A Functional Quantum Programming Language

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based on joint work with Jonathan Grattage

and discussions with V.P. Belavkin

supported by EPSRC grant GR/S30818/01

Alternative title

What you always wanted to know about quantum computation but never dared to ask.

Another alternative title

Quantum programming

for the

lazy functional programmer

Background

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- Simulation of quantum systems is expensive: Exponential time to simulate polynomial circuits.

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Assumption: Nature is fair. . .

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- Richard Josza, QPL 2004: *We need to develop quantum thinking!*

QML

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- Important issue: **control of decoherence**
- Compiler under construction (Jonathan)

Example: Hadamard operation

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Matrix

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QML

$$\begin{aligned} & had : Q_2 \multimap Q_2 \\ & had\ x = \mathbf{if}^{\circ}\ x \\ & \qquad \mathbf{then}\ \{\mathrm{qfalse} \mid (-1)\ \mathrm{qtrue}\} \\ & \qquad \mathbf{else}\ \ \{\mathrm{qfalse} \mid \mathrm{qtrue}\} \end{aligned}$$

Something we know well ...

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- Quantum mechanics is time-reversible. . .
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- **However:** Newtonian mechanics, Maxwellian electrodynamics are also time-reversible. . .
- . . . hence classical computation **should be** based on reversible operations.

Classical computation (FCC)

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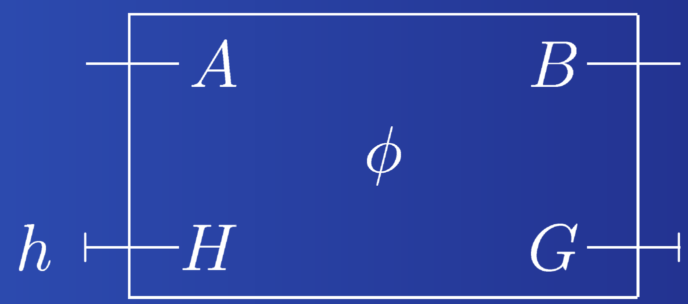
Given finite sets A (input) and B (output):



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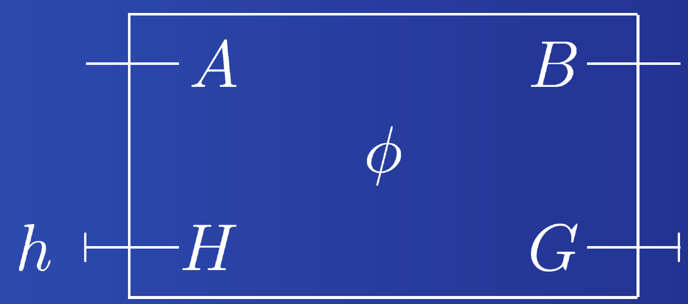
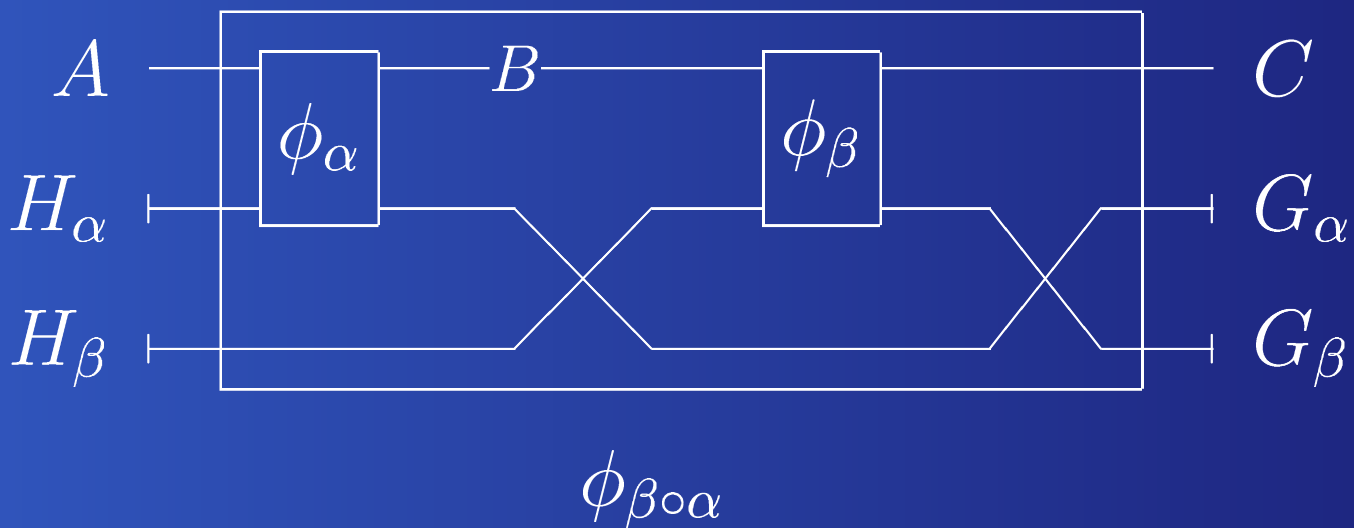


- a finite set of initial heaps H,
- an initial heap $h \in H$,
- a finite set of garbage states G,
- a bijection $\phi \in A \times H \simeq B \times G$,

Composing computations

Composing computations



Extensional equality

Extensional equality

- A classical computation $\alpha = (H, h, G, \phi)$ induces a function $\mathsf{U}\alpha \in A \to B$ by

$$\begin{tikzcd}
A \times H \arrow[r, "\phi"'] & B \times G \arrow[d, "\pi_1"] \\
A \arrow[u, "(-{,}h)"] \arrow[r, "\mathsf{U}\,\alpha"'] & B
\end{tikzcd}$$

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- We say that two computations are **extensionally equivalent**, if they give rise to the same function.

Extensional equality ...

- **Theorem:**

$$\mathbf{U}\,(\beta \circ \alpha) = (\mathbf{U}\,\beta) \circ (\mathbf{U}\,\alpha)$$

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- **Hence, classical computations upto extensional equality give rise to the category FCC.**
- **Theorem: Any function $f \in A \to B$ on finite sets A, B can be realized by a computation.**
- ***Translation for Category Theoreticans:***
  U is full and faithful.

Example π_1 :

function

$$\pi_1 \in (2,2) \to 2$$
$$\pi_1\ (x,y) = x$$

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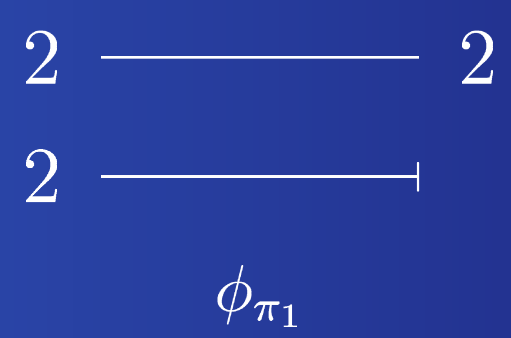
computation



Example δ :

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$$\delta \in 2 \to (2, 2)$$
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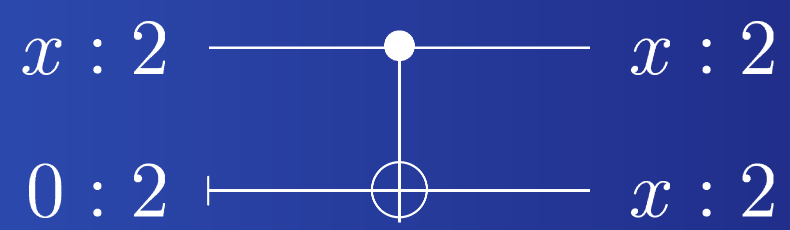

ϕ_δ

$$\phi_\delta \in (2,2) \to (2,2)$$
$$\phi_\delta\ (0, x) = (0, x)$$
$$\phi_\delta\ (1, x) = (1, \neg\ x)$$

Classical vs quantum

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classical ($\mathbf{FCC}$)	quantum ($\mathbf{FQC}$)

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Classical vs quantum

classical (**FCC**)	quantum (**FQC**)
finite sets	finite dimensional Hilbert spaces
bijections	unitary operators
cartesian product ($\times$)	tensor product ($\otimes$)
functions	superoperators
projections	partial trace

$\pi_1 \circ \delta$, classically

$$\pi_1 \circ \delta : 2 \to 2$$

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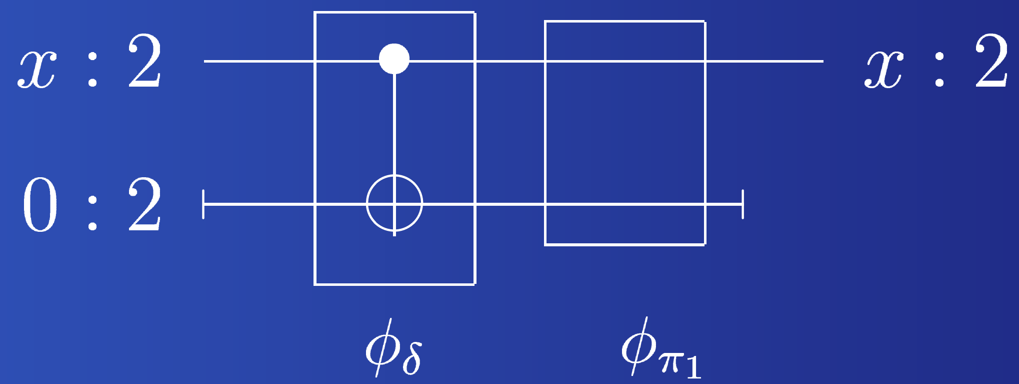

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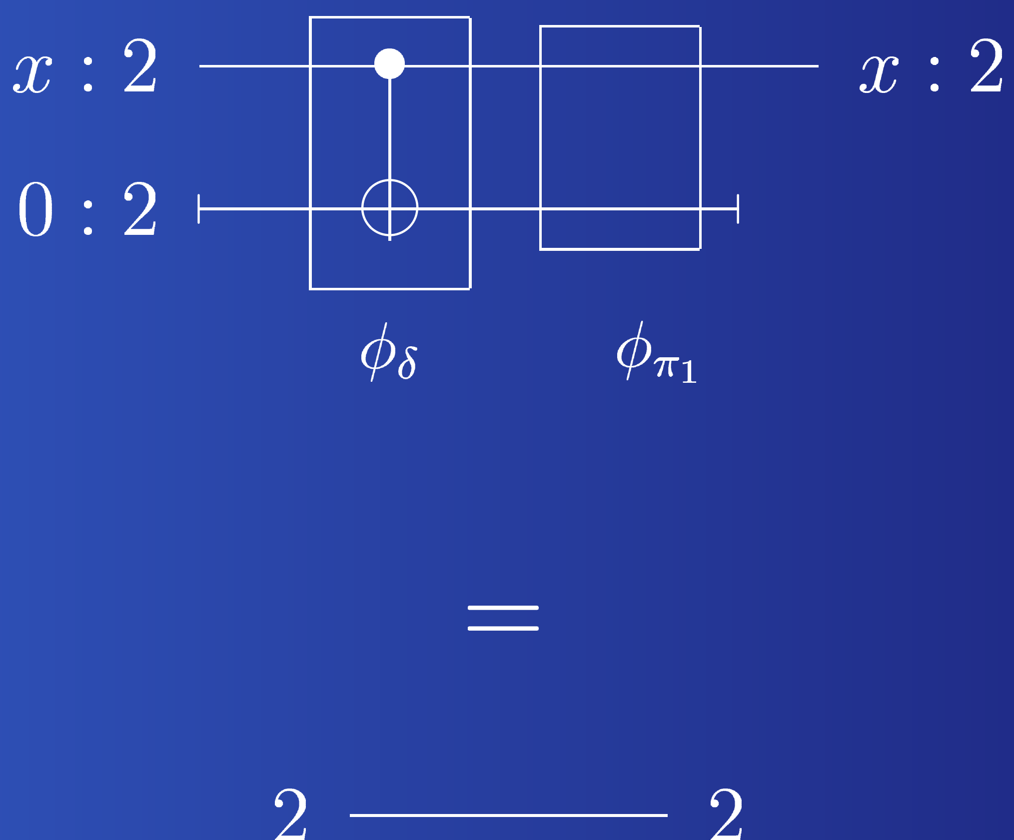
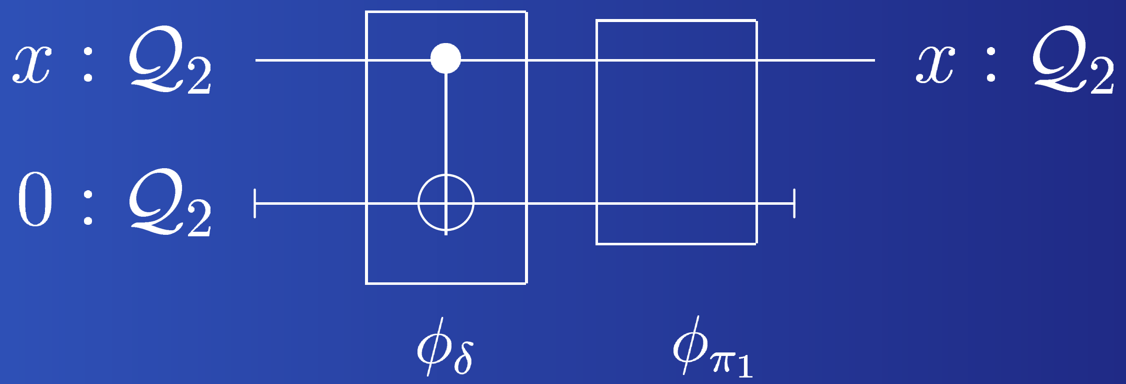


$\pi_1 \circ \delta$, quantum



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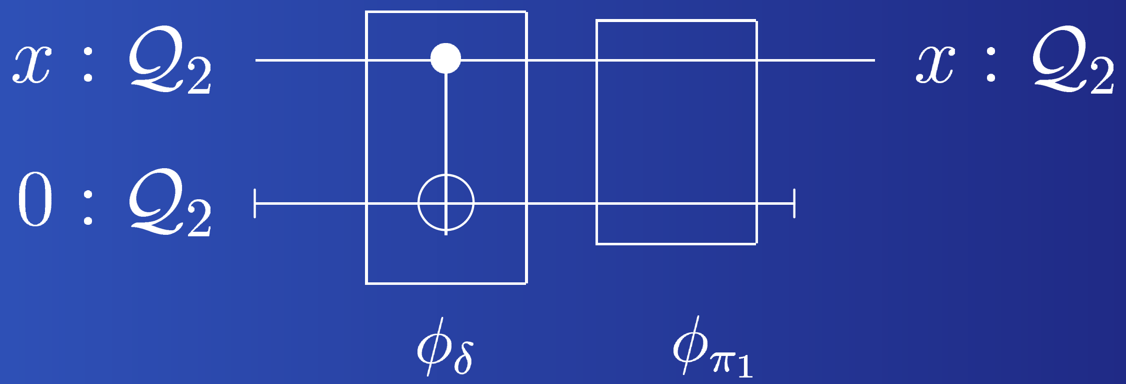

input: $\left\{\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle\right\}$

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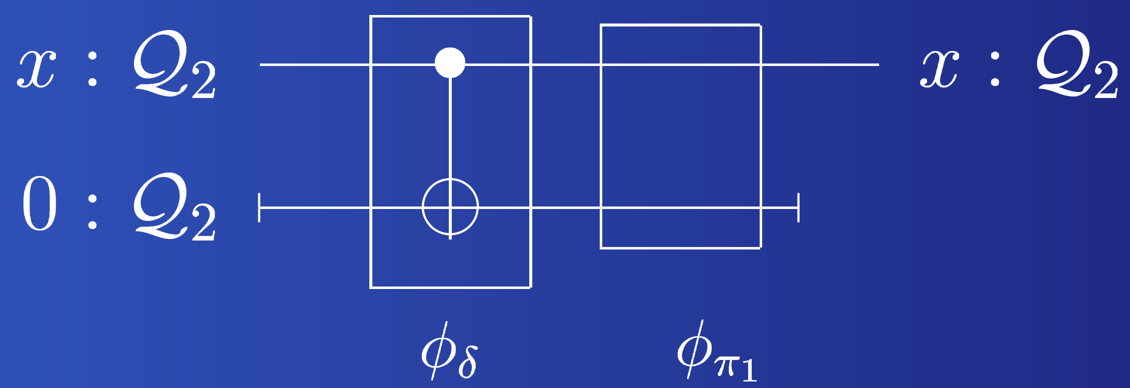

input: $\{\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\}$

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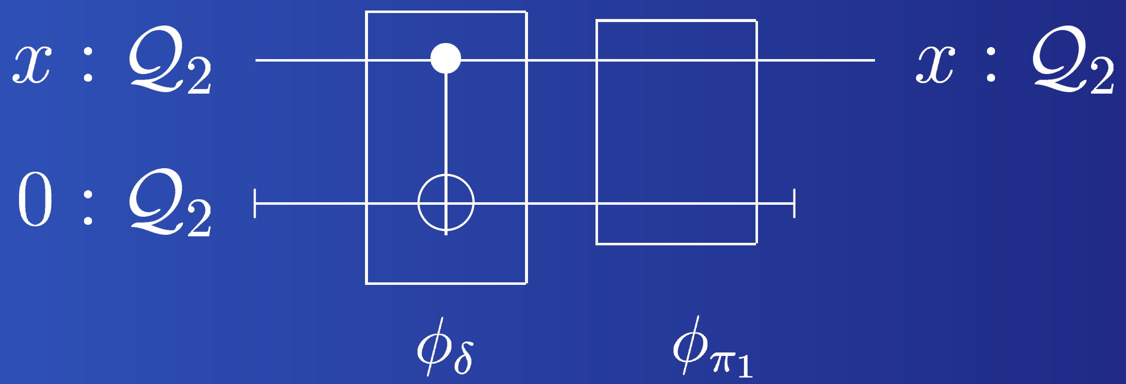

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Decoherence!

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- Contraction is implicit and realized by ϕ_δ.
- Weakening is explicit and leads to decoherence.

QML overview

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Types

$$\sigma = 1 \mid \sigma \otimes \tau \mid \sigma \oplus \tau$$

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$$\sigma = 1 \mid \sigma \otimes \tau \mid \sigma \oplus \tau$$

Terms

$$\begin{aligned} t &= x \mid \mathbf{let}\ x = t\ \mathbf{in}\ u \mid x \uparrow xs \\ &\mid () \mid (t, u) \mid \mathbf{let}\ (x, y) = t\ \mathbf{in}\ u \\ &\mid \mathrm{qinl}\ t \mid \mathrm{qinr}\ u \\ &\mid \mathbf{case}\ t\ \mathbf{of}\ \{\mathrm{qinl}\ x \Rightarrow u \mid \mathrm{qinr}\ y \Rightarrow u'\} \\ &\mid \mathbf{case}^\circ\ t\ \mathbf{of}\ \{\mathrm{qinl}\ x \Rightarrow u \mid \mathrm{qinr}\ y \Rightarrow u'\} \\ &\mid \{(\kappa)\ t \mid (\iota)\ u\} \end{aligned}$$

Qbits

$$Q_2 = 1 \oplus 1$$

$$\mathrm{qtrue} = \mathrm{qinl}\ ()$$

$$\mathrm{qfalse} = \mathrm{qinr}\ ()$$

$$\mathbf{if}\ t\ \mathbf{then}\ u\ \mathbf{else}\ u'$$

$$= \mathbf{case}\ \{\mathrm{qinl}\ \_ \Rightarrow u \mid \mathrm{qinr}\ \_ \Rightarrow u'\}$$

$$\mathbf{if}^{\circ}\ t\ \mathbf{then}\ u\ \mathbf{else}\ u'$$

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Typing judgements

$\Gamma \vdash t : \sigma$	programs
$\Gamma \vdash^{\circ} t : \sigma$	strict programs

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Typing judgements

$\Gamma \vdash t : \sigma$ programs

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Semantics

$$\frac{\Gamma \vdash t : \sigma}{[\![t]\!] \in \mathbf{FQC}[\![\Gamma]\!][\![\sigma]\!]} \qquad \frac{\Gamma \vdash^\circ t : \sigma}{[\![t]\!] \in \mathbf{FQC}^\circ[\![\Gamma]\!][\![\sigma]\!]}$$

The let-rule

$$\frac{\Gamma \vdash t : \sigma \qquad \Delta, x : \sigma \vdash u : \tau}{\Gamma \otimes \Delta \vdash \mathtt{let}\ x = t\ \mathtt{in}\ u : \tau}\ \text{let}$$

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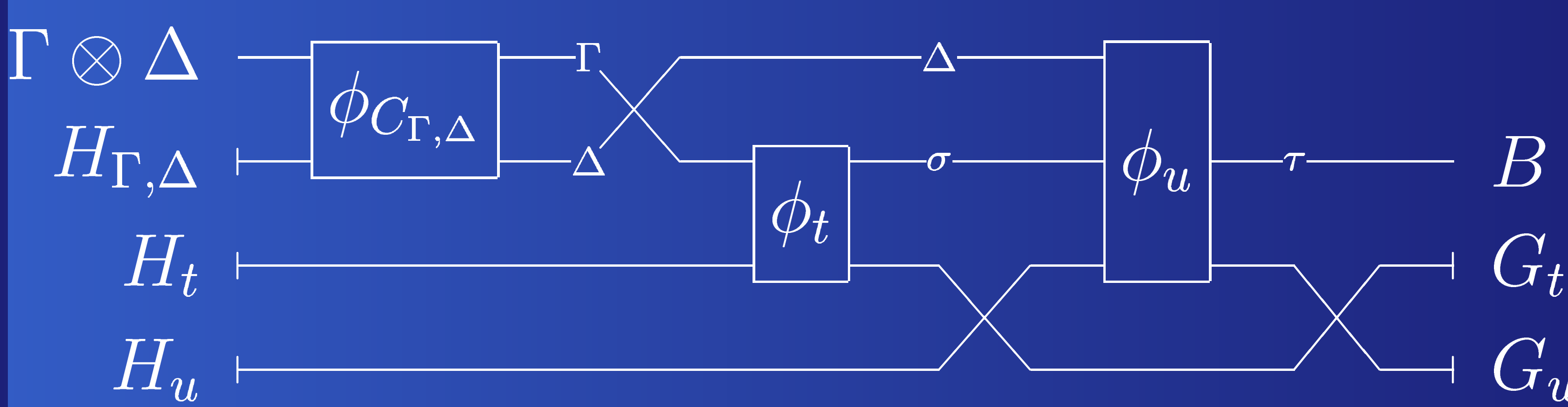

$\otimes$ on contexts

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$$
\begin{array}{lcll}
\Gamma, x:\sigma \otimes \Delta, x:\sigma & = & (\Gamma \otimes \Delta), x:\sigma & \\
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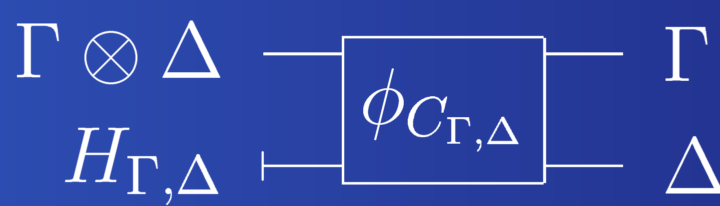

Another source of decoherence

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- *forget* mentions x

$$forget : 2 \multimap 2$$

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- but doesn't use it.
- Hence, it **has** to measure it!

$\oplus$-elim

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$$\frac{\Gamma \vdash c : \sigma \oplus \tau \qquad \Delta, x : \sigma \vdash t : \rho \qquad \Delta, y : \tau \vdash u : \rho}{\Gamma \otimes \Delta \vdash \texttt{case}\ c\ \texttt{of}\ \{\texttt{inl}\ x \Rightarrow t \,|\, \texttt{inr}\ y \Rightarrow u\} : \rho} + \mathrm{elim}$$

$\bigoplus$-elim

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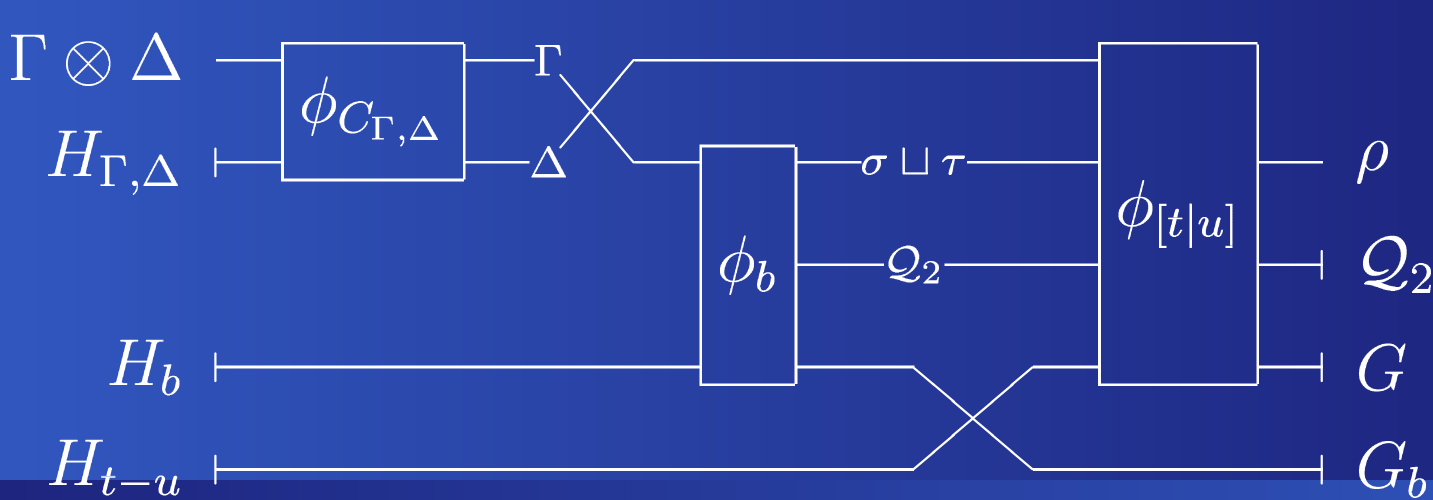

$\oplus$-elim decoherence-free

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$$\frac{\Gamma \vdash^{a} c : \sigma \oplus \tau \qquad \Delta,\ x : \sigma \vdash^{\circ} t : \rho \qquad \Delta,\ y : \tau \vdash^{\circ} u : \rho \quad t \perp u}{\Gamma \otimes \Delta \vdash^{a} \texttt{case}^{\circ}\ \ c\ \texttt{of}\ \ \{\texttt{inl}\ x \Rightarrow t \mid \texttt{inr}\ y \Rightarrow u\} : \rho} \bigoplus - \mathrm{elim}^{\circ}$$

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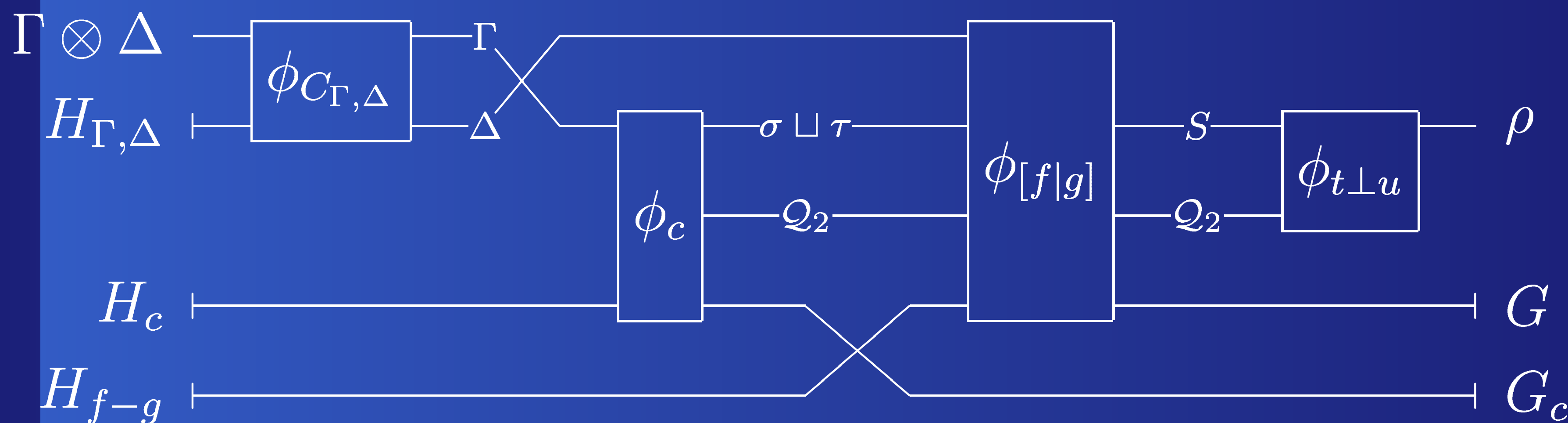

if°

$\mathbf{if}^\circ$

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- Our analysis also highlights the differences between classical and quantum programming.
- Quantum programming introduces the problem of *control of decoherence*, which we address by making forgetting variables explicit and by having different if-then-else constructs.

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- How to deal with higher order programs?
- How to deal with infinite datatypes?
- Investigate the similarities/differences between FCC and FQC from a categorical point of view.

The end

Thank you for your attention.