Isomorphisms on inductive types

Thorsten Altenkirch

based on discussions with
Wouter Swierstra and Peter Morris

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Spine trees $\mu X.B \times (A \times X)^* = \mu X.B \times \mu Y.1 + A \times X \times Y$

Fibred . . .

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Simple slice $\mathbf{C}/\!\!/\Gamma$ $(\Gamma \in \mathrm{Obj}\,\mathbf{C})$

$\mathrm{Obj}\,\mathbf{C}/\!\!/\Gamma$	$A, B \in \mathrm{Obj}\,\mathbf{C}$
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In CCCs: Coproducts and initial algebras are always fibred.

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Fibred initial algebras:
$\mu X.A \times X + B \simeq (\mu X.A \times X + 1) \times B \simeq A^* \times B$

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$$(A + B)^* \simeq (A^* \times B)^* \times A^*$$

$\omega + \{\omega\}$

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Full subcategory of **Set**:

$\mathrm{Obj}\,(\omega + \{\omega\}) \qquad \alpha, \beta \in \omega + \{\omega\}$

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Modifications:

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- Consider muliplicities instead of acceptance.
  Replace $- \to$ **Bool** by $- \to (\omega + \{\omega\})$.

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$$\delta\, x\, y = \begin{cases} 1 & \text{if } x \equiv y \\ 0 & \text{otherwise} \end{cases}$$

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[\![\sigma^*]\!]^{\mathrm{M}}\, w & = & \Sigma_{\{v,v' \mid v+v'=w, v \neq \vec{0}\}} [\![\sigma]\!]^{\mathrm{M}}\, v \times [\![\sigma^*]\!]^{\mathrm{M}}\, v' \\
 & & +(\delta\, \vec{0}\, w) + \omega \times ([\![\sigma]\!]^{\mathrm{M}}\, \vec{0})
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$[\![\sigma]\!]^{\mathrm{F}}$ vs $[\![\sigma]\!]^{\mathrm{M}}$

$\sigma \simeq \tau$ **iff** $[\![\sigma]\!]^{\mathrm{M}} = [\![\tau]\!]^{\mathrm{M}}$???

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as pow $f\,\vec{X} = \Sigma_{g \in \mathrm{I} \rightarrow \omega}(f\,g) \times \Pi p \in \mathrm{I}.(g\,p) \rightarrow (\vec{X}\,p)$

Observe that pow $[\![\sigma]\!]^{\mathrm{M}} \simeq [\![\sigma]\!]^{\mathrm{F}}$
because pow $-$ preserves $0, +, 1, \times, -^{*}$.

Proof idea: **only if**

Using ideas from:

Abbott,A.,Ghani 05 *Containers - Constructing Strictly Positive Types*, *Theoretical Computer Science*, special issue on Applied Semantics (APPSEM).

we define a notion of morphisms on the multiset semantics. Using our representation theorem we can show that $f = g$, if $\mathrm{pow}\, f \simeq \mathrm{pow}\, g$.

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- What about context-free types in general? (No idea, maybe undecidable).
- What is the relation to recursive types (cf. Marcello's work).
- Can we use $\mathbb{R}$ (or $\mathbb{C}$) to decide the iomorphism problem for regular expressions? E.g. interpret $x^* = \frac{1}{1-x}$.