On the Length of Cardinality Statements

1. *Introduction: Nominalism and Fictionalism*

Nominalism is the claim that there are no abstract, and, in particular, no mathematical entities: numbers, functions, sets and structures. A formulation of this claim occurs in the opening of a *locus classicus* in the recent nominalist tradition:

> We do not believe in abstract entities. Noone supposes that abstract entities—classes, relations, properties, etc.—exist in space-time; but we mean more than this. We renounce them altogether. (Quine & Goodman 1947 (in Goodman 1972), p. 173).

Sometimes this view is called 'fictionalism'. On this view, our mathematical theories form something like a system of statements, perhaps useful and convenient in various ways, but, as one famous fictionalist—about everything, apparently—put it, 'there is nothing outside the text'. Fictionalism has been advocated in several domains, most surprisingly concerning science. Fictionalism about science (and usually accompanied with indignant animosity towards a host of related notions, such as truth, objectivity, evidence, reason, facts, freedom, democracy, Enlightenment, etc.) has been *de rigeur*, particularly in academic departments of the social science and humanities, for the last 40 years or so. This fashion appears to have died down somewhat of late, particularly since Sokal's amusing hoax. However, and to be fair, few serious philosophers have ever succumbed to the weirdness of fictionalism about *science*. But for domains of inquiry which lack the epistemic status which science clearly enjoys, fictionalism remains an option. In the cases of modality and ethics, fictionalism is a popular, perhaps viable, view. But for mathematics the situation is different. If fictionalism is untenable for science, then how could it be tenable for mathematics? For, unlike ethics (and perhaps modality) *mathematics forms an integral part of our scientific understanding of the world*.

The resulting problem for nominalism was stressed by Quine in 1948 and by Putnam in 1971. If mathematics is a fiction, as the nominalist suggests, how are we to account for its *application* in science? As Putnam put it,

> … quantification over mathematical entities is indispensable for science … therefore we should accept such [talk]; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the *indispensability* of [talk about] mathematical entities and the *intellectual dishonesty* of denying the existence of what one daily presupposes. (Putnam 1971 (1979), p. 347).

> … mathematics and physics are integrated in such a way that it is not possible to be a realist with respect to physical theory and a nominalist with respect to mathematical theory. (Putnam 1975 (1979), p. 74).

Fictionalism about mathematics comes in two varieties, which I shall call '*de*constructionism' and '*re*constructionism'.[1] The first view, rather more fashionable of late, resembles instrumentalism about scientific theories ('our theories are

[1] What I'm calling 'deconstructionist fictionalism' corresponds loosely to what John Burgess calls '*instrumentalist* nominalism' in his 1983, and his 1997 book with Gideon Rosen. What I'm calling 'reconstructionist fictionalism' corresponds to what Burgess calls '*revolutionary* nominalism' and '*hermeneutic* nominalism'.

convenient instruments for coping with sense data'), and faces a rather similar objection: given that 'there is nothing outside the text', it appears to be a *miracle* that characters from the mathematical 'fiction'—numbers, functions, sets, vector spaces, manifolds, fibre bundles, path integrals, etc.—find their way into scientific theories.[2] The comparison with fiction is more miraculous since, to my knowledge, no characters from the novels of Cervantes or Dickens have found their way into scientific theories and explanations. On this deconstructionist view, both the *applicability* of mathematics, and the *non-applicability* of the rest of (genuine) fiction, are both inexplicable miracles.[3] This might be called a No Miracles Argument against deconstructionalist fictionalism, since it is entirely analogous to the No Miracles Argument used against certain kinds of fictionalism and instrumentalism about science. Moreover, this form of deconstructionist fictionalism about mathematics was dismissed many years ago by a leading fictionalist, Hartry Field, as 'doublethink':

> If one just advocates fictionalism about a portion of mathematics, without showing how that part of mathematics is dispensable in applications, then one is engaging in *intellectual doublethink*: one is merely taking back in one's philosophical moments what one asserts in doing science. (Field 1980, p. 2.)

Setting aside deconstructionist fictionalism, there remains a constructive approach, which attempts to provide a constructive explanation of how mathematics functions in our scientific theories. In particular, this explanation involves a *reconstruction* of mathematics and mathematicized science *avoiding* quantification over mathematical entities. Such reconstructions are sometimes called 'nominalizations'. On one view, mathematical statements about the world provide a 'compressed' way of expressing what otherwise *could* have been expressed nominalistically, modulo sufficient physical, temporal and psychological resources. The leading fictionalist Hartry Field has developed a widely discussed programme of 'nominalizing' applicable mathematics in science (Field 1980). Field's account of the application of mathematics turns, in part, on the idea that,

> … any inference from nominalistic premises to a nominalistic conclusion that can be made with the help of mathematics could be made (usually more long-windedly) without it. (Field 1980, p. x).
>
> … the conclusions we arrive at [by mathematics] are not genuinely new, they are already derivable in a more long-winded fashion from the [nominalistic theory] without recourse to mathematical entities. (Field 1980, pp. 10-11).

[2] In his celebrated article, Eugene Wigner makes a similar point. However, he does not take it to be a *reductio ad absurdum* of the fictionalist view. The essential point itself find expression scattered throughout the literature, particularly when scientists reflect on this problem: the classic statement was, of course, made by Galileo. But recent examples include James Jeans, Arthur Eddington, Albert Einstein, Richard Feynman, Paul Dirac, Roger Penrose, Steven Weinberg, and Paul Davies. For a nice discussion of 'The Miracle of Applied Mathematics', see Colyvan 2001.

[3] Let me stress that there is *much* more to be said about the nature of genuine fiction, and its role in our engagement with the world. For example, although nobody believes that fictional characters such as Sancho Panza or Madame Bovary exist, it is nonetheless true that, as Noam Chomsky has remarked, fiction can play an important and perhaps indispensable role in developing our empathetic understanding of the world, particularly the social world. In fiction we do seem to create a mental representation of a world which is unreal, and we can engage with the world as though it were real. However, no one takes this simple fact to imply the absurdity that Madame Bovary exists, or that Mr Darcy exists.

Spelt out a little more precisely, this is a *conservation claim*: the idea is that adding mathematics to non-mathematically formulated theories always results in conservative extensions, and thus anything proved using the mathematics could be proved without it, but perhaps 'more long-windedly', as Field puts it. The viability of such a programme would thereby illustrate the reconstructionist fictionalist's view that mathematics is merely an 'efficient tool of sharpened logical thought' (Newton 2001, p. 76).

Suppose that it is true that we *could in principle* re-express our scientific theories without mathematics, and *could in principle* prove the requisite non-mathematical results without using mathematics, but that these re-expressions and proofs are 'more long-winded'.[4] How much more 'long-winded' would these re-expressions and nominalistic proofs be? Presumably, a mild constraint on this approach is that the 'decompression' of a mathematical claim to its nominalization, and the decompression of a mathematical proof of a nominalistic conclusion to its 'nominalistic' proof, is at least *possible*, and preferably *feasible*. Here we give a examples of how unfeasible it can be. In general, the phenomenon discussed is a kind of 'speed-up' phenomenon.[5]

2. *Length of Cardinality Statements*

Consider a cardinality statement, such as,

(1) The numbers of *F*s is at least 3.

The conventional 'nominalization' of (1) is,

(2) There are *x*, *y*, and *z*, all different and all *F*s.

If the predicate *Fx* is nominalistic, then the statement (2) counts as 'nominalistic', since it does not refer to numbers. The nominalistic idea is that one *could* write down such nominalizations, and that it's a matter only of convenience to use a compressed statement like (1), which is merely an abbreviation for the nominalistic claim (2).[6]

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[4] Field's conservation claim has been disputed. For example, by Shapiro in 1983, where he shows that if our scientific theory is sufficiently rich for Gödel's theorems to apply, then there will be non-mathematical physical claims which can only be proved by adding enough mathematics. For such theories, adding mathematics is generically non-conservative. See Burgess and Rosen 1997 for more discussion. But we shall set this issue aside in what follows.

[5] The idea goes back to Gödel's length-of-proof paper (Gödel 1936). If S^+ is a conservative extension of S, then the shortest proof of a statement φ in S^+ may be much shorter than the shortest proof of φ in S. One example concerns Peano arithmetic PA and its weak second-order extension ACA_0. Every arithmetic statement φ provable in ACA_0 is already provable in PA, but the shortest proof of an arithmetic statement φ in PA may be vastly longer than the shortest proof in ACA_0. Related is the 'Parikh phenomenon': there are sentences φ such that we can (meta-)prove that *there exists* a proof of φ in S, despite the fact that the shortest such proof of φ in the theory S is unfeasibly huge. The phenomenon of *non-conservation* occurs when a richer theory S^+ proves a statement with has *no* proof in S. For some accessible illustrations of such phenomena, see Boolos 1984 and Boolos 1987. Note also that a similar point to the one made here, about the astonishing length of decompressing certain abbreviations, is made in Mathias 2002, where the decompression of Bourbaki's definition of '1' is examined.

[6] The corresponding case of 'infinite speed-up' occurs with a statement like 'The number of *F*s is $\aleph_0$' which has no finite nominalization, for it would require an 'infinitely long' nominalization. Note the similarity with the indefinablity of *truth*. A sufficiently rich interpreted language cannot define its own truth predicate. However, by moving to an *infinitary* language of the next 'order', the truth predicate becomes definable (the truth predicate is an infinitary conjunction of formulas of the form ($x = \ulcorner\varphi\urcorner \wedge$

Consider the inductive definition for the finite cardinality quantifiers $\exists_{\geq n}x$ ('there are at least n x such that …') and $\exists_n x$ ('there are exactly n x such that …'):[7]

(3a) $\exists_0 \mathbf{v}\varphi(\mathbf{v})$ for $\neg\exists\mathbf{v}\varphi(\mathbf{v})$

(3b) $\exists_{\geq 1}\mathbf{v}\varphi(\mathbf{v})$ for $\exists\mathbf{v}\varphi(\mathbf{v})$

(3c) $\exists_{\geq n+1}\mathbf{v}\varphi(\mathbf{v})$ for $\exists\mathbf{v}(\varphi(\mathbf{v}) \wedge \exists_{\geq n}\mathbf{w}(\mathbf{w} \neq \mathbf{v} \wedge \varphi(\mathbf{w})))$

(3d) $\exists_{\mathrm{n}}\mathbf{v}\varphi(\mathbf{v})$ for $\exists_{\geq \mathrm{n}}\mathbf{v}\varphi(\mathbf{v}) \wedge \neg\exists_{\geq n+1}\mathbf{v}\varphi(\mathbf{v})$

where **v** is any variable, and **w** is a variable not appearing in $\varphi(\mathbf{v})$. We use x, x', x'', x''', etc., as our variables. Write $\sigma_{\geq n}(F)$ for $\exists_{\geq n}xFx$. Here are the first few cases:

$\sigma_{\geq 1}(F)$: $\exists xFx$

$\sigma_{\geq 2}(F)$: $\exists x(Fx \wedge \exists x'(x' \neq x \wedge Fx'))$

$\sigma_{\geq 3}(F)$: $\exists x(Fx \wedge \exists_{\geq 2}x'(x' \neq x \wedge Fx'))$

$\Rightarrow$ $\exists x(Fx \wedge \exists x'(x' \neq x \wedge Fx' \wedge \exists x''(x'' \neq x' \wedge x'' \neq x \wedge Fx'')))$

$\sigma_{\geq 4}(F)$: $\exists x(Fx \wedge \exists_{\geq 3}x'(x' \neq x \wedge Fx'))$

$\Rightarrow$ $\exists x(Fx \wedge \exists x'(x' \neq x \wedge Fx' \wedge \exists x''(x'' \neq x' \wedge x'' \neq x \wedge Fx'' \wedge \exists x'''(x''' \neq x'' \wedge x''' \neq x' \wedge x''' \neq x \wedge Fx'''))))$

$\sigma_{\geq 5}(F)$: $\exists x(Fx \wedge \exists_{\geq 4}x'(x' \neq x \wedge Fx'))$

$\Rightarrow$ $\exists x(Fx \wedge \exists x'(x' \neq x \wedge Fx' \wedge \exists x''(x'' \neq x' \wedge x'' \neq x \wedge Fx'' \wedge \exists x'''(x''' \neq x'' \wedge x''' \neq x' \wedge x''' \neq x \wedge Fx''' \wedge \exists x''''(x'''' \neq x''' \wedge x'''' \neq x'' \wedge x'''' \neq x' \wedge x'''' \neq x \wedge Fx'''')))))$

Write $\#\varphi$ for the *number of symbols* in φ. Count F as a single symbol. Let $k(n)$ be $\#\sigma_{\geq n}(F)$. Starting with $n = 1$, the first few terms are: 4, 20, 44, 81, 134, 206, 300 and 419. For $n > 1$, we have the recursion relation $k(n+1) = k(n) + 7 + 5.5n + 1.5n^2$. This recursion is solved, for $n > 1$, by $k(n) = -1 + 4.5n + 2n^2 + 0.5n^3$. The limiting growth goes like $^1/_2 n^3$. This growth is explained by the number of occurrences of the prime symbol $'$ on the variables appearing in $\sigma_{\geq n}(F)$. Let $l(n)$ be the number of occurrences of $'$ in $\sigma_{\geq n}(F)$. It's easy to see that $l(n)$ is given by the series 0, 3, 12, 30, 60, … So, $l(n) = {}^1/_2 n(n+1)(n+2)$, which tends to $^1/_2 n^3$ as n gets large. In any case, the point is that, for any positive integer n, $k(n) >> n$.

Consider the mundane cardinality statement,

(4) The number of books in Cambridge University Library is at least 10^6.

Take the predicate 'book in Cambridge University Library' to be represented by the predicate symbol F. We may express (4) in an extension of the first-order language of arithmetic, with a predicate binding operator #, the relation symbol $\leq$, the successor symbol S and a binary function symbol E (for the exponential function) as follows:[8]

φ)). Here, the limiting case of 'infinite speed-up' is *indefinability*. By comparison, the limiting case of (Gödelian) proof-theoretic speed-up is *non-conservation*.

[7] Using the inductive definition, the cardinality quantifiers can be proven 'correct', in the sense that given an interpretation X of the predicate F, $\exists_{\geq n}xFx$ is true just in case the cardinality of X is at least n.

[8] The system known as Frege arithmetic FA contains such a primitive cardinality operator # governed by Hume's Principle. FA has the property that, for any number n, FA $\vdash \exists_n xFx \leftrightarrow (\#F = \underline{n})$.

(5) $E(SSSSSSSSSS0, SSSSSS0) \leq \#F$.

This has 25 symbols. However, on the present analysis, the number of symbols in the nominalization of (5) is roughly 5 x 10^{17}. If we guess at approximately 10 million symbols per book, this nominalistic statement could only be expressed by a huge sentence token filling a vast hyper-library equivalent to some fifty thousand Cambridge University Libraries. Such a library would cover a huge square plot with sides about 10km long. Alternatively, it would require a huge computer hard drive with around 10^{17} bytes storage space, around a hundred million Gigabytes (the equivalent to, say, a million personal computers).

Iterating, consider the claim,

(6) The number of symbols in the nominalization of (5) is at least 10^{17}.

Let *G* represent 'symbol in the nominalization of (5)'. Then, (6) becomes,

(7) $E(SSSSSSSSSS0, SSSSSSSSSSSSSSSSS0) \leq \#G$.

This has 36 symbols. But the number of symbols in the nominalization of (7) is roughly 5×10^{50}. The corresponding sentence token wouldn't even fit on the Earth, or even the solar system. It would comprise an astronomical hyper-library cube with edges roughly 10^{11} km long (the distance light travels in about 10 hours).

3. *'Nominalistic' Proofs Concerning Cardinality*

A further illustration of the problem concerns not just the length of statements involved, but the *length of proofs* involving cardinality assertions. For example, consider the following argument,

(8a) There are at least one billion people on Earth.

(8b) There are one hundred and eighty countries.

(8c) Every person on Earth is in exactly one country.

(8d) *Hence*, there are at least two people in the same country.

This argument can be understood mathematically as an application of the Pigeonhole Principle, which says that if you have *m* pigeons and *n* holes, with $m>n$, then there must be at least two pigeons in the same hole. For example,

*	*	*	*	4 pigeons
[]	[]	[]		3 holes

Clearly, if every pigeon goes into exactly one hole, then at least two pigeons must go into the same hole.

As a statement about sets, their cardinality and functions, the Pigeonhole Principle says that, if $\#X < \#Y$, then any function $f : Y \to X$ is non-injective.

The above argument (8a)-(8d) is, in fact, valid in first-order logic, although the proof as astronomically long. However, there is a quick mathematical proof. Take the Pigeonhole Principle, which itself admits a reasonably short proof (say, in informal set theory). Assume that there is a *set* P of people on Earth, a *set* C of countries and a *function* $f : P \to C$ represented by "is in". Then, by the premises, plus the quick proof that $180 < 10^9$, we conclude that $\#C < \#P$, and thus that the function f is non-injective. So, there exist $a, b \in P$, with $a \neq b$ and $f(a) = f(b)$. Hence, there are at least two people in the same country. End of proof.

This proof is a *few lines long*. However, the number of symbols (and lines of proof) in the 'nominalistic' derivation of the conclusion will be astronomically huge. To see this, consider the (implicitly quantified) second-order formula,

$$[\#X \geq m \wedge \#Y = n \wedge m > n \wedge \forall x(X(x) \rightarrow Y(f(x))] \rightarrow \exists x \exists y(x \neq y \wedge f(x) = f(y))$$

which expresses a version of the Pigeonhole Principle. Its ordinary mathematical proof (in, say, Frege arithmetic or in informal set theory) is not too long. Related to this is the first-order formula,

$$[\exists_{\geq m} xAx \wedge \exists_n xBx \wedge \forall x(Ax \rightarrow Bf(x))] \rightarrow \exists x \exists y(x \neq y \wedge f(x) = f(y)).$$

This may be simplified a little to,

$$[\exists_{\geq m} x(x = x) \wedge \exists_n xBx \wedge \forall xBf(x))] \rightarrow \exists x \exists y(x \neq y \wedge f(x) = f(y)).$$

Call this formula PHP(m, n).

If $m > n$, then PHP(m, n) is a theorem of first-order logic. In austere notation, the statement PHP(m, n) has length of around $\max\{^1/_2 m^3, n^3\}$, by our earlier results. A *proof* of PHP(m, n) in a deductive system for first-order logic would be significantly larger. For example, using the refutation tree method, we have a 'trunk' which contains a huge number of symbols (we need to instantiate the premises $\exists_{\geq m} x(x=x)$, $\exists_n xBx$, $\forall xBf(x)$) and $\neg\exists x \exists y(x \neq y \wedge f(x) = f(y))$ with appropriately many distinct constants), plus a huge number of branches, each of which closes when $m > n$. By a rough calculation, I estimate the number of nodes in the branching part of the proof of PHP(m, n) to be around $2n^m$. Clearly, this is huge if m and n are. For example, the above argument concerns the statement PHP(10^9, 180). Its tree proof has something like $180^{1,000,000,000}$ nodes. In practice, the only reason a nominalist could give for believing that PHP(10^9, 180) is true is its mathematical or 'platonistic' proof. And given the premises (8a)-(8c), the nominalist cannot, in practice, provide a *nominalistically acceptable reason* for believing that the conclusion (8d) is true.

Using the Windows-based SPASS Theorem Prover on a desktop PC, the proof of PHP(m, n) was examined for very small values of the parameters m and n. That is, it was set up to verify the unsatisfiability of the four axioms $\exists_{\geq m} x(x = x)$, $\exists_n xBx$, $\forall xBf(x)$ and $\forall x \forall y(f(x) = f(y) \rightarrow x = y)$. The proof for PHP(3, 2) took 5 seconds. For PHP(4, 3), the proof took 1 hour and 16 minutes! And yet this represents the situation,

*	*	*	*	4 pigeons
[]	[]	[]		3 holes

where it is *intuitively mathematically clear* that no injective map from the pigeons to the holes can exist. Whatever 'intuitive mathematical clarity' involves, it seems to me that it cannot involve constructing a 'combinatorial' proof of PHP(4, 3) in the form discussed above.[9]

[9] A possible reply, along Fieldian lines, is that the nominalist can appeal to some conservation theorem. For example, the result that if PHP(m, n) is provable in FA, then it has a proof in first-order logic. However, it is not clear to me that this conservation theorem itself has a *feasible* 'nominalistic' proof. Consider the statement,

(*) For any positive integers m, n, if $m > n$, then there is a proof Γ of PHP(m, n) in FOL.

This can be expressed in the language of arithmetic, and no doubt may be proved in a fragment of PA. I suspect that it can be proved in the fragment $I\Sigma_1$, and thus in PRA. But while this is of comfort to the *finitist*, I do not see how this could be of any help to the radical fictionalist (or strict finitist), who doesn't even accept the existence of numbers (or sufficiently large numbers).

4. *Statements Asserting a Finite Number of Objects*

Let $Id(x)$ be the self-identity predicate $x = x$ and let S_n be the sentence $\neg\sigma_{\geq n+1}(Id)$, which says 'there are at most n objects in the world'. Then, $\#S_n >> n$. For a nominalist, the content of the assertion,

(9) The number of objects in the world is at most n,

is given by its nominalization S_n. (9) appears to be a meaningful sentence, with just 11 words or so. If the nominalist thinks that (9) is meaningful *because* its genuine content given by S_n, then presumably he must think that a token of S_n *exists*. But S_n has more than n symbols! So, a nominalist who accepts (9), but also believes that the correct representation of this claim is given by its nominalization S_n, also thinks that there are more than n objects in the world (i.e., at least $^1/_2n^3$ objects in the world). In short, he seems to be saying the number of objects is at most n but also more than n. This works for any finite number. So, a consistent nominalist who holds that *tokens* of nominalizations of ordinary cardinality assertions exist, cannot accept the ordinary mathematical assertion 'the number of objects in the world is at most n', a most peculiar conclusion.

5. *Summary*

To summarize. First, on our method of formalizing cardinality statements and counting symbols, the nominalization of 'the number of *F*s at least n' has roughly $^1/_2n^3$ symbols, and consequently, rather mundane cardinality claims are not, at least in practice, nominalistically expressible (at least not if we insist on the existence of relevant tokens). Second, there's a problem with cardinality explanations, in that the 'nominalistic' computations are unfeasible. We illustrated with what might be called 'Pigeonhole explanations', which are in fact formalizable in first-order logic as proofs of formulas of the form PHP(m, n). But though theorems of first-order logic, their proofs are astronomically large, even for small values of m and n. In any case, it seems that the 'moral' reason why a conclusion like 'There are at least two people in the same country' are true, given the factual premises, is because the Pigeonhole Principle is true. Finally, I have suggested that there may be a problem for the nominalist in asserting 'there are at most n objects in the world', since its austere nominalization has more than n symbols.[10]

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