & P(T,\ldots,T) \mid \cdots \mid \forall X.A \mid \exists X.A \\ T & :: & X \mid F(T,\ldots,T) \end{array}$$

We identify formulas upto renaming of bound variables $\forall x.P(x) \equiv \forall y.P(y)$. Given a formual A, a variable x and a term t we write $A[x := t]$ for capture avoiding substitution. We write $\mathrm{FV}(A)$ for the set of variables free in A and extend this to finite sets of formulas $\mathrm{FV}(\Gamma)$.

1 Hilbert style (H)

$$\frac{\vdash_{\mathrm{H}} A}{\vdash_{\mathrm{H}} \forall x.A}\,\mathrm{G}$$

$$\begin{array}{ll} \vdash_{\mathrm{H}} (\forall x.A) \Rightarrow A[x := t] & \\ \vdash_{\mathrm{H}} A[x := t] \Rightarrow (\exists x.A) & \\ \vdash_{\mathrm{H}} (\forall x.B \Rightarrow A) \Rightarrow B \Rightarrow \forall x.A & x \notin \mathrm{FV}(B) \\ \vdash_{\mathrm{H}} (\forall x.A \Rightarrow B) \Rightarrow (\exists x.A) \Rightarrow B & x \notin \mathrm{FV}(B) \end{array}$$

2 Natural deduction (N)

$$\frac{\Gamma \vdash_{\mathrm{N}} A \qquad x \notin \mathrm{FV}(\Gamma)}{\Gamma \vdash_{\mathrm{N}} \forall x.A}\,\forall_{\mathrm{I}} \qquad \frac{\Gamma \vdash_{\mathrm{N}} \forall x.A}{\Gamma \vdash_{\mathrm{N}} A[x := t]}\,\forall_{\mathrm{E}}$$

$$\frac{\Gamma \vdash_{\mathrm{N}} A[x := t]}{\Gamma \vdash_{\mathrm{N}} \exists x.A}\,\exists_{\mathrm{I}} \qquad \frac{\Gamma.A \vdash_{\mathrm{N}} C \qquad \Gamma \vdash \exists x.A \qquad x \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(C)}{\Gamma \vdash_{\mathrm{N}} C}\,\exists_{\mathrm{E}}$$

3 Sequent calculus (G)

$$\frac{\Gamma \vdash_{\mathrm{G}} \Delta . A \qquad x \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\Delta)}{\Gamma \vdash_{\mathrm{G}} \Delta . \forall x . A} \forall_{\mathrm{R}} \qquad \frac{\Gamma . A[x := t] \vdash_{\mathrm{G}} \Delta}{\Gamma . \forall x . A \vdash_{\mathrm{G}} \Delta} \forall_{\mathrm{L}}$$

$$\frac{\Gamma \vdash_{\mathrm{G}} \Delta . A[x := t]}{\Gamma \vdash_{\mathrm{G}} \Delta . \exists x . A} \exists_{\mathrm{R}} \qquad \frac{\Gamma . A \vdash_{\mathrm{G}} \Delta \qquad x \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\Delta)}{\Gamma . \exists x . A \vdash_{\mathrm{G}} \Delta} \exists_{\mathrm{I}}$$