Normalisation by Completeness

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- Helmut Schwichtenberg needed to implement $\beta\eta$ -conversion for his MINLOG system.
- The implementation language was SCHEME.
- He wondered how he could exploit SCHEME's evaluator...
- This lead to the LICS 91 paper by Berger and Schwichtenberg.

How NBE should have been discovered...

- Derive normalisation from intuitionistic completeness proofs.
- Simpler then NBE because we ignore equality.
- Minimal logic (\approx simply typed λ calculus).
- Investigate disjunction (\approx coproducts).

References:

- CTCS 95 A.,Hofmann, Streicher Reconstruction of a reduction-free normalisation proof
 - LICS 01 A.,Dybjer, Hofmann, Scott Normalization by evaluation for typed lambda calculus with coproducts

$\frac{\Gamma \vdash A}{\Gamma.A \vdash A} \quad \frac{\Gamma \vdash A}{\Gamma.B \vdash A}$ $\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \quad \frac{\Gamma.A \vdash B}{\Gamma \vdash A \rightarrow B}$

with:

Propositions $A :: X | \cdots | A \rightarrow A$ with $X = \{P, Q, R, \dots\}$ atoms. Contexts $\Gamma ::$ empty $| \Gamma.A$

Exercise

Show that $\not\vdash (P \rightarrow P) \rightarrow P$.

Solution

Use truthtable semantics: if $\vdash A$ then $[\![A]\!]_{\rho}$ = true for any truth assignment. However

$$\llbracket (P \to P) \to P \rrbracket_{P \mapsto \text{false}} = \text{false}$$

hence $eq (P \rightarrow P) \rightarrow P$.

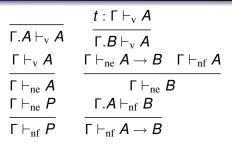
Exercise

Show that
$$\not\vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P$$
.

Solutions

- Use Normalisation...
- Ose Kripke semantics...

Normal derivations



Lemma : $\nvdash_{\mathrm{nf}}((P \to Q) \to P) \to P$

Proof: Analyze possible derivations.

Normalisation theorem:

$$\frac{\Gamma \vdash A}{\Gamma \vdash_{\mathrm{nf}} A} \text{hence } \not\vdash ((P \to Q) \to P) \to P$$

But how do we prove normalisation?

Kripke model

A Kripke model $K = (W, \leq, \Vdash)$ is given by

• A preordered set of worlds (W, \leq) .

• A monotone forcing relation $\Vdash \subseteq W \times X$: $\frac{w' \leq w \quad w \Vdash P}{w' \Vdash P}$

Forcing

We recursively extend the forcing relation to: propositions $w \Vdash A \rightarrow B = \forall w' \leq w.w' \Vdash A \rightarrow w' \Vdash B$ contexts $w \Vdash A_0 \dots A_n = w \Vdash A_0 \wedge \dots \wedge w \Vdash A_n$

Lemma

Monotonicity holds for all propositions:

$$\frac{w' \le w \qquad w \Vdash A}{w' \Vdash A}$$

Soundness

$$\frac{\Gamma \vdash A}{\forall w.w \Vdash \Gamma \to w \Vdash A}$$
 sound

$\not\vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P$ using a Kripke model

A countermodel

$$0 \not\Vdash ((P \to Q) \to P) \to P$$

hence using soundness

$$\not\vdash ((P \to Q) \to P) \to P$$

How good are Kripke models ?

- We can refute some unprovable propositions using truthtables.
- We can refute more unprovable propositions using Kripke models.
- Are all unprovable propositions refutable by Kripke models?
- Or positively: are all propositions which hold in all Kripke models, provable.
- Even better there is one universal Kripke model *U* in which precisely the derivable propositions hold:

$$\frac{\forall w.w \Vdash \Gamma \to w \Vdash A}{\Gamma \vdash A}$$

Define:
$$\Gamma \vdash^* A_1 \dots A_n = \Gamma \vdash A_1 \wedge \dots \Gamma \vdash A_n$$
, we can show:

 $\Gamma \vdash^* \Gamma$
 $\Gamma \vdash^* \Delta \quad \Delta \vdash A$
 $\Gamma \vdash A$
 $\Gamma \vdash^* \Theta$
 $\Gamma \vdash^* \Theta$

The universal model

$$U = (\text{Contexts}, \vdash^*, \vdash)$$

- (Contexts, ⊢*) is a preorder by 1,3
- ⊢ is monotone by 2

quote and unquote

$$\frac{\Gamma \Vdash A}{\Gamma \vdash A} quote \qquad \frac{\Gamma \vdash A}{\Gamma \Vdash A} unquote$$

Proof: mutual induction over A.

Completeness

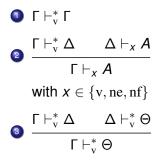
$$\frac{\forall \Delta. \Delta \Vdash^* \Gamma \to \Delta \Vdash A}{\Gamma \vdash A} \operatorname{Compl}$$

Proof: Combine quote and unquote.

Going back and forth

$$\frac{\Gamma \vdash A}{\forall \Delta . \Delta \Vdash^* \Gamma \to \Delta \Vdash A} \text{ sound} \\ \frac{\Gamma \vdash A}{\Gamma \vdash A}$$

- What have we achieved?
- We would like to obtain $\Gamma \vdash_{nf} A$.
- Let's shrink the model...
- and revisit completeness.



The universal model (with normal forms)

$$U = (\text{Contexts}, \vdash_{v}^{*}, \vdash_{ne} (=\vdash_{nf}))$$

- (Contexts, \vdash_v^*) is a preorder by 1,3
- \vdash_{ne} is monotone by 2

quote and unquote

$$\frac{\Gamma \Vdash A}{\Gamma \vdash_{\mathrm{nf}} A} quote \qquad \frac{\Gamma \vdash_{\mathrm{ne}} A}{\Gamma \Vdash A} unquote$$

Completeness

$$\frac{\forall \Delta. \Delta \Vdash^* \Gamma \to \Delta \Vdash A}{\Gamma \vdash_{\mathrm{nf}} A} \operatorname{Compl}$$

Proof: Combine quote and unquote.

Normalisation from completeness

$$\frac{\Gamma \vdash A}{\forall \Delta . \Delta \Vdash^* \Gamma \to \Delta \Vdash A} \operatorname{sound}_{\Gamma \vdash_{\mathrm{nf}} A} \operatorname{compl}$$

- Normalisation is a consequence of completeness!
- We adjust the model and check the proof to show that completeness always produces normal forms.
- Once we have normalisation we don't need the models anymore!

NBC	NBE
mininal logic	λ -calculus (CCC)
preorder	category
monotone	functorial
Kripke model	presheaf model
soundness	presheaves are cartesian closed

Adding connectives

Conjunction

$$w \Vdash A \land B = w \Vdash A \land w \Vdash B$$

 $w \Vdash \top = \top$

Soundness ok Completeness ok

Disjunction

$$w \Vdash A \lor B = w \Vdash A \lor w \Vdash B$$

 $w \Vdash \bot = \bot$

Soundness ok Completeness ???

$$U = (\text{Contexts}, \vdash^*, \vdash)$$

 $\frac{P \lor Q \vdash P \lor Q}{P \lor Q \Vdash P \lor Q} \text{ unquote}$ $\frac{(P \lor Q \Vdash P) \lor (P \lor Q \vDash Q)}{(P \lor Q \vdash P) \lor (P \lor Q \vdash Q)} \text{ quote}$

- But aren't Kripke models complete for intuitionistic logic?
- Yes, but the universal model has to be constructed differently.
- Contexts are replaced by saturated contexts...
- The construction of the universal model now requires decidability:

$$\Gamma \vdash \mathbf{A} \lor \Gamma \not\vdash \mathbf{A}$$

- Indeed, completeness for Kripke models for intuitionistic predicate logic is **not** provable intuitionistically.
- Instead, we will consider a different class of models.

Beth model

A Beth model $B = (W, \leq, \Vdash, \triangleleft)$ is given by

- A Kripke model (W, \leq, \Vdash) .
- A covering relation $\lhd \subseteq W \times \mathcal{P}W$ such that:

trivial $w \triangleleft \{w' \mid w' \leq w\}$ monotone $\frac{w \triangleleft P \quad w' \leq w}{w' \triangleleft P}$ union $\frac{w \leq P \quad \forall w' \in P.w' \triangleleft Q}{w \triangleleft Q}$ paste $\frac{w \triangleleft P \quad \forall w' \in P.w' \Vdash Q}{w \Vdash Q}$

Forcing

We extend the forcing relation:

$$w \Vdash A \lor B = \exists P.w \lhd P \land \forall w' \in P.w' \lhd A \lor w' \lhd B$$
$$w \Vdash \bot = w \lhd \{\}$$

Lemma:

Monotonicity and paste hold for all formulas:

$$\frac{w \triangleleft A \quad \forall w' \in P.w' \Vdash A}{w \Vdash A} \qquad \frac{w' \leq w \qquad w \Vdash A}{w' \Vdash A}$$

Soundness:

$$\frac{\Gamma \vdash A}{\forall w.w \Vdash \Gamma \rightarrow w \Vdash A}$$
 sound

The universal Beth model

$$U = (\text{Contexts}, \vdash^*, \vdash, \triangleleft)$$

- (Contexts, \vdash^* , \vdash) is the universal Kripke model.
- ⊲ is defined inductively:

$$\ \ \, \square \ \ \Gamma \lhd \{\Delta \mid \Delta \le \Gamma\}$$

$$\int \frac{\Gamma }{\Gamma } dP$$

Completeness

$$\frac{\forall \Delta. \Delta \Vdash^* \Gamma \to \Delta \Vdash A}{\Gamma \vdash A} \operatorname{Compl}$$

Proof: Extend quote and unquote.

- Left as an exercise.
- First step: come up with a good notion of normal form...

NBCNBE⊲Grothendieck topologyBeth modelsheaf model

- We have solved simpler problems: the existence of normal forms.
- We have ignored equality of derivations.
- We have shown that normalisation can be obtained by a modified universal model.
- NBE can be recovered by moving to corresponding proof-relevant constructions.
- Now for something completely different: Why is it hard to formalize Type Theory in Type Theory?