Containers

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University of Nottingham

The gang

Joint work with

- Michael Abbott, Neil Ghani (Leicester)
- Conor McBride (Durham)

FOSSACS 03 Michael,Neil and me
Categories of Containers

TLCA 03 Michael,Neil, Conor and me
Derivatives of Containers

See my publications page for electronic copies:
`http://www.cs.nott.ac.uk/˜txa/publ/`

Health warning!

I am going to illustrate our results using **Sets**. However, they are applicable to a much wider range of categories.

What is a container?

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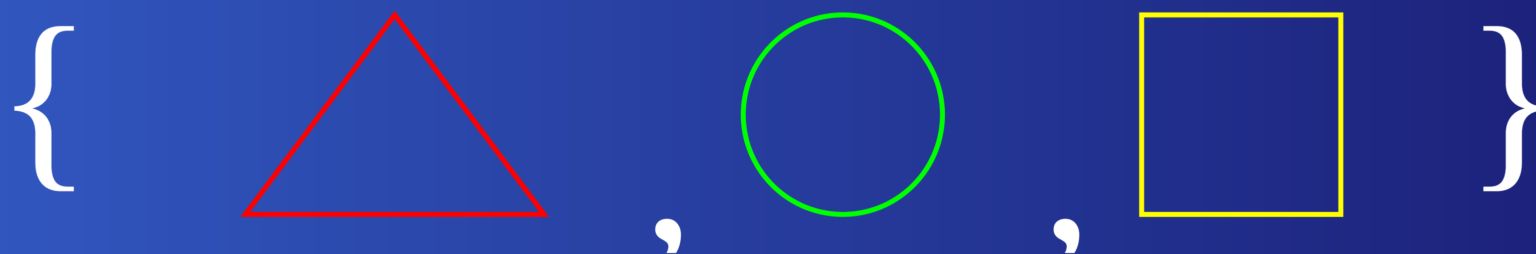
- A set S of shapes, e.g.



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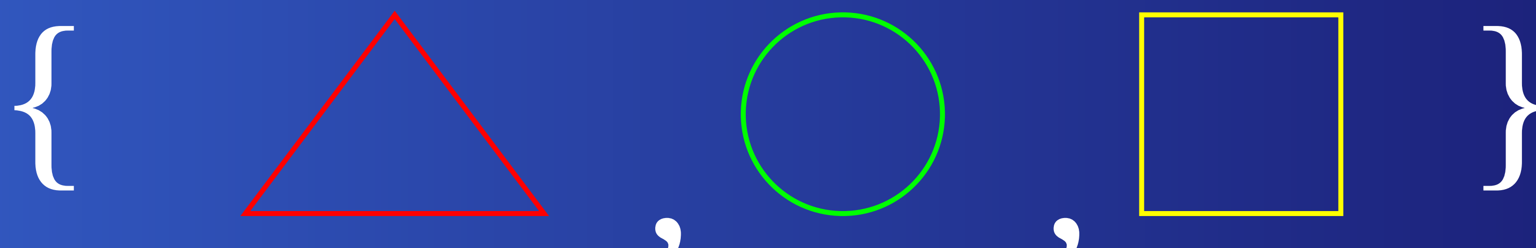
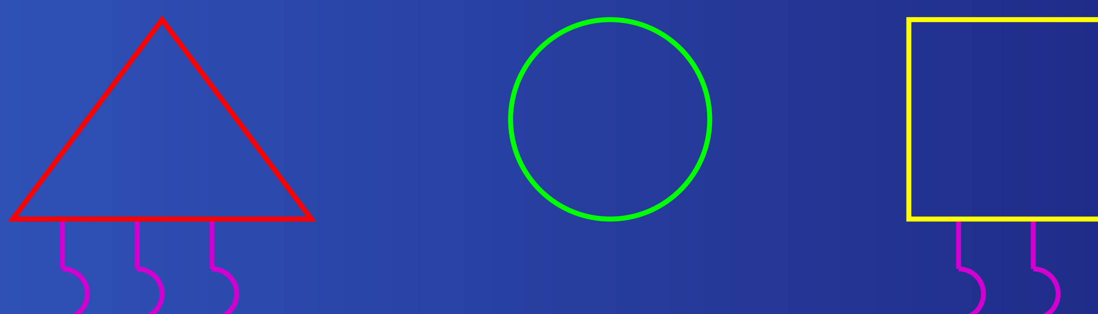


- For any shape $s \in S$ a set of positions $P(s)$,

e.g. 

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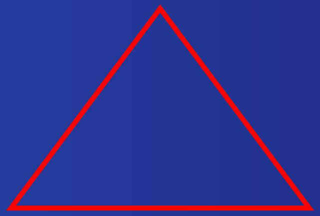
- Choosing a shape, e.g. 

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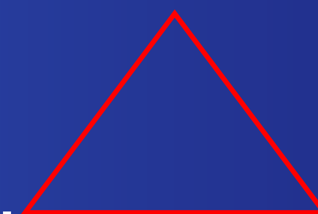
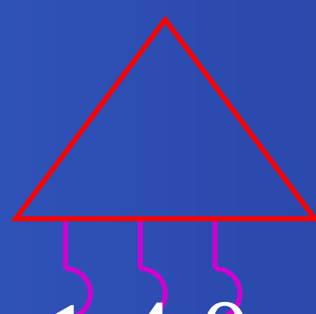
- Choosing a shape, e.g. 
- Filling the positions with payload, e.g.



e.g. 1 4 0

Extension of a container type

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$$\Sigma a \in \{0, 1\}.B(a) = B(0) + B(1)$$

Containers are functors

$$T_{S \triangleright P}(X) = \Sigma s \in S.P(s) \to X$$

$$\begin{aligned} T_{S \triangleright P}(f \in X \to Y) &\in T_{S \triangleright P}(X) \to T_{S \triangleright P}(Y) \\ &= (s, h) \mapsto (s, f \circ h) \end{aligned}$$

Example of a container type: $\mathsf{List}(X)$

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Coproducts of containers

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$$F + G(X) \simeq \Sigma u \in S + T.\left(\begin{cases} P(s) & \text{if } u = \mathrm{inl}(s) \\ Q(t) & \text{if } t = \mathrm{inr}(s) \end{cases}\right) \to X$$

Products of containers

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$$F \times G(X) \simeq \Sigma(s,t) \in S \times T.(P(s) + Q(t)) \to X$$

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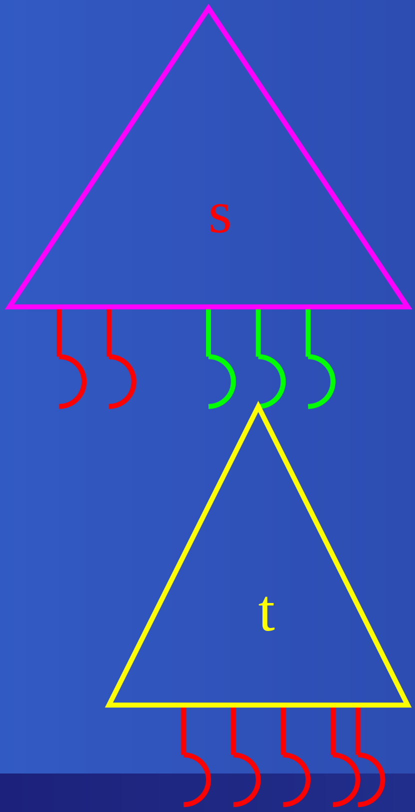
$$\begin{aligned} \mu Y.H(X,Y) &\simeq \Sigma t \in T.R(t) \to X \\ T &= \mu Y.\Sigma s \in S.Q(s) \to Y \\ &= W(S,Q) \\ R(s,f) &\simeq P(s) + \Sigma q \in Q(s).R(f(q)) \end{aligned}$$

(Least) Fixpoints of containers

$$R(s,f) \;\simeq\; P(s) + \Sigma q \in Q(s).R(f(q))$$



Strictly positive types

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Theorem: All strictly positive type expressions give rise to containers.

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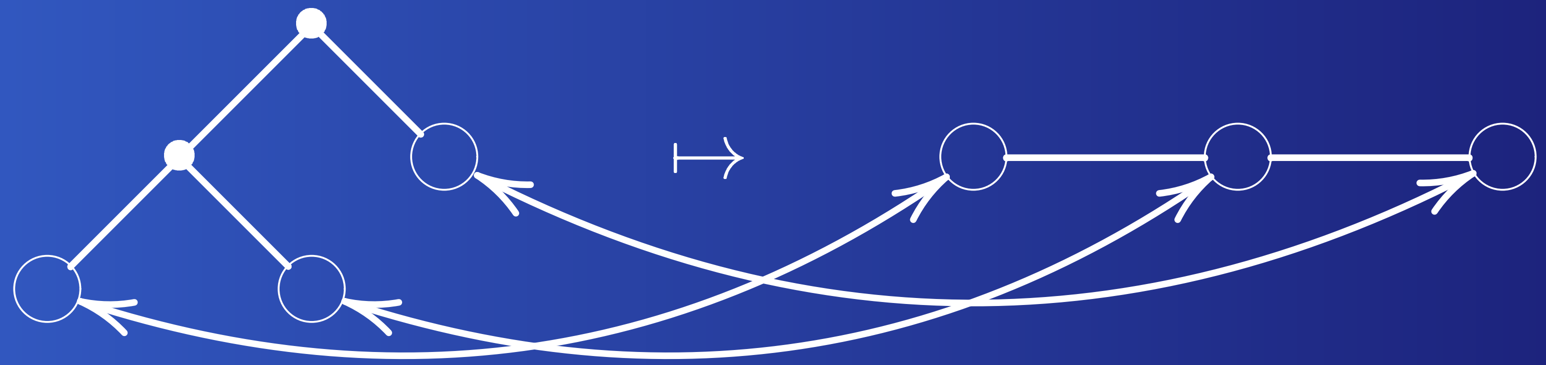

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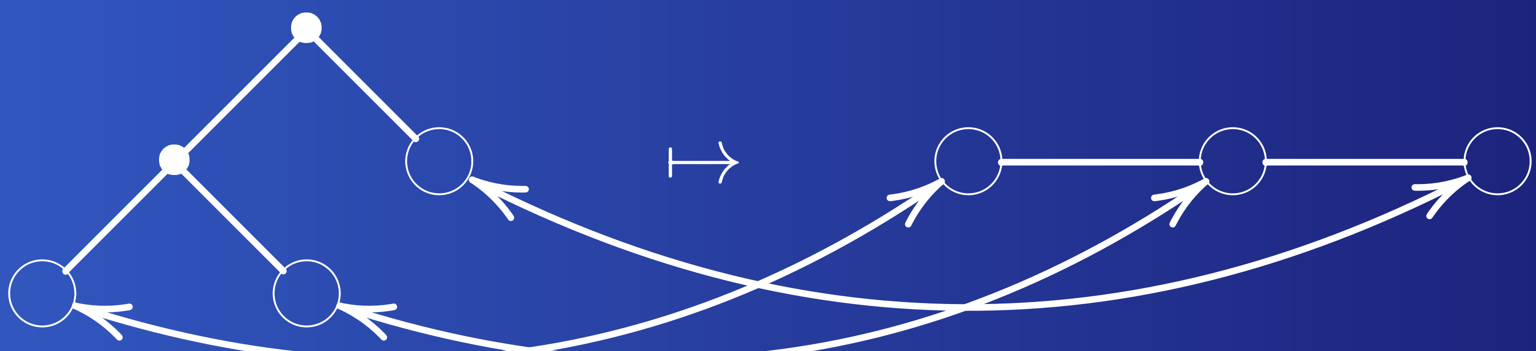

$$\text{tail} : \text{List}\ X \to \text{List}\ X$$

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Corollary: All natural transformations $\alpha_X \in \mathsf{List}(X) \to \mathsf{List}(X)$ are given by

- a function on the length $f \in \mathbb{N} \to \mathbb{N}$
- a *repotting function* $u \in \Pi n \in \mathbb{N}. f(n) \to n$.

Containers with holes

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Can we construct F' more systematically?

Making holes ($\times$)

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$$T'(X) = F'(X) \times G(X) + F(X) \times G'(X)$$

Making holes (chain)

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Conor’s observation

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The type of one-hole contexts of a datatype is its formal derivative.

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H(X,Y) &= 1 + X \times Y \\
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H'_2(X,Y) &= X
\end{aligned}
$$

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$$F' = I \multimap F$$

Theorem: Containers with a decidable equality on positions are differentiable and this class of containers is closed under derivaties.

What is e^x

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The answer can be found in our TLCA paper.

Generalizing Sets

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We should be able to replace **LFP** by something nicer . . .

Related work

- André Joyal
  Analytical functors and their derviatives.
- Paul Hoogendijk and Oege de Moor
  Containers in categories of relations.
- Bary Jay
  Shapely types.
- Peter Dybjer
  Inductive types as W-types.

What is this good for?

- Generic programming (e.g. the zipper)
- Reasoning about datatypes
- Reasoning about parametric programs
- Combinatorical representations of datatypes
- Optimizing functional programs

Forthcoming

Michael Abbott's PhD thesis
Categories of Containers