Inductive Types for Free

Representing Nested Inductive Types using W-types

Michael Abbott (U. Leicester)

Thorsten Altenkirch (U. Nottingham)

Neil Ghani (U. Leicester)

Ideology

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- David Turner:
  Elementary Strong Functional Programming

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- Dependent types to avoid accidental partiality (e.g. hd).
- E.g.: Conor McBride's Epigram system.

Plan of the talk

- Inductive and coinductive types.
- Container types for dummies.
- Properties of container types.
- W-types are sufficent for inductive types.
- Further work and applications
- Related work

Inductive and coinductive types

- Widespread in functional programming (e.g. Haskell)

$$\mathbf{data}\, \mathit{Lam} = \mathrm{Var}\, \mathit{Int} \mid \mathrm{App}\, \mathit{Lam}\, \mathit{Lam} \mid \mathrm{Lam}\, \mathit{Lam}$$

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$$\mathbf{data}\,\textit{Lam} = \mathrm{Var}\,\textit{Int}\,|\,\mathrm{App}\,\textit{Lam}\,\textit{Lam}\,|\,\mathrm{Lam}\,\textit{Lam}$$

- Categorically: Initial algebra of a functor

$$\textit{Lam} = \mu X.1 + X \times X + X$$

μ vs. ν

- In a total setting $\mu \neq \nu$ (terminal coalgebras):

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- **Application**: Small trusted cores
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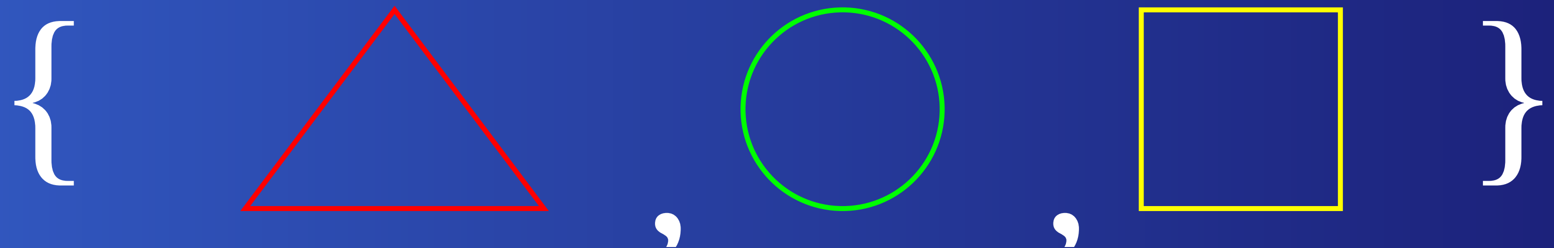

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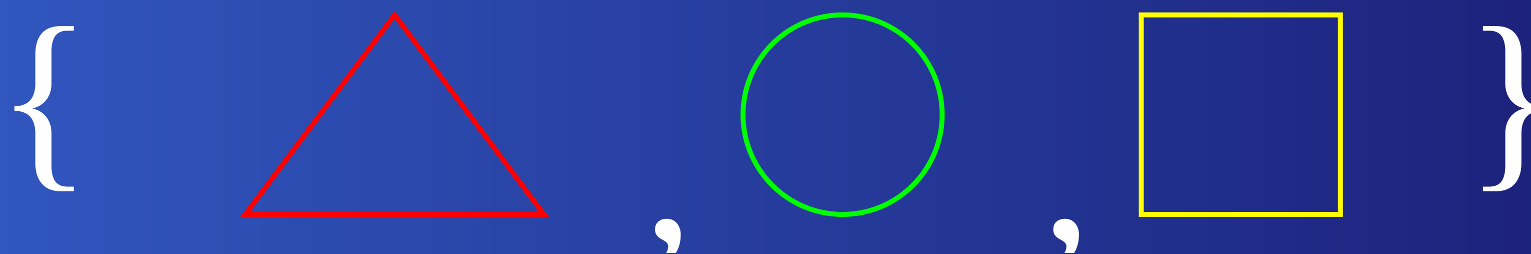
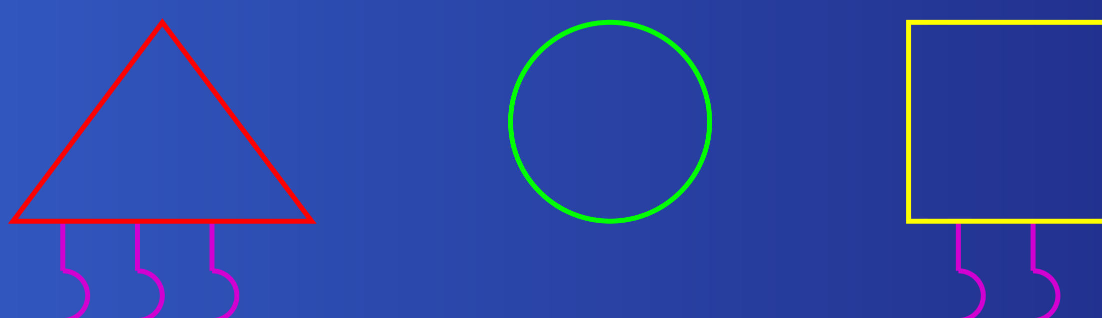


- An assignment of positions to shapes P, e.g.



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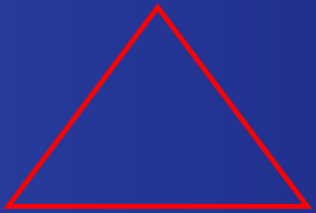
- Choosing a shape, e.g. 

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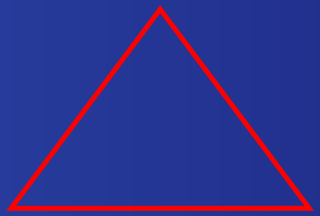
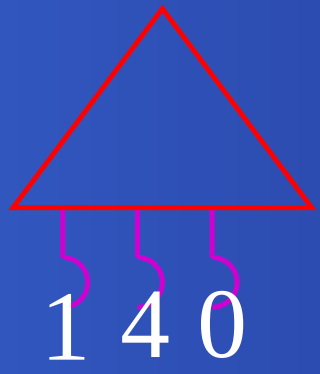
- Choosing a shape, e.g. 
- Filling the positions with payload (here natural numbers), e.g.



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A container type $s \in S \triangleright P\, s$ is given by

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The extension $[\![s \in S \triangleright P\,s]\!]$ of a container is the endofunctor

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Straightforward extension to n-ary containers $s \in S \triangleright P_1\,s, P_2\,s, \ldots, P_n\,s$.

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its extension is the natural transformation

$$\begin{aligned} [\![f \rhd u]\!] &\in [\![S \rhd P]\!] \stackrel{\cdot}{\to} [\![T \rhd Q]\!] \\ [\![f \rhd u]\!](s,h) &= (f\;s, h \circ u\;s) \end{aligned}$$

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Theorem (AAG,FOSSACS 03)
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- A length transformer $f \in \mathbb{N} \to \mathbb{N}$,
- A where-did-you-come-from function $u \in \Pi n \in \mathbb{N}.P\,(f\,n) \to P\,n$.

Closure properties

Containers are closed under (*)

- Constant functors,
- Coproducts ($+$)
- Products ($\times$)
- Constant exponentation $F\,X = C \to G\,X$
- Composition of functors
- initial algebras (μ) [ICALP 04]
- terminal coalgebras (ν) [Journal paper]

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(*) In any Martin-Löf category = LCCC (locally cartesian closed category) + W-types.

Coproducts of containers

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F\,X &= \Sigma s \in S.P\,s \to X \\
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T &= \mu Y.\Sigma s \in S.Q(s) \to Y \\
&= W\,S\,Q \\
R\,(s, f) &\simeq P\,s + \Sigma q \in Q\,s.R\,(f\,q)
\end{aligned}$$

Reply to referee comment

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Thus the corollary mentioned in the proof of 4.1 would be wrong and as a result the entire argument collapses.

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Reply: *Since there are no infinite paths in a finite tree, there is only one solution to this isomorphism, the initial one.*

This is reflected in the proof of proposition 4.1!

Constructing the iso

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$$\begin{aligned}\hat{R} &= \mathrm{List}\,(\Sigma s \in S.(Q\,s \to T) \times Q\,s) \\ &\quad \times \Sigma s \in S.(Q\,s \to T) \times P\,s\end{aligned}$$

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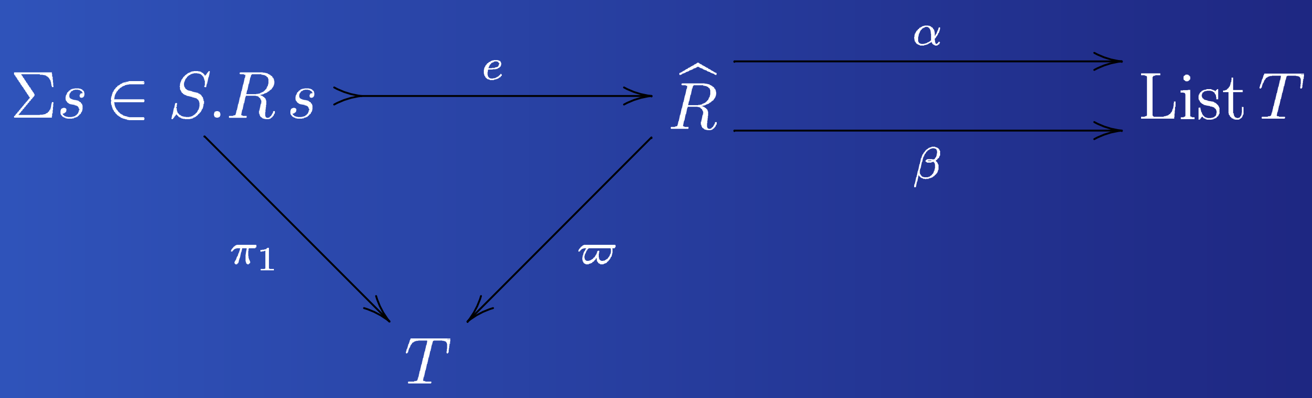

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See: *Containers - Constructing Strictly Positive Types* on my publication page.

Further work

- Quotient containers to model types like bags.
  First steps, see our MPC paper.
  Constructing Polymorphic Programs with Quotient Types
- Dependent containers
  Work in progress.

Related work

Joyal 86 *Foncteurs Analytiques et Espèces de Structures*

Jay 95 *A semantics for shape*

Dybjer 97 *Representing inductively defined sets by wellorderings in Martin-Löf's type theory*

Hoogendijk and de Moor 00 *Container Types Categorically*

Moerdijk and Palmgren 00 *Wellfounded Trees in Categories*

Hasegawa 02 *Two applications of analytic functors*

Gambino and Hyland 03 *Wellfounded Trees and Dependent Polynomial Functors*