Polytypic Functional Programming and Data Abstraction

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ABSTRACT

Structural polymorphism is a generic programming technique known within the functional programming community under the names of *polytypic* or *datatype-generic* programming. In this thesis we show that such a technique conflicts with the principle of data abstraction and propose a solution for reconciliation. More concretely, we show that popular polytypic extensions of the functional programming language Haskell, namely, *Generic Haskell* and *Scrap your Boilerplate* have their genericity limited by data abstraction. We propose an extension to the Generic Haskell language where the 'structure' in 'structural polymorphism' is defined around the concept of interface and not the representation of a type.

More precisely, polytypic functions capture families of polymorphic functions in one single template definition. Instances of a polytypic function for specific algebraic types can be generated automatically by a compiler following the definitional structure of the types. However, the definitional structure of an *abstract* type is, for maintainability reasons, logically hidden and, sometimes, even physically unavailable (*e.g.*, precompiled libraries). Even if the representation is known, the semantic gap between an abstract type and its representation type makes automatic generation difficult, if not impossible. Furthermore, if it were possible it would nevertheless be impractical: the code generated from the definitional structure of the internal representation is rendered obsolete when the representation changes. The purpose of an abstract type is to minimise the impact of representation changes on client code.

Data abstraction is upheld by client code, whether polytypic or not, when it works with abstract types through their *public interfaces*. Fortunately, interfaces can provide enough description of 'structure' to guide the automatic construction of two polytypic functions that extract and insert data from abstract types to concrete types and vice versa. Polytypic functions can be defined in this setting in terms of polytypic insertion, polytypic extraction, and ordinary polytypic functions on concrete types. We propose the extension of the Generic Haskell language with mechanisms that enable programmers to supply the necessary information. The scheme relies on another proposed extension to support polytypic programming with type-class constrained types, which we show are not supported by Generic Haskell.

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Chapter 1
Introduction

> In order to get to where I want to be from here, I would not start from here. [Moo02]

1.1 General theme and contribution

Structural polymorphism is a Generic Programming technique known within the functional programming community under the names of *polytypic* or *datatype-generic* programming. In this thesis we show that such a technique conflicts with the principle of data abstraction and propose a solution for reconciliation. More concretely, we show that popular polytypic extensions of the functional programming language Haskell, namely, *Generic Haskell* [HJ02, Hin02, Hin00, Löh04] and *Scrap your Boilerplate* [LP03, LP04, LP05] have their genericity limited by data abstraction. We propose a solution for Generic Haskell where the 'structure' in 'structural polymorphism' is defined around the concept of interface and not the representation of a type. Section 1.3 describes the research problem in more detail. Section 1.4 lists the thesis' contributions. Section 1.5 provides a detailed list of contents.

1.2 Notes to the reader

Style of presentation. The present thesis has been written in a discursive and 'reader-friendly' style where the tension between rigour and readability has been eased often in favour of the latter. Naturally, doctoral theses are not textbooks and a discursive style could fall into an excess of verbosity. However, theses should be meant to be read by someone other than the author and the examiners. Communicating ideas to a wide audience is also an essential aspect of scholarship and research.

I have had several types of reader in mind during the writing. I hope to have been able to balance their dissenting expectations. The first type is that of graduate students, like myself, who would like to make use of this work but may not be entirely familiar with the background material and cannot indulge in fetching and studying the cited papers,

often less recommended as a first exposure and by their very nature less comprehensive. I have tried to spell out the prerequisites for understanding and to be as self-contained as possible. Inexorably, the organisation and exposition of background material is personal. I hope the reader finds it useful and interesting.

The second type of reader is an stereotyped practitioner for whom C++ is the only language supporting expressive Generic Programming features. Such a reader praises the language for its 'efficiency' and backward-compatibility while neglecting its theoretical and practical flaws. It may strike that a work concerned with functional programming should care about those who mistakenly regard functional programming as "toy recursive programming with lists". Certainly, by comparison functional programming is practiced by a minority, and functional polytypic programming by an even smaller minority. Consequently, I have described Generic Haskell and Scrap your Boilerplate in considerable detail in Chapter 6, so we can thereafter explore whether there is life beyond the C++ Standard Template Library that may be of interest to programmers for whom data abstraction is a *sine qua non*.

The last type of reader is the functional programmer for whom the world of algebraic types is not deemed low-level. To my surprise, during a workshop discussion I found amusing that the Haskell type:

```
data Ord a ⇒ Tree a = Empty | Node a (Tree a) (Tree a)
```

was considered as *defining* an ordered bag. Why not an ordered set, or a priority queue, or what have you? Some functional programmers despise object-oriented languages because of orthogonal unsafe features such as downcasting. But object-orientation is not only about objects passing messages, but also about programming with first-class, re-usable, and extensible abstractions, an aspect which is found wanting in Haskell. Chapters 7 and 8, as well as parts of Chapter 4, have been written with this reader in mind.

Floating boxes will appear scattered throughout the text following a sequential numbering within each chapter. Boxes 1.1 and 1.2 on the next pages are two examples. Boxes expand on particular topics or discuss issues cross-cutting several sections.

Cited work. I have made an effort to cite original authors and papers but, in some cases, instead of standard or 'classic' references I have opted for references that I have

BOX 1.1: About Functional Programming

We assume the reader is familiar with functional programming in general and the Haskell language in particular. Let us recall that Functional Programming [Rea89, BW88, Mac90] is based on two central ideas: (1) computation takes place by evaluating applications of functions to arguments and (2) functions are first-class values. In particular, functions are *higher-order* (can be passed to or be returned by other functions) and can be components of data structures.

Functional languages differ on whether they are strongly type-checked, weakly type-checked, or untyped; whether they are dynamically type-checked or statically type-checked; whether they are pure or impure; and finally whether they are strict or non-strict.

In pure functional languages, an expression produces the same value independently of when it is evaluated—a property called ***referential transparency***. Side effects like input-output are carefully controlled and separated at the type level by so-called *monads* [Mog91, Nog05] or *uniqueness types* [PVV93]. Pure languages usually have non-strict semantics for functions and their evaluation order is typically *lazy* (*i.e.*, *call-by-need*). In contrast, *impure* functional languages allow side effects like imperative languages, they have strict semantics, and evaluation order is *eager* (*i.e.*, *call-by-value*). Purity and non-strictness are not just a matter of style. Programs in impure, strict languages will look and work quite differently than their pure counterparts. The main benefit of purity is referential transparency. The main benefits of non-strictness are higher modularity and lower coupling from evaluation concerns [Hug89].

In the rest of the thesis, an unqualified ***function*** refers to a typed and pure function that is a first-class value.

BOX 1.2: The Haskell Language

Haskell is a strongly type-checked, pure, and non-strict functional language which has become pretty much the *de facto* standard lazy language. The reader will find information about the Haskell language in `www.haskell.org`.

Haskell's syntax is sugar for a core language similar to System F_ω with type classes and nominal type equivalence. It supports rank-n polymorphism with the help of type annotations [OL96, SP04]. (We explain what all this means in Chapters 2 and 4.)

In Haskell, types and values are separated and its designers deliberately overloaded notation at both levels. Examples are expressions like `(a,b)` or `[a]` which can be interpreted as value or type expressions. At the value level the expressions denote, respectively, a pair of values and a singleton list value, where the values are given by variables `a` and `b`. At the type level the expressions denote, respectively, the type of pairs with elements of type `a` and elements of type `b`, and the type of lists with elements of type `a`. The overloading of parentheses for products and bracketing can also lead to confusion. Since we cannot redesign Haskell, we have to stick to its actual syntax and common conventions.

studied in more detail or that may be of better help to unacquainted readers.

1.3 The problem in a nutshell

This section explains the research problem in a nutshell. Part II of the thesis provides all the details.

The state-of-the-art. Generic Programming is often associated with varieties of polymorphism (parametric, subtype, etc). Structural polymorphism or polytypism is one such variety in which programs (functions) can be obtained *automatically* from the definitional structure of the types on which they work.

Equality is an archetypical example of polytypic function: it can be defined automatically for algebraic data types that lack function components (recall that function equality is, in general, not computable [Cut80]). Some Haskell examples:

```
data Nat    = Zero | Succ Nat
data List a = Nil  | Cons a (List a)
```

Type `Nat` is the hoary type of natural numbers with its well-known value constructor `Zero` and the rather rude `Succ`. Type `List` is the hoary type of lists.[1] It is a parametric type, *i.e.*, `List` takes a non-parametric type through type variable `a` and yields the type of lists with data of type `a`.

Polytypic programming is founded on the idea that the structure of a function is determined by the structure of its input type. Look at Figure 1.1. The equality function for natural numbers takes two natural-number arguments and returns a boolean value. Because `List` is parametric, the equality function for lists needs the function that computes equality on list elements as an extra argument. The body of equality for `Nat` and `List` is defined by pattern-matching on the value constructors. Differing value constructors are unequal. Identical value constructors are equal only if their components are all equal. The structure of the functions clearly follows the structure of the types.

Because of this, it is possible for a compiler to generate the types and bodies of equality functions automatically. The fact that equality's function name is overloaded is a somewhat orthogonal, yet important, issue: the type-class mechanism employed in resolving overloading [Blo91, WB89, Jon92, Jon95b, MJP97] has several limitations

[1] Admittedly, looking at its constructors the type `List` is really the type of stacks.

```
eqNat :: Nat → Nat → Bool
eqNat Zero     Zero     = True
eqNat Zero     _        = False
eqNat _        Zero     = False
eqNat (Succ n) (Succ m) = eqNat n m

eqList :: ∀ a. (a → a → Bool) → (List a → List a → Bool)
eqList eqa Nil          Nil        = True
eqList eqa Nil          _          = False
eqList eqa _            Nil        = False
eqList eqa (Cons x xs) (Cons y ys) = (eqa x y) && (eqList eqa xs ys)
```

Figure 1.1: Equality for `Nat`s and `List`s.

which restrict the sort of types for which equality can be automatically derived.

Generic Haskell is a language extension in which programmers can define a polytypic function (a generic template) which is used by the Generic Haskell compiler in the automatic generation of instances of the polytypic function for every type in the program (Section 6.1). Equality is one such example.

Scrap your Boilerplate combines polytypic programming and strategic programming techniques. The Glasgow Haskell Compiler supports the necessary extensions to generate instances of special functions, called one-layer traversals, following the structure of types. Programmers can define generic functions in terms of one-layer traversals (Section 6.2)

Conflict with data abstraction. Functions on *abstract* data types cannot be obtained automatically following the definitional structure of a type. For one thing, the definitional structure (*i.e.*, the internal representation) of an abstract type is, for maintainability reasons, logically hidden and, sometimes, even physically unavailable (*e.g.*, precompiled libraries). Even if the representation is known, the semantic gap between an abstract type and its representation type makes automatic generation difficult, if not impossible. Furthermore, if it were possible it would nevertheless be impractical: the code generated from the definitional structure of the internal representation is rendered obsolete when the representation changes. The purpose of an abstract type is to minimise the impact of representation changes on client code.

Let us illustrate this point with a particular example of abstract type: ordered sets

implemented as ordered lists:

```
data Ord a ⇒ Set a = MkSet (List a)
```

An equality function can be obtained from the definitional structure of the type:

```
eqSet :: ∀ a. Ord a ⇒ (a → a → Bool) → (Set a → Set a → Bool)
eqSet eqa (MkSet xs) (MkSet ys) = eqList eqa xs ys
```

However, this definition would consider `MkSet [1]` and `MkSet [1,1]` unequal sets, which is not the case.

The ordered-set type is more restricted than the ordered-list type, *i.e.*, it is subject to more laws: no repetition. Consequently, its equality function has to reflect that restriction somehow. If we change the representation from lists to binary search trees, say, a new definition of `eqSet` has to be generated.

Ordinary functions on abstract types typically access the latter's information content via an *interface* of operators that enable the observation and construction of values of the type. In this setting, equality for sets would be programmed thus:

```
eqSet :: ∀ a. Ord a ⇒ (a → a → Bool) → (Set a → Set a → Bool)
eqSet eqa s1 s2
  | isEmpty s1 && isEmpty s2       = True
  | isEmpty s1 && not (isEmpty s2) = False
  | not (isEmpty s1) && isEmpty s2 = False
  | otherwise  = let m1 = smallest s1
                     m2 = smallest s2
                     r1 = remove m1 s1
                     r2 = remove m2 s2
                 in (eqa m1 m2) && (eqSet eqa r1 r2)
```

This definition uses interface operators and, therefore, is not affected by changes of representation. The question to address is whether we can generate such definitions following the 'structure' provided by an interface, and how to put it to work. This is the topic of this thesis.

1.4 Contributions

1. We provide a survey of Generic Programming in general (Chapter 4) and of Generic Haskell and Scrap your Boilerplate in particular (Chapter 6), discussing their features, expressibility, limitations, differences, and similarities.

2. We show that polytypism conflicts with data abstraction (Chapter 7). This should not be surprising: a function that is defined in terms of the definitional structure of the type *implementing* an abstract type can wreak havoc, whether it is an ordinary function or the instance of a polytypic function. However, it is important to drive home the point for those lured by the 'generic' adjective, and there are also conflicts specific to the nature of Generic Haskell and Scrap your Boilerplate. We also explain why polytypic extension is an unsatisfactory solution.

3. Abstract types are often implemented in terms of type-class constrained types, *i.e.*, parametric algebraic data types with some or all of their argument types constrained by type classes (Sections 5.6 and 5.8.2). We show that Generic Haskell does not support constrained types (Section 6.1.10) and propose a solution in which polykinded types are made context-parametric (Section 6.1.11). The proposal entails an extension to the Generic Haskell *compiler*, not the *language*. We discuss the wider implications of constraints in abstraction in Chapter 5.

4. We provide a formal introduction to the syntax and semantics of algebraic specifications with partial operators and conditional equations, which for us provide the meaning to the word 'abstract type'. Algebraic specifications have *equational laws* which are important to us because they *specify* a type and, more relevantly, are needed in our approach to polytypic programming with abstract types (Chapter 5 and Appendix A).

5. We define the concept of *unbounded* and *bounded* abstract types and explain the conditions that both classes of types must satisfy to be functors, *i.e.*, to have a map function (Section 5.10).

6. Polytypic functions and their instances are defined by pattern matching, and pattern matching conflicts with data abstraction. There are several proposals for reconciling pattern matching with data abstraction and the first thing that comes to mind is to investigate whether they can be of any use in reconciling polytypic programming

with data abstraction. We survey the most popular and promising proposals and argue that their applicability to polytypic programming is unsatisfactory and limited (Chapter 8).

7. We propose an extension to the Generic Haskell language for supporting polytypic programming with abstract types. The key idea is to provide 'definitional structure' in terms of interfaces (algebraic specifications), not type representations. Working with interfaces leads to a form of *Extensional Programming*. We show that *Generic Extensional Programming* is possible (Chapter 9).

   More precisely, we introduce *functorial views* or *F-views*, which specify the functorial structure of operator interfaces, and *named signature morphisms*, which specify the conformance of particular abstract types or concrete types to particular F-views. Equational laws have to be used by the programmer when declaring signature morphisms.

   Observation and construction in abstract types may not be inverses. Polytypic functions on abstract types cannot be programmed without the help of insertion and extraction functions from/to the abstract type to/from some concrete type. We show that these functions can be defined polytypically, *i.e.*, instances for particular abstract and concrete types can be obtained automatically following the structure of F-views and using the operators provided by signature morphisms. We show that polytypic functions on abstract types can be defined in terms of polytypic insertion, polytypic extraction, and ordinary polytypic functions on concrete types. We show that polytypic extension is supported. Finally, we introduce the notion of *exporting* in order to support polytypic programming with *non-parametric* abstract types.

1.5 Structure and organisation

This thesis is organised in three parts. Part I explains background material used by later chapters. Part II surveys polytypic functional programming, describes the conflict with data abstraction, and proposes a solution for reconciliation. Part III is an appendix with technical details from Chapter 5.

Part I. Background

Chapter 2: Language Games introduces conceptual terminology and notational conventions. It also contains a brief account of types and typed programming, pinpointing their relevance to Generic Programming. Several families of Lambda Calculi are then overviewed which are necessary for a full understanding of Chapters 4 and 6. The overview is not meant to be a tutorial but a brushing up. Bibliographic references are provided in the relevant sections.

Chapter 3: Bits of Category Theory. Category Theory provides a general, abstract, and uniform meta-language in which to express many ideas that have different concrete manifestations. This chapter spells out several category-theoretical concepts that are used in Chapters 6, 5, and 9.

Chapter 4: Generic Programming overviews the manifestations of genericity in programming. Section 4.1 opens the chapter with a discussion on the two variants of abstraction (control and data) and defines Generic Programming as the judicious integration of *parametrisation*, *instantiation* and *encapsulation*. Section 4.2 overviews the concept of data abstraction, which is expanded and formalised in Chapter 5. Section 4.3 discusses the role of Generic Programming in the wider context of Software Engineering. Section 4.4 talks briefly about the role of Generic Programming in Generative Programming and vice versa. Sections 4.5 and 4.6 discuss the importance of typed programming in Generic Programming, a topic resumed from Chapter 2. Section 4.7 provides a coarse classification of genericity and its different manifestations in programming. Finally, Section 4.8 winds up discussing where the present thesis stands in the described setting.

Chapter 5: Data Abstraction. Data abstraction corresponds with the principle of representation independence. But what are abstract types, really? How should we formalise them?

We believe *algebraic specifications* are the best route to the formalisation and understanding of abstract types. Algebraic specifications have several advantages beyond the mere specification of a formal object; in particular, they provide an interface for client code, they can be used in the formal construction and verification of client code, there

is a formal relation between the specification and the implementation, and prototype implementations can be obtained automatically [LEW96, Mar98, GWM+93].

Mainstream languages do not support algebraic specifications or equational laws for operators. However, we assume that algebraic specifications have been used in the design and implementation of abstract types. In particular, the presence of equational laws is important to motivate and describe our approach to Generic Programming.

The chapter starts discussing the advantages and disadvantages of data abstraction in Sections 5.1 and 5.2 respectively, underlining the impact of parametricity constraints on maintainability, a recurring issue whose import to Generic Programming is discussed in Chapter 6.

Sections 5.3, 5.4, and 5.5 introduce algebraic specifications with partial operators and conditional equations. Partial operators are those that may produce run-time errors. They are common in strongly-typed languages that separate values from types (a simple example is the list function **head**). Conditional equations are needed to cope with partiality. For readability, the formal and technical details have been moved to Appendix A.

The formalism presented is *first order*. Our aim is to explore Generic Programming on classic abstract types which can be described perfectly well in a first-order setting. Higher-order functions such as catamorphisms will be written as *generic* programs outside the type using the latter's first-order operators (Chapter 9).

The chapter presents several examples of algebraic specifications that are used by subsequent chapters (*e.g.*, Chapters 7 and 9). Section 5.6 illustrates with an algebraic specification example the problems of *constrained* abstract types, which were discussed in the context of the Haskell language in Section 5.8.

The mechanisms available in Haskell and Standard ML for supporting abstract data types are described in Section 5.8.

The chapter concludes with a classification of abstract types and their operators that is assumed by subsequent chapters (Sections 5.9 and 5.10).

Part II. Functional Polytypic Programming and Data Abstraction

Chapter 6: Structural Polymorphism in Haskell examines the two most popular polytypic language extensions of Haskell: *Generic Haskell* [Hin00, Hin02, HJ02] and *Scrap your Boilerplate* [LP03, LP04, LP05]. The latter combines polytypic and Strategic Programming techniques [VS04, LVV02], which are also examined.

The version of Generic Haskell studied is the so-called *classic* one supported by the Beryl release of the Generic Haskell compiler (version 1.23, Linux build). However, its syntax has been sugared to fit some of the notational conventions of Chapter 2. The differences with *Dependency-style* Generic Haskell [Löh04] are also outlined. We do not go into much detail concerning Dependency-style Generic Haskell because there is an excellent presentation [Löh04] and, more importantly, because it is based on the same idea (structural polymorphism) as classic Generic Haskell and is therefore subject to the same problems we study in Chapter 7.

Section 6.2 describes the Scrap your Boilerplate approach paper by paper after an initial exposure to the ideas of Strategic Programming.

The material for this chapter has been used in several talks. It is self-contained and follows a tutorial style. The following topics are of special interest:

- The impact of nominal versus structural type systems (Sections 6.1.3 and 6.1.4), and the question of expressibility (Section 6.1.5). It is also not clear what the most general polykinded type of a function is (Section 6.1.2). There are polytypic functions that are not expressible in Generic Haskell (Section 6.1.7). Finally, we argue in favour of parameterising polytypic functions on the cases for manifest types (Section 6.1.12).

- Section 6.1.10 shows that Generic Haskell does not support *constrained* data types, which play a major role in the implementation of abstract data types. (The reader may want to read Sections 5.2, 5.6, 5.7, and 5.8.1 in order to put the problem in context.) Section 6.1.11 proposes a solution based on making polykinded types parametric on type-class constraints. The moral of this contribution is that "polytypic functions possess *constrained-parametric* polykinded types".

- Polytypic function definitions are described categorially[2] in Section 6.1.13. The categorial rendition is used later in Chapter 9 to justify that it is not possible to express polytypic functions on abstract types in the same way as on concrete types.

Chapter 7: Polytypism and Data Abstraction shows that data abstraction limits polytypism's genericity because polytypic functions manipulate *concrete* representations. The problems are outlined at the beginning of Section 7.1 and the rest of the chapter elaborates with examples. Section 7.1.3 explains when a map function can be programmed for an abstract type and discusses the obstacles involved in programming it polytypically in terms of concrete representations. Section 7.2 argues that abstracting over data contents is not a satisfactory way of dealing with manifest abstract types, making the case for 'exporting', a mechanism to be introduced in Chapter 9. Section 7.3 summarises: buck the representations.

Chapter 8: Pattern Matching and Data Abstraction. Pattern matching is another language feature that conflicts with data abstraction. There are several proposals for reconciling pattern matching and data abstraction. This chapter overviews them and argues that their application to reconciling polytypic programming and data abstraction is unsatisfactory and limited. Resistance to bucking the representations is futile.

Chapter 9: F-views and Extensional Programming begins with an examination of some possible ways of reconciling polytypic programming with data abstraction, and narrows down the list after analysing the pros and cons. The remaining sections introduce and develop our proposal in detail.

Bucking the representations means programming with abstract types has to be done through their interfaces. This leads to a form of *Extensional Programming* where client functions are concerned with the data contents of a type and ignore its representation. Some notion of structure is needed for polytypism to be possible. The *clients* of an abstract type can provide a definition of structure in terms of F-views. Observation and construction must be separated. Observation is performed by a polytypic extraction function that extracts payload from an abstract type into a concrete type that conforms to the same F-view. Correspondingly, construction is performed by a polytypic

[2]Following [Gol79] we use *categorial* instead of *categorical* in order to distinguish the technical from the ordinary use of the adjective.

insertion function that inserts payload from the concrete type to the abstract type.

In Section 9.8 we demonstrate that many of the ideas can be encoded in Haskell for particular families of abstract types. From Section 9.9, we generalise and show how insertion and extraction can be defined polytypically: their types are polytypic on the structure provided by an F-view and the signature morphisms are used in the generation of their bodies by a compiler. Section 9.11 shows how polytypic functions on abstract types can be defined in terms of polytypic insertion and extraction. Section 9.12 shows how polytypic extension or specialisation can be done in our system. Section 9.13 introduces the idea of exporting in order to support polytypic programming with manifest abstract types. Sections 9.14 and 9.15 discuss some applications of polytypic extension.

Chapter 10: Future Work discusses future lines of research and other design choices and the challenges they present.

Appendix

Details from Chapter 5. The appendix contains the technical details of the formal syntax and semantics of the algebraic specification formalism of Chapter 5. Of particular importance is the concept of Σ-Algebra (Definition A.1.7) which involves the notion of *symbol-mapping*. This is an important but often obviated ingredient that is also present in the definition of *signature morphisms*, Σ-*homomorphisms*, and partiality. Signature morphisms and F-algebras (Section A.3.1) are essential for understanding and justifying our approach to Generic Programming (Chapter 9).

Part I

Prerequisites

Chapter 2
Language Games

> "In order to recognise the symbol in the sign we must consider the significant use." Ludwig Wittgenstein. *Tratactus Logico-Philosophicus*, Proposition 3.326

This chapter presents some concepts, terminology, and notational conventions used throughout the thesis. It is not meant as an introductory exposition but as a brushing up. Bibliographic references are provided in the relevant sections. The treatment of Category Theory is postponed to Chapter 3.

2.1 Object versus meta

We assume familiarity with the distinction between ***object language***, a particular formal language under study, and ***meta-language***, the notation used when talking about the object language.

2.2 Definitions and equality

It is common practice in mathematics to use equality as a definitional device. Since equality is also used as a relation on already defined entities, we distinguish equality from definitional equality and use the symbol $\stackrel{\text{def}}{=}$ for the latter. A definition induces an equality in the sense that if $X \stackrel{\text{def}}{=} Y$ then $X = Y$; the converse need not be true.

In some chapters we make heavy use of inductive definitions expressed as *natural deduction rules* and *rule schemas* which have the following shape:

$$\frac{\text{antecedent}_1 \quad \ldots \quad \text{antecedent}_n}{\text{consequent}_1 \quad \ldots \quad \text{consequent}_n}$$

where $n \geq 0$. Rules can be read forwards (when all antecedents are the case *then* all the consequents are the case) or backwards (all the consequents are the case *if* all the antecedents are the case). Inductive rules will be used for expressing conditional

definitions and inference rules of formal deduction systems. Variables in antecedents and consequents are assumed universally quantified unless indicated otherwise.

2.3 Grammatical conventions

We use EBNF notation to express the syntax of several formal languages. Non-terminals are written in capitalised slanted. Any other symbol stands for a terminal with the exception of the postfix meta-operators `?`, $^+$, and *. Their meaning is as follows: `X?`, `X`*, and `X`$^+$ denote, respectively, zero or one `X`, zero or more `X`, and one or more `X`, where `X` can be a terminal or non-terminal. Parentheses are also used as meta-notation for grouping, *e.g.*, `(X Y)`*.

The following EBNF example is a snippet of C++'s grammar [Str92]:

CondOp `::= if` *Cond Block* `(else` *Block*`)?`
Cond `::= (` *Expression* `)`
Block `::=` *Stmt* `|  {` *Stmt*$^+$ `}`

In the first production parentheses are meta-notation. In the second they are object-level symbols because they are not followed by a postfix meta-operator.

2.4 Quantification

We follow the widely used and well-known 'quantifier-dot' convention when denoting quantification in logical formulae. For example, in $\forall x.P$ the scope of bound variable x starts from the dot to the end of the expression P. Also,

$\forall x{\in}S.\ P$ abbreviates $\forall x.\ x{\in}S \Rightarrow P$

2.5 The importance of types

We deliberately use ***term*** to refer to English terms (for instance, 'overloading' is a term) and to program terms. Following the convention of most strongly type-checked programming languages, we distinguish between value-level terms and type-level terms, called ***type-terms*** or just ***types***. Originally types were introduced as a mechanism for optimising storage in early programming languages like FORTRAN [CW85, p7ff]. Types

classify syntactically well-formed terms.[1] Every well-formed term is associated with one or perhaps more type-terms. Thus, types introduce a new level of description that is reflected grammatically and semantically. This has important repercussions on language design:

- The classification of terms by type-terms provides a syntactic *method* for proving the *absence* of particular classes of execution (run-time) errors, generically called ***type errors***. A ***type system*** is a precise *specification* of such a method and a ***type checker*** a feasible and hopefully efficient *implementation*.

  Type errors typically include incompatibility errors—which arise when operators are applied to terms of the wrong type—and those related to enforcing abstraction, *e.g.*, scoping, visibility, etc. A precise definition of what constitutes a type error is determined, amongst other factors, by the expressibility of the type language (Box 2.1).

  Type systems usually come in the guise of logical proof systems, and type checkers in the guise of specialised proof-checking algorithms. Type systems must be able to *decidably* prove or disprove propositions, here called ***judgements***, which assert that a well-formed term t has a particular type-term σ in a type-assignment Γ, the whole judgement typically written as $\Gamma \vdash t : \sigma$. A type-assignment contains the type-terms of the term's free variables that are in scope. Terms with no free variables are called ***closed terms***.

  Type-terms and terms are defined by means of context-free grammars, but their association is established in a context-sensitive fashion by ***inference rules***, usually written in natural deduction style, which establish compositional implications between judgements. Compositionality means that the type of a term can be determined from the types of its constituent subterms and associated type-assignments. Type checking is the process of proving a judgement by deduction, *i.e.*, of providing a derivation of the judgement from some axioms by the application of the type inference rules.

  We define some type languages and systems in Section 2.7. The following references are excellent introductions to types and type systems in relation to program-

[1] This sentence is deliberately ambiguous; both interpretations are true: types classify terms syntactically and these terms are syntactically well-formed.

ming: [Car97, CW85, Mit96, Pie02, Pie05, Sch94].

BOX 2.1: Type Soundness Can Be Deceptive

A type system comes with a definition of what constitutes a type error. A program is ***well-typed*** if it passes the type checker. Type ***soundness*** means well-typed programs are well-behaved, where ***well-behaved*** programs are those that don't crash because of a type error. A formal semantics is needed to prove type soundness [Car97, Pie02].

Type soundness can be deceptive: a well-behaved program may still crash if the source of the error is not included in the type system's definition of type error. One must be careful when spouting the old chestnut 'well-typed programs cannot go wrong'. In many popular type systems it is disproved at the first counterexample—like computing the head of an empty list in languages of the ML family or downcasting to the wrong class in C++ or Java.

- Type checking may influence term-language design; for instance, its feasibility may restrict the permitted recursion schemes (*e.g.*, structural, generative, polymorphic, general, etc). In practice, the language of types is designed with a particular type-checking algorithm in mind [CW85, p11]. However, *type reconstruction*,[2] also known as *type inference*, need not restrict the language of types, for disambiguating annotations can always be given by the programmer, *e.g.* [OL96]

- Value-level terms are evaluated at run time; type-level terms are usually evaluated at compile time (Box 2.2). A powerful and sophisticated language of types can become pretty much a static (compile-time) mini-programming language, with more effort and computation performed by the type checker. By 'effort' we not only mean that type checking or type reconstruction may require substantial computation, but that some form of compile-time 'execution' of type-level terms is also taking place. How involved this execution is depends on the complexity of the type language. (Section 2.7.4 shows a trivial example.)

  Type-level computation has an impact not only on software development (*i.e.*, being able to widen the definition of type error and catch more correctness errors stat-

[2] The process of finding automatically the most general type of a term that has no type annotations.

BOX 2.2: Strong/Weak, Static/Dynamic

Static type-checking is typically distinguished from ***dynamic type-checking*** in that programs are type checked *without* evaluating them, whereas in the latter they are type checked *while* evaluating them. With modern languages this account of static type-checking is somewhat imprecise; we should rather say that programs are type checked without evaluating the whole of them.

The difference between ***strong*** and ***weak type-checking*** hinges upon the definition of type error or, in other words, on whether the language of types is sophisticated enough to guarantee that well-typed terms don't crash.

Strong/Weak is orthogonal to Static/Dynamic. For instance, Lisp[†] is strongly and dynamically type-checked. C is statically and weakly type-checked.

From the standpoint of program development, the advantages of strong *and* static type-checking should be clear after reading the previous definitions in a different light: *dynamic type-checking puts run-time errors and type errors at the same level.* Weak type-checking is about being happy with narrower notions of type error and passing the hot potato to the programmer. Programmers of the C era revel on their bestowed responsibility, but "the price of freedom is hours spent hunting errors that might have been caught automatically" [Pau96, p7].

In this thesis we take strong, static type-checking for granted.

[†] Lots of Insidious and Silly Parenthesis.

ically), but also on aspects related to Generic Programming such as the ability to define typed language extensions within the language itself, automatic program generation, and meta-programming, *e.g.*, [CE00, LP03, KLS04, Läu95, MS00]. Types are also essential for Generic Programming for other reasons that not only have to do with *typed* programming: they are a necessary precondition for genericity in a typed world (Chapter 4).

- Type-level languages vary in complexity according to their term language. For instance, Haskell has some sort of 'prologish' language at the type level due to its *type class* mechanism [HHPW92, Blo91, Jon92]. C++ Templates are Turing-complete: it is possible to write programs that 'run' at compile time [VJ03, CE00]. In dependently-typed languages like Epigram [MM04], there is no separation between type-terms and terms and the type checker also deals with (normalising) values of computations. In Epigram, the semantic properties of programs are encoded in the language directly as types, which express relationships between values and other types—for example, one can define the type `List a n`, that is, the type of lists of payload `a` and length `n`.

  Of course, not all semantic properties are decidable statically, for they may depend on dynamic information. After all, we have to run programs in order to compute; compiling them is not enough. However, many 'interesting' properties can be approximated by types. To the author's knowledge, a sort of Rice's Theorem on type-language expressibility has not been enunciated; the range of semantic properties that are decidable and feasible via type approximations is still a matter of research in type languages, the theoretical limit being the halting problem. However, it has yet to be elucidated whether programming in that fashion is more convenient. What is certain is that Generic Programming techniques will be essential [AMM05].

- Since types provide a conservative, static, and decidable *approximation* of program semantics, they also play an important role in program specification and construction. Of course, program values are not fully understood just by looking at their types, but the more sophisticated the type language, the more properties captured by them. For instance, many properties of functions can be obtained from just their types, *e.g.* [Wad89], and function construction can be interactively guided by type information in languages with rich type systems, *e.g.* [MM04].

- Finally, types are useful in documentation, security, efficient compilation, and optimisation, *e.g.*: [HM95, Wei02] [DDMM03, GP03]

2.6 Denoting functions

In mathematics, the application of unary function f to x is written $f(x)$ and f's definition is expressed as $f(x) \stackrel{\text{def}}{=} E$. Here E abbreviates an expression where x may occur free. The notation generalises naturally to n-ary functions. We follow this convention at the meta-level. For us, variables may be strings, not just characters, and therefore notations such as fx or FX are deemed confusing. In functional languages supporting currying, function application is denoted by whitespace, with parentheses breaking the convention. For example, `f(x)`, `f x`, and `f (x)` are all valid applications. Inexorably, we follow this convention at the object-level.

2.7 Lambda Calculi

The ***Lambda Calculus*** [Chu41, Mit96, Bar84] introduces a uniform and convenient notation for manipulating *unnamed* first-class functions. Initially a formal (*i.e.*, symbolic) language of untyped functions that was part of a proof system of functional equality, it has developed into a family of systems that model different aspects of computation. Typed extensions with polymorphism, recursion, built-in primitives, plus naming and definitional facilities at value and type level make up the *core languages* of functional languages [Lan66, Pie02, Mit96, Rea89]; in fact, many functional language constructs are syntactic sugar or *derived forms* [Pie02, p51]. Improvements to the core language's operational aspects form the basis of functional language implementations.

We assume the reader is familiar with the Lambda Calculus. In the following pages we gloss over the syntactic, context-dependent, and operational aspects of the family of calculi that make up most of Haskell's core language. These are necessary for a full understanding of Chapter 4 and Chapter 6. For axiomatic and denotational semantic aspects the reader is referred to [Mit96, Sch94, Sto77, Ten76]. We only mention in passing that, commonly, types and functional programs are taken to be objects and arrows in the category of Complete Partial Orders [Mit96], but such interpretation is slightly inaccurate [DJ04]. Category Theory (Chapter 3) provides, among other things, a uniform meta-language for talking and moving about semantic universes.

2.7.1 Pure Simply Typed Lambda Calculus

$$
\begin{array}{lll}
\mathit{Type} & ::= & * \qquad\qquad\qquad\qquad\qquad\qquad -- \text{ base type} \\
& | & \mathit{Type} \rightarrow \mathit{Type} \qquad\qquad -- \text{ function type} \\
& | & (\mathit{Type}) \qquad\qquad\qquad\qquad -- \text{ grouping} \\
\\
\mathit{Term} & ::= & \mathit{TermVar} \qquad\qquad\qquad\qquad -- \text{ term variable} \\
& | & \mathit{Term}\ \mathit{Term} \qquad\qquad\qquad -- \text{ term application} \\
& | & \lambda\ \mathit{TermVar}\ :\ \mathit{Type}\ .\ \mathit{Term} \qquad -- \text{ term abstraction} \\
& | & (\mathit{Term}) \qquad\qquad\qquad\qquad -- \text{ grouping}
\end{array}
$$

$$
\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash t_1 : \sigma \rightarrow \tau \qquad \Gamma \vdash t_2 : \sigma}{\Gamma \vdash (t_1\ t_2) : \tau} \qquad \frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash (\lambda x{:}\sigma\,.\,t) : \sigma \rightarrow \tau}
$$

$$
\frac{}{(\lambda x{:}\tau\,.\,t)\ t' \rhd t[t'/x]}\ \beta \qquad \frac{t_1 \rhd t_1'}{t_1\ t_2 \rhd t_1'\ t_2}\ \text{LI1} \qquad \frac{t_2 \rhd t_2'}{x\ t_2 \rhd x\ t_2'}\ \text{LI2}
$$

$$
\frac{}{t \rhd t}\ \text{REF} \qquad \frac{t_1 \rhd t_2 \qquad t_2 \rhd t_3}{t_1 \rhd t_3}\ \text{TRS}
$$

Figure 2.1: The Pure Simply Typed Lambda Calculus.

Figure 2.1 shows the syntax, type rules, and operational semantics of the Pure Simply Typed Lambda Calculus (PSTLC). There are terms carrying type annotations and for that reason it is dubbed 'explicitly typed'—or *á la* Church, who first proposed it [HS86]. The following paragraphs elaborate.

Terms and types. The PSTLC has a language of types (non terminal *Type*) for expressing the types of functions inductively from a unique base or ground type $*$, and a language of terms (non-terminal *Term*) which consists of variables, ***lambda abstractions*** (unnamed functions), and ***applications*** of terms to other terms.[3] Variables stand for formal parameters or yet-to-be-defined primitive values when not bound by any λ. In a lambda abstraction $\lambda x{:}\tau.t$, the λ symbol indicates that x is a bound

[3]That application is denoted by whitespace is not quite deducible from the grammar alone. In order for the two terms to be distinguished there must be some separator token between them which is assumed to be whitespace.

variable (*i.e.*, a formal parameter), τ is the type of x, and t abbreviates an expression where x may occur free.

In the rest of the chapter, we stick to the meta-variable conventions shown in Table 2.2. Table 2.1 lists the symbols whose notation at the meta-level and object level (*i.e.*, Haskell) differ. An exception is the 'has kind' symbol (Sections 2.7.3 and 2.7.4) which is not standard Haskell 98. (The Glasgow Haskell Compiler supports 'has kind', written '`::`', but we use '`:`' instead to differentiate kind from type signatures.)

σ, τ, ...	range over types.
x, y, ...	range over term variables.
t, t', ...	range over terms.
Γ	ranges over type-assignments.
α, β, ...	range over type variables (Section 2.7.3).
κ, ν	range over kinds (Section 2.7.3).

Figure 2.2: Meta-variable conventions for Lambda Calculi.

Notion	Meta-level symbol	Haskell symbol
Definition	$\stackrel{\text{def}}{=}$	`=`
Equality	$=$	`==`
'Has type'	:	`::`
'Has kind'	:	`:`
Type variable	α, β, ...	`a`, `b`, ...
Lambda abstraction	$\lambda x{:}\tau\,.\,x$	$\lambda x \rightarrow x$

Table 2.1: Differences in notation at meta- and object level. Type annotations are provided separately from terms in Haskell.

Types and terms are separated with the only exception that types can appear as annotations in lambda abstractions. The type of a function is also called its ***type signature***. It describes the function's arity, order, and the type of its arguments. The ***arity*** is the number of arguments it takes. The ***order*** is determined from its type signature as follows:

$$\begin{aligned} \text{order}(*) &\stackrel{\text{def}}{=} 0 \\ \text{order}(\sigma \rightarrow \tau) &\stackrel{\text{def}}{=} \max\,(\,1 + \text{order}(\sigma),\ \text{order}(\tau)\,) \end{aligned}$$

Let τ be the type of a lambda abstraction and suppose $\text{order}(\tau) = n$. If $n = 1$ then the lambda abstraction may either return a manifest (non-function) value of type $*$ or another lambda abstraction of order 1 as result. If $n > 1$, then it is a higher-order

abstraction that either takes or returns a lambda abstraction of order n.

Occasionally we blur the conceptual distinction between manifest values and function values by considering the former as ***nullary functions*** and the latter as ***proper functions***.

The ***fixity*** of a function is an independent concept. It determines the syntactical denotation of the application of the function to its arguments. In some functional languages, functions can be infix, prefix, postfix, mixfix, and have their precedence and associativity defined by programmers. In the PSTLC, lambda abstractions are prefix, application associates to the left—for example, $t_1\ t_2\ t_3$ is parsed as $(t_1\ t_2)\ t_3$—and consequently arrows in type signatures associate to the right—for example, $* \to * \to *$ is parsed as $* \to (* \to *)$.

Multiple-argument functions are represented as ***curried*** higher-order functions that take one argument but return another function as result. For example, the term:

$$\lambda x{:}*\,.\,\lambda y{:}* \to *\,.\,y\ x$$

is a higher-order function that takes a manifest value x and returns a function that takes a function y as argument and applies it to x.

Related terminology. An ***operator*** is a term whose value is a function (a precise definition of 'value' is given on page 26). It also has a more specific use in relation to abstract data types and algebra (Chapter 5). Sometimes ***operation*** is used interchangeably with operator. The term ***method*** has wider connotations than operation and is used in its object-oriented sense [Bud02]. A ***call site*** is another name for an ***application*** of an operator to an operand.

An ***operand*** is a term that plays the role of a ***parameter*** or ***argument***. A ***formal*** parameter or argument appears in a definition whereas an ***actual*** parameter or argument appears in an application. The following are synonyms: X is a parameter of Y (or Y is parameterised by X), Y is *indexed* by X (or Y is X-indexed), Y is *dependent* on X (or Y is X-dependent).

We use the word 'type' not only in reference to type-terms but also in reference to ***data types***, *i.e.*, a concrete realisation of the type in an implementation design or actual code. We use ***data structure*** for data of more elaborate structural complexity, usually

involving not only type operators but perhaps other linguistic constructs (*e.g.*, modules). In a purely functional setting, data-type values are immutable and ***persistent***: operations on values of the type produce new values. Occasionally, however, it is convenient to treat all these values as a "unique identity invariant under changes" [Oka98a, p3]. This figure of speech is the ***persistent identity***.

Type rules. The type rules listed in Figure 2.1 can be employed to check the type of a term compositionally from the type of its subterms. The type of a term depends on the type of its free variables. This context-dependent information is captured by a type-assignment function $\Gamma : \mathit{TypeVar} \to \mathit{Type}$ which acts as a symbol table of sorts that stores the types of free variables in scope. The operation $\Gamma, x : \tau$ denotes the construction of a new type-assignment and has the following definition:

$$(\Gamma, x : \tau)(y) \stackrel{\text{def}}{=} \text{if } x = y \text{ then } \tau \text{ else } \Gamma(y)$$

The type rules are rather intuitive. Notice only that Γ is enlarged in the last rule because x may occur free in t.

Operational semantics. The call-by-name operational semantics is shown in the last box of Figure 2.1. A reduction relation $\triangleright$ is defined between terms. Briefly, Rule β captures the reduction of an application of a lambda abstraction to an argument. The free occurrences of the parameter variable are substituted (avoiding variable capture) by the argument in the lambda abstraction's body. This is what the operation $t[t'/x]$ means, which reads "t where t' is substituted for free x" [Bar84]. Rule LI1 specifies that an application $t_1\ t_2$ can be reduced to the term $t_1'\ t_2$ when t_1 can be reduced to t_1'. Rule LI2 specifies that reduction must proceed to the argument of an application when the term being applied is a free variable. Together, these rules specify a leftmost-outermost reduction order. Rules REF and TRS specify that $\triangleright$ is a reflexive and transitive relation.

A ***value*** is a program term of central importance. Operationally, the set of values V is a subset of the set of normal forms N, which is in turn a subset of the set of terms T, that is, $V \subseteq N \subseteq T$. These sets are to be fixed by definition. A term is in normal form if no reduction rule, other than reflexivity, is applicable to it. In the PSTLC, all normal forms are values and they are defined by the following grammar:

$$\mathit{NF} \ \texttt{::=}\ \mathit{TermVar} \mid \lambda\ \mathit{TermVar}\ \texttt{:}\ \mathit{Type}\ \texttt{.}\ \mathit{Term}$$

That is: variables and lambda abstractions are normal forms, which means that function bodies are evaluated only after the function is applied to an argument. This is reflected in the operational semantics by the deliberate omission of the following rule:

$$\frac{t \ \rhd \ t'}{\lambda x{:}\tau \,.\, t \ \rhd \ \lambda x{:}\tau \,.\, t'}$$

It can be the case in other languages that there are normal forms that are not values. Examples are ***stuck terms*** which denote run-time errors.

Example. The following derivation proves a reduction:

$$\dfrac{\dfrac{\dfrac{}{(\lambda x{:}* \to * \,.\, x)\ (\lambda x{:}* \,.\, x) \ \rhd \ (\lambda x{:}* \,.\, x)}\,\beta}{(\lambda x{:}* \to * \,.\, x)\ (\lambda x{:}* \,.\, x)\ y \ \rhd \ (\lambda x{:}* \,.\, x)\ y}\,\textsc{l1} \quad \dfrac{}{(\lambda x{:}* \,.\, x)\ y \ \rhd \ y}\,\beta}{(\lambda x{:}* \to * \,.\, x)\ (\lambda x{:}* \,.\, x)\ y \ \rhd \ y}\,\textsc{trs}$$

The following is an example reduction of a well-typed PSTLC term to its normal form. The subterm being reduced at each reduction step is shown underlined.

$$\begin{aligned}
&\underline{(\lambda y{:}* \to * \,.\, y \ z)\ ((\lambda y{:}* \to * \,.\, y)\ (\lambda x{:}* \,.\, x))} \\
&\quad \rhd \underline{((\lambda y{:}* \to * \,.\, y)\ (\lambda x{:}* \,.\, x))}\ z \\
&\quad \rhd \underline{(\lambda x{:}* \,.\, x)\ z} \\
&\quad \rhd z
\end{aligned}$$

2.7.2 Adding primitive types and values.

The PSTLC is impractical as a programming language. Given a term t, its free variables have no meanings. The PSTLC *extended* with various primitives has been given specific names. In particular, the language PCF (Programming Computable Functions) is a PSTLC extended with natural numbers, booleans, cartesian products, and fixed points [Sto77, Mit96].

In Figure 2.3 we *extend* the grammar of terms and types of Figure 2.1 to include some primitive types. The base type $*$ is now removed from the language of types. Of particular interest are cartesian product and disjoint sum types that endow the Extended STLC (referred to as STLC from now on) with algebraic types roughly similar

to those supported by functional languages.

We only show a tiny sample of type and reduction rules for primitives, the latter called ***δ-rules*** in the jargon, to illustrate how the extension goes. Consult [Car97, CW85, Pie02, Mit96] for more detail. Primitive types are all manifest and therefore their order is 0.

Example. The following is an example reduction of a well-typed STLC term:

$$\begin{aligned}
&\underline{(\lambda x{:}\texttt{Nat}\,.\,\texttt{if}\ x > 0\ \texttt{then}\ 1\ \texttt{else}\ x+1)\ ((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)}\\
&\quad\rhd\ \texttt{if}\ \underline{((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)} > 0\ \texttt{then}\ 1\ \texttt{else}\ ((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)+1\\
&\quad\rhd\ \texttt{if}\ \underline{(4+4)} > 0\ \texttt{then}\ 1\ \texttt{else}\ ((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)+1\\
&\quad\rhd\ \texttt{if}\ \underline{8 > 0}\ \texttt{then}\ 1\ \texttt{else}\ ((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)+1\\
&\quad\rhd\ \underline{\texttt{if true then}\ 1\ \texttt{else}\ ((\lambda y{:}\texttt{Nat}\,.\,y+y)\ 4)+1}\\
&\quad\rhd\ 1
\end{aligned}$$

2.7.3 Adding parametric polymorphism: System F

The STLC is not polymorphic. For example, the identity function for booleans and naturals is expressed by two syntactically different lambda abstractions:

$$\begin{aligned}
(\lambda x{:}\texttt{Nat}\,.\,x) &\ :\ \texttt{Nat} \to \texttt{Nat}\\
(\lambda x{:}\texttt{Bool}\,.\,x) &\ :\ \texttt{Bool} \to \texttt{Bool}
\end{aligned}$$

However, they only differ in type annotations. ***System F*** [Gir72, Rey74] extends the STLC with *universal parametric polymorphism* (see also Chapter 4). It adds new forms of abstraction and application where types appear as terms, not just annotations. The new syntax can be motivated using the above identity functions. A parametrically polymorphic identity is obtained by abstracting over types (***universal abstraction***), *e.g.*:

$$\Lambda\alpha{:}*\,.\,\lambda x{:}\alpha\,.\,x$$

This term has type:

$$\forall\alpha{:}*\,.\,\alpha \to \alpha$$

(We explain the role of $*$ in a moment.) A capitalised lambda 'Λ' is introduced to distinguish universal abstraction over types from term abstraction. Dually, there is

```
Type ::= Nat                    -- naturals
       | Bool                   -- booleans
       | Type × Type            -- products
       | Type + Type            -- disjoint sums
       | 1                      -- unit type

Term ::= Num                          -- natural literals
       | true                         -- boolean literals
       | false
       | + | − | ...                  -- arithmetic functions
       | not Term                     -- boolean functions
       | if Term then Term else Term
       | ...
       | (Term , Term)                -- pairs
       | fst Term
       | snd Term
       | Inl Term                     -- sums
       | Inr Term
       | case Term of Inl TermVar then Term ; Inr TermVar then Term
       | unit                         -- unit value
```

$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{Bool}} \qquad \frac{}{\Gamma \vdash \texttt{unit} : \texttt{1}}$$

$$\frac{\Gamma \vdash t : \texttt{Bool} \quad \Gamma \vdash t_1 : \tau \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\texttt{if}\ t\ \texttt{then}\ t_1\ \texttt{else}\ t_2) : \tau}$$

$$\frac{}{(\texttt{if true then}\ t_1\ \texttt{else}\ t_2) \rhd t_1} \qquad \frac{}{(\texttt{if false then}\ t_1\ \texttt{else}\ t_2) \rhd t_2}$$

$$\frac{t \rhd t'}{(\texttt{if}\ t\ \texttt{then}\ t_1\ \texttt{else}\ t_2) \rhd (\texttt{if}\ t'\ \texttt{then}\ t_1\ \texttt{else}\ t_2)}$$

Figure 2.3: The (Extended) Simply Typed Lambda Calculus. The base type $*$ is removed from the language of types and new productions are added for terms and type-terms to those in Figure 2.1. Only a small sample of type and reduction rules are shown.

universal application, *e.g.*:

$$(\Lambda\alpha{:}*\,.\,\lambda x{:}\alpha\,.\,x)\ \texttt{Nat} \ \rhd\ (\lambda x{:}\texttt{Nat}\,.\,x)$$

A universal application is evaluated by substituting the type-term argument for the free occurrences of the bound type-variable in the body of the universal abstraction. Another example:

$$(\Lambda\beta{:}*\,.\,(\Lambda\alpha{:}*\,.\,\lambda x{:}\alpha\,.\,x)\ \beta)\ \texttt{Nat} \ \rhd\ (\Lambda\alpha{:}*\,.\,\lambda x{:}\alpha\,.\,x)\ \texttt{Nat}$$

Figure 2.4 shows the additions to the grammar, to the type rules, and to the operational semantics. Because of the introduction of type variables, rules for type-term well-formedness are provided (a few are shown in the first row of the second box).

We reintroduce $*$ at a new level and call it a base or ground *kind*. Kinds classify types and are explained in detail in Section 2.7.4; for the moment, type variables in universal abstractions always have kind $*$, for they can only be substituted for base types which are all manifest, but we use in advance the kind meta-variable κ because the language of kinds is extended in Section 2.7.4. Type-assignments now also store the kinds of type variables.

The type rules for universal abstraction and application are shown in the second row of the second box. Notice that a type-level, capture-avoiding substitution operation is assumed which replaces type variables for types in terms. The last box in Figure 2.4 enlarges the reduction relation to account for universal applications. Universal abstractions are normal forms like regular term abstractions.

2.7.4 Adding type operators: System $\mathbf{F}_\omega$

System F_ω extends System F with ***type operators***, *i.e.*, functions at the type level. They are also called ***type constructors***, but we prefer to use 'constructor' at the value level when referring to the terms associated with type operators, called ***value constructors***. (For personal reasons, we deprecate the term ***data constructor***.) An example of type operator is `List` which when applied to a manifest type τ returns the type of lists of type τ. Its associated value constructors are:

```
Nil  :: ∀ a:∗. List a
Cons :: ∀ a:∗. a → List a → List a
```

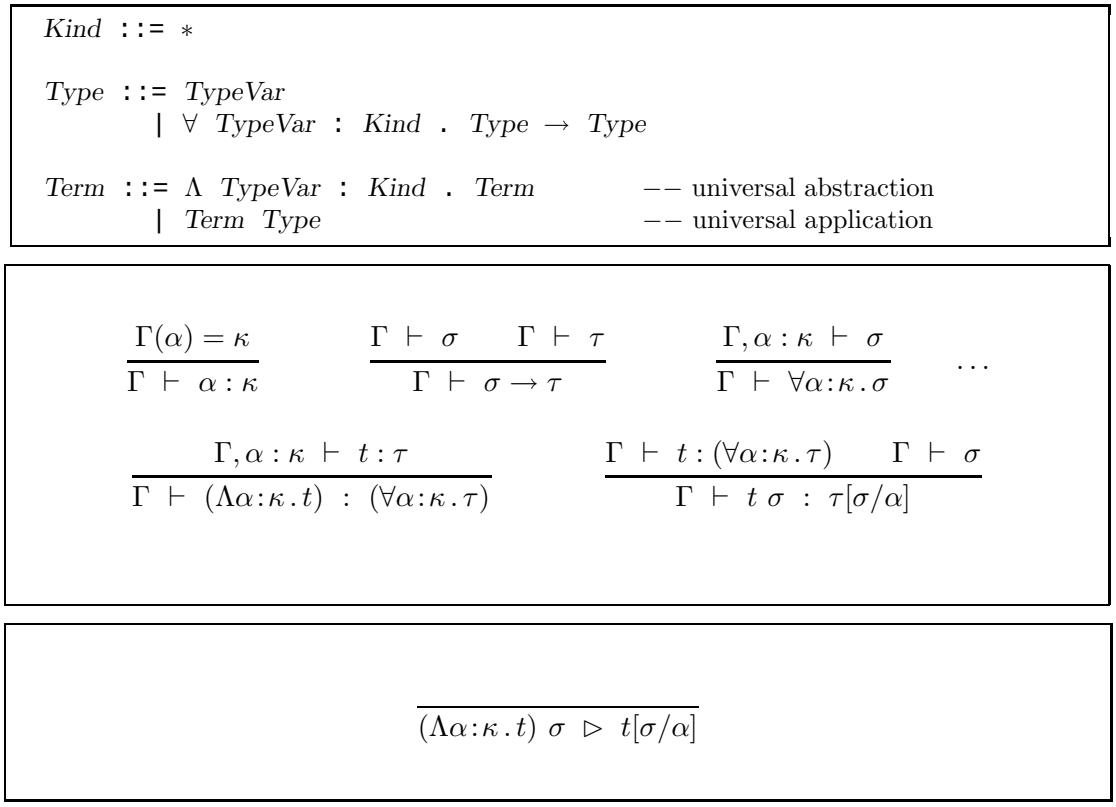


Figure 2.4: System F extensions.

which are names for primitive constants without δ-rules—*e.g.*, a term like `Cons` t `Nil` is in normal form, whatever the t (Section 6.1.1). Manifest types such as `Nat` or `Bool` are 'values' at the type level. A fully applied (*i.e.*, closed) type operator also constitutes a manifest type, *e.g.*: `List Nat`. Occasionally, we blur the distinction between manifest types and type operators by considering the former as ***nullary type operators*** and the latter as ***proper type operators***.

To model type-level functions, the PSTLC of Figure 2.1 is lifted to the type level as shown in the first box of Figure 2.5, so that terms such as α, $\lambda\alpha{:}\kappa\,.\,\tau$, and $\tau\ \sigma$ (that is, ***type variables***, ***type-level abstractions***, and ***type-level applications***) are defined as legal type-terms.

The ***kind*** of a type-term is somewhat inaccurately described as the 'type' of a type-term. *But kinds only describe the arity and order of type operators.* The kind of a nullary type operator (a manifest type) is $*$. The kind of a proper type operator is denoted as $\kappa \to \nu$, where κ is the kind of its argument and ν the kind of its result. The order of a type operator is determined from its ***kind signature*** as follows:

$$\begin{array}{lcl} \mathrm{order}(*) & \stackrel{\mathrm{def}}{=} & 0 \\ \mathrm{order}(\kappa \to \nu) & \stackrel{\mathrm{def}}{=} & \max\ (\ 1 + \mathrm{order}(\kappa),\ \mathrm{order}(\nu)\ ) \end{array}$$

Kinds do not have a status as the 'types' of types when there are orthogonal features in the type language (*e.g.* qualified types [Jon92]) that render them inaccurate as such. For instance, the following two Haskell definitions of the type operator `List` have the same kind, but the second is constrained on the range of type arguments:

```
data List a = Nil | Cons a (List a)
data Ord a ⇒ List a = Nil | Cons a (List a)
```

Type checkers *kind-check* applications of type operators to arguments to make sure the latter have the right expected kind. Kind-checking rules are shown in the second box of Figure 2.5. The first three lines establish the kinds of manifest types and also depict the kind-checking rules for primitive type operators such as $+$ and $\times$, etc. The last line contains the type rules of the PSTLC but lifted as kind rules (compare with Figure 2.1).

The third box in Figure 2.5 shows a *type-level reduction* relation $\blacktriangleright$ between type-terms that is reflexive, transitive, and compatible with all ways of constructing type-terms.

$$
\begin{array}{l}
\mathit{Kind} \ \texttt{::=} \ * \\
\qquad \texttt{|} \ \mathit{Kind} \ \rightarrow \ \mathit{Kind} \\
\\
\mathit{Type} \ \texttt{::=} \ \mathit{Type} \ \ \mathit{Type} \qquad\qquad -- \text{ type application} \\
\qquad \texttt{|} \ \lambda \ \mathit{TypeVar} \ \texttt{:} \ \mathit{Kind} \ \texttt{.} \ \mathit{Type} \quad -- \text{ type abstraction}
\end{array}
$$

$$
\frac{}{\Gamma \ \vdash \ \texttt{Nat} : * } \qquad \frac{}{\Gamma \ \vdash \ \texttt{Bool} : *} \qquad \frac{\Gamma \ \vdash \ \tau_1 : * \qquad \Gamma \ \vdash \ \tau_2 : *}{\Gamma \ \vdash \ (\tau_1 \rightarrow \tau_2) \ : \ *}
$$

$$
\frac{\Gamma \ \vdash \ \tau_1 : * \qquad \Gamma \ \vdash \ \tau_2 : *}{\Gamma \ \vdash \ (\tau_1 \times \tau_2) \ : \ *} \qquad \frac{\Gamma \ \vdash \ \tau_1 : * \qquad \Gamma \ \vdash \ \tau_2 : *}{\Gamma \ \vdash \ (\tau_1 + \tau_2) \ : \ *}
$$

$$
\frac{\Gamma, \alpha : \kappa \ \vdash \ \tau : *}{\Gamma \ \vdash \ \forall \alpha{:}\kappa \, . \, \tau \ : \ *}
$$

$$
\frac{\Gamma(\alpha) = \kappa}{\Gamma \ \vdash \ \alpha \ : \kappa} \qquad \frac{\Gamma \ \vdash \ \tau_1 : \kappa \rightarrow \nu \qquad \Gamma \ \vdash \tau_2 : \kappa}{\Gamma \ \vdash \ (\tau_1 \ \tau_2) \ : \nu} \qquad \frac{\Gamma, \alpha : \kappa \ \vdash \ \tau : \nu}{\Gamma \ \vdash \ (\lambda \alpha{:}\kappa \, . \, \tau) \ : \kappa \rightarrow \nu}
$$

$$
\frac{}{(\lambda \alpha{:}\kappa \, . \, \tau) \ \tau' \ \blacktriangleright \ \tau[\tau'/\alpha]} \qquad \frac{\tau_1 \ \blacktriangleright \ \tau_1'}{\tau_1 \ \tau_2 \ \blacktriangleright \ \tau_1' \ \tau_2} \qquad \frac{\tau_2 \ \blacktriangleright \ \tau_2'}{\texttt{P} \ \tau_2 \ \blacktriangleright \ \texttt{P} \ \tau_2'}
$$

$$
\frac{\tau_1 \ \blacktriangleright \ \tau_1' \qquad \tau_2 \ \blacktriangleright \ \tau_2'}{\tau_1 \ + \ \tau_2 \ \blacktriangleright \ \tau_1' \ + \ \tau_2'} \qquad \frac{\tau_1 \ \blacktriangleright \ \tau_1' \qquad \tau_2 \ \blacktriangleright \ \tau_2'}{\tau_1 \ \times \ \tau_2 \ \blacktriangleright \ \tau_1' \ \times \ \tau_2'} \qquad \frac{\tau_1 \ \blacktriangleright \ \tau_1' \qquad \tau_2 \ \blacktriangleright \ \tau_2'}{\tau_1 \ \rightarrow \ \tau_2 \ \blacktriangleright \ \tau_1' \ \rightarrow \ \tau_2'}
$$

$$
\frac{\tau \ \blacktriangleright \ \tau'}{\forall \alpha{:}\kappa \, . \, \tau \ \blacktriangleright \ \forall \alpha{:}\kappa \, . \, \tau'} \qquad \frac{}{\tau \ \blacktriangleright \ \tau} \qquad \frac{\tau_1 \ \blacktriangleright \ \tau_2 \qquad \tau_2 \ \blacktriangleright \ \tau_3}{\tau_1 \ \blacktriangleright \ \tau_3}
$$

$$
\frac{}{\tau \ \equiv \ \tau} \qquad \frac{\tau_1 \ \equiv \ \tau_2}{\tau_2 \ \equiv \ \tau_1} \qquad \frac{\tau_1 \ \equiv \ \tau_2 \qquad \tau_2 \ \equiv \ \tau_3}{\tau_1 \ \equiv \ \tau_3}
$$

$$
\frac{\tau_1 \ \equiv \ \sigma_1 \qquad \tau_2 \ \equiv \ \sigma_2}{\tau_1 \rightarrow \tau_2 \ \equiv \ \sigma_1 \rightarrow \sigma_2} \qquad \frac{\tau \ \equiv \ \sigma}{\forall x{:}\kappa \, . \, \tau \ \equiv \ \forall x{:}\kappa \, . \, \sigma} \qquad \frac{\tau \ \equiv \ \sigma}{\lambda \alpha{:}\kappa \, . \, \tau \ \equiv \ \lambda \alpha{:}\kappa \, . \, \sigma}
$$

$$
\frac{\tau_1 \ \equiv \ \sigma_1 \qquad \tau_2 \ \equiv \ \sigma_2}{\tau_1 \ \tau_2 \ \equiv \ \sigma_1 \ \sigma_2} \qquad \frac{}{(\lambda \alpha{:}\kappa \, . \, \tau) \ \tau' \ \equiv \ \tau[\tau'/\alpha]}
$$

Figure 2.5: System F_ω adds type operators, a reduction relation ($\blacktriangleright$), and an equivalence relation ($\equiv$) on type-terms.

The symbol `P` stands for a primitive type, *i.e.*, `Bool`, `Nat`, or **1**, which is in normal form. The reduction relation is static: type-level applications are reduced by the type-checker at *compile* time. This relation is a trivial example of type-level computation.

The last box in Figure 2.5 defines a relation of ***structural type equivalence*** which specifies that two type-terms are equal when their structure is equal. The relation is reflexive, symmetric, transitive, and compatible with all ways of constructing types.

Normal forms of type-terms are type variables, primitive types, type-level abstractions, and type-terms of the form $\tau_1 \times \tau_2$, $\forall\alpha{:}\kappa\,.\,\tau_1$, $\tau_1 + \tau_2$, and $\tau_1 \to \tau_2$, when τ_1 and τ_2 are themselves in normal form.

Notice that there are three sorts of substitutions, two at the term level (one replacing term variables for terms in term abstractions and another replacing type variables for type-terms in universal abstractions) plus one at the type level (replacing type variables for type-terms in type-level abstractions).

2.7.5 Adding general recursion

All the languages described so far are strongly normalising, *i.e.*, terms and type-terms always reduce to normal form [Pie02, Mit96]. However, in order to use System F_ω for real programming we need to introduce some form of recursion. In this section we extend the language of terms and types to cater for *general recursive* functions and type operators. In functional languages, functions (term-level or type-level) are recursive when the function name is applied to another term within its own body. For instance, the recursive definition of the list type and the factorial function can be written in Haskell as follows:

```
data List a = Nil | Cons a (List a)
factorial n = if n==0 then 1 else n * factorial(n-1)
```

Which can be translated to the lambda notation of System F_ω as follows:

```
List = λa:*. 1 + (a × (List a))
factorial = λn:Nat. if n==0 then 1 else n * factorial (n-1)
```

But term and type lambda abstractions are unnamed; the naming mechanism above is meta-notation. Recursion must be achieved indirectly. Let us abstract in both cases over the name of the function to remove the recursion:

```
List° = λf:∗ → ∗. λa:∗. 1 + (a × f a)
factorial° = λf:Nat→Nat. λn:Nat. if n==0 then 1 else n ∗ f (n-1)
```

It is typical at this point to resort to meta-level arguments or denotational semantics to explain that the equations:

```
List = List° List
factorial = factorial° factorial
```

have a least solution in some semantic domain that gives meaning to System F_ω syntax. Such solution is the *least fixed point* of the equation. Fortunately, there is no need to resort to meta-level arguments. Operationally, recursive functions are reduced by unfolding their body at each recursive call. This unfolding can be carried out *at the object level* by two new primitive term and type constants `fix` and `Fix` respectively. They are called ***fixed-point operators*** because, semantically, they return the least fixed point of their argument.[4] Their type- and kind-signatures are respectively:

```
fix : ∀τ:κ. (τ → τ) → τ
Fix : ∀κ. (κ → κ) → κ
```

Their type-checking and reduction rules are shown in Figure 2.6. The intuition is that at the meta-level, the following equations must hold:

```
(fix f°) = f° (fix f°)
(Fix F°) = F° (Fix F°)
```

which turned into reduction rules give:

```
(fix f°) ▷ f° (fix f°)
(Fix F°) ▶ F° (Fix F°)
```

but since `F°` and `f°` abbreviate respectively type and term lambda abstractions, we have:

```
(fix (λx:α.t)) ▷ (λx:α.t) (fix (λx:α.t))
               ▷ t[x/(fix (λx:α.t))]

(Fix (λα:κ.τ)) ▶ (λα:κ.τ) (Fix (λα:κ.τ))
               ▶ τ[α/(Fix (λα:κ.τ))]
```

[4] Fixed-point operators are also denoted Y and μ in the literature.

Type ::= Fix *Type*
Term ::= fix *Term*

$$\frac{\Gamma \vdash \tau : \kappa \to \kappa}{\Gamma \vdash \texttt{Fix}\,\tau : \kappa} \qquad \frac{\Gamma \vdash t : \tau \to \tau}{\Gamma \vdash \texttt{fix}\,t : \tau}$$

$$\frac{}{\texttt{Fix}\,(\lambda\alpha{:}\kappa\,.\,\tau) \blacktriangleright \tau[\alpha/\texttt{Fix}\,(\lambda\alpha{:}\kappa\,.\,\tau)]} \qquad \frac{}{\texttt{fix}\,(\lambda x{:}\alpha\,.\,t) \rhd t[x/\texttt{fix}\,(\lambda x{:}\alpha\,.\,t)]}$$

Figure 2.6: Extension of System F_ω with fixed-point operators.

Figure 2.6 collects these steps in a single reduction rule. Call-by-name guarantees that recursive calls are unfolded only when their value is required.

Going back to our examples, the terms:

```
Fix (λf:∗ → ∗. λa:∗. 1 + (a × f a))
fix (λf:Nat→Nat. λn:Nat. if n==0 then 1 else n ∗ f (n-1))
```

are both legal System F_ω syntax that represent the recursive list type operator and the factorial function. The following is an example reduction that demonstrates the unfolding (ellipsis abbreviate some subexpressions and reduction steps):

```
(fix (λf:Nat→Nat. λn:Nat. if n==0 then 1 else n ∗ f (n-1))) 2
▷ (λn:Nat. if n==0 then 1 else n ∗ (fix ...) (n-1)) 2
▷ if 2==0 then 1 else 2 ∗ (fix ...) (2-1)
▷ if false then 1 else 2 ∗ (fix ...) (2-1)
▷ 2 ∗ (fix ...) (2-1)
▷ 2 ∗ (λn:Nat. if n==0 then 1 else n ∗ (fix ...)) (2-1)
▷ 2 ∗ if (2-1)==0 then 1 else (2-1) ∗ (fix ...) ((2-1)-1)
...
▷ 2 ∗ if false then 1 else (2-1) ∗ (fix ...) ((2-1)-1)
▷ 2 ∗ (2-1) ∗ (fix ...) ((2-1)-1)
...
▷ 2 ∗ (2-1) ∗ 1
...
```

```
▷ 2
```

Et Voilà.

Chapter 3
Bits of Category Theory

> [C]ategory theory is not specialised to a particular setting. It is a basic conceptual and notational framework in the same sense as set theory ... though it deals with more abstract constructions. [Pie91, p.xi]

Category Theory is heavily used in programming language theory, especially in denotational semantics, algebraic specification, and program construction. The central concepts of these disciplines are usually wielded in their categorial formulation because Category Theory provides a general, abstract, and uniform meta-language in which to express many ideas that have different concrete manifestations.

Abstraction is of special interest to us and certain category-theoretical concepts will be used when talking about polytypic programs and abstract data types in Chapters 6, 5, and 9. More precisely, the categories of particular interest to us are the category of types and the category of algebras and partial algebras. Further references on category theory are [SS03, Fok92, BBvv98, BW99, Pie91].

3.1 Categories and abstraction

For mathematical structures to constitute categories one needs to identify 'entities with structure', called ***objects***, and 'structure-preserving' mappings between them, called ***arrows***. Preserving structure means preserving the property of being a valid object of the category. Due to their often graphical presentation, a collection of objects and a collection of arrows is called a ***diagram***. (We use the word 'collection' in a technical sense: a collection is an homogeneous set.)

The axioms describing what constitutes a category are rather general and wildly different mathematical structures can be 'categorised as categories'. Only arrows need satisfy minimal requirements: there must be an arrow composition operation that is partial, closed, associative, and has unique neutral element—the identity arrow, which must exist for every object.

Category Theory is *constructive* in the sense that witnesses (arrows) are always constructed in terms of compositions of other arrows rather than have their existence posited. Category Theory is *coherent* in the sense that such arrows must satisfy *universal properties* (also known as *naturality* properties) expressed as equations involving universally-quantified arrows and their compositions. More precisely, many properties of diagrams do not depend on the internal structure of the particular objects under consideration and can be studied abstractly and independently of them. These universal properties are expressible *externally*, that is, purely in terms of composition of arrows.

As a contrasting illustration, Set Theory is concerned with the internal structure of sets and mappings. For instance, injective functions are characterised in terms of a property held by the elements of their domain and codomain sets:

$$\frac{f : A \to B \qquad a \in A \qquad a' \in A \qquad f(a) = f(a') \Rightarrow a = a'}{f \text{ is injective}}$$

The categorial approach abstracts away from this detail and concentrates on the external relationships between arrows. Sets considered as objects and set-theoretic total functions considered as arrows make up a category where arrow composition is function composition. The equivalent concept of injective function, namely, *monic* arrow, is defined in terms of its properties under composition:

$$\frac{f : A \to B \qquad g : C \to A \qquad h : C \to A \qquad f \circ g = f \circ h \;\Rightarrow\; g = h}{f \text{ is monic}}$$

This definition is a generalisation that applies in all categories and therefore a monic arrow in some categories may have nothing to do with the notion of injective function (Section 3.4).

3.2 Direction of arrows

Arrows will be written forwards, whether in type signatures or categorial diagrams. That is, we will write $f : A \to B$ and not $f : B \leftarrow A$. Good reasons have been given in favour of the latter style. In particular, the type of an applied function composition reads swiftly from the types of the functions involved when read from right to left—*i.e.*,

in the same direction as that of the arrows—as shown in Figure 3.1(1). This is not the case when arrows and types are read from left to right, as shown in Figure 3.1(2)(3), for composition applies its right argument first.

$$\frac{g : C \leftarrow B \qquad f : B \leftarrow A}{g \circ f : C \leftarrow A}\ (1) \qquad \frac{g : B \rightarrow C \qquad f : A \rightarrow B}{g \circ f : A \rightarrow C}\ (2)$$

$$\frac{f : A \rightarrow B \qquad g : B \rightarrow C}{g \circ f : A \rightarrow C}\ (3) \qquad \frac{f : A \rightarrow B \qquad g : B \rightarrow C}{f ; g : A \rightarrow C}\ (4)$$

Figure 3.1: Arrows and composition.

Furthermore, writing the target type on the left and the source type on the right is consistent with the normal notation for function application, where the arguments appear to the right of the function name, *i.e.*: $(g \circ f)\ x = g\ (f\ x)$. "[In] the alternative, so-called diagrammatical forms, one writes $x\ f$ for application and $f ; g$ for composition, where $x\ (f ; g) = (x\ f)\ g$" [BdM97, p2]. Nonetheless, there are also good reasons for writing arrows forwards:

1. It is the standard notation in mathematics and functional programming languages. It requires practice to get used to the backwards notation and we risk confusing readers unfamiliar with it. The choice is between flipping some compositions around in order to read types naturally versus flipping all arrows in order to make composition read naturally.

2. Only aesthetics or rigidity proscribes the use of a diagrammatical notation for composition alongside the normal notation for function application. There is no reason why composition cannot be used at will both in its diagrammatical or traditional form.

3. Only in the diagrammatical form does the type of composition itself read naturally, as it is only there that f is the first argument:

$$\text{Figure 3.1(1)}, \quad \circ \ : \ (C \leftarrow A) \leftarrow (B \leftarrow A) \leftarrow (C \leftarrow B)$$
$$\text{Figure 3.1(2)(3)}, \quad \circ \ : \ (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$$

Figure 3.1(4), $\quad ; : \quad (A \to B) \to (B \to C) \to (A \to C)$

Unless, of course, one defines $g \circ f$ as the infix way of writing $\circ(f, g)$ instead of the expected $\circ(g, f)$.

3.3 Definition of category

A category is identified by defining the objects, the arrows, what is composition, what is an identity arrow, and checking that categorial axioms are satisfied.

DEFINITION 3.3.1 A ***category*** $\mathbf{C}$ is a collection of ***objects*** $Obj(\mathbf{C})$ and a collection of arrows $Arr(\mathbf{C})$, such that:

1. For every pair of objects A and B there might be zero or more arrows from A to B. These arrows can be collected into a set which we denote by $Arr(A, B)$. Notice that $Arr(\mathbf{C})$ denotes the collection of all arrows of $\mathbf{C}$ whereas $Arr(A, B)$ denotes the collection of arrows from A to B. It is common practice to write $f : A \to B$ when $f \in Arr(A, B)$. It is also common practice to call A the *source* of f and B the *target* of f. Arrows have unique sources and targets. This is usually represented neatly in a diagram:

$$A \xrightarrow{\quad f \quad} B$$

2. There is an arrow-composition operation (denoted by ; or by $\circ$ as shown in Figure 3.1) with the following properties:

   (a) It is partial: two arrows f and g compose if the target of f equals the source of g.

   (b) It is closed: the resulting arrow is in $Arr(\mathbf{C})$: $\dfrac{f : A \to B \qquad g : B \to C}{f ; g \ : \ A \to C}$

   (c) It is associative: $\dfrac{f : A \to B \qquad g : B \to C \qquad h : C \to D}{f ; (g ; h) \ = \ (f ; g) ; h}$

(d) It has a left and right identity arrow for every object:

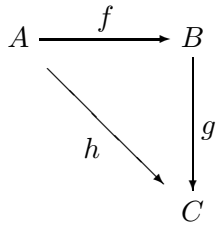
$$\frac{f : A \to B \qquad id_A : A \to A \qquad id_B : B \to B}{id_A ; f = f \qquad f ; id_B = f}$$

Identities are unique:

$$u ; f = f \;\Rightarrow\; u = id_A$$
$$f ; u = f \;\Rightarrow\; u = id_B$$

□

Category Theory is an algebra of 'typed' arrows. By composing arrows we obtain new arrows; but arrows with the same source and target need not be equal. It is only when $h = f ; g$ that we say the following diagram ***commutes***:



Universal properties are equations involving arrows expressible in terms of diagrams that commute.

3.4 Example categories

Categories are named after their objects. A typical category is **Set**, where objects are sets, arrows are total set-theoretic functions, and composition is function composition, which satisfies the categorial requirements for arrow composition. Another typical category is **Pre**, where objects are the members of a pre-ordered set $(A, \leq)$, arrows are pairs $(x, y) : x \to y$ such that $x \in A$, $y \in A$, $x \leq y$, and arrow composition is defined as follows:

$$\frac{(x, y) \;:\; x \to y \qquad (y, z) \;:\; y \to z}{(x, y) \;;\; (y, z) \;:\; x \to z \qquad (x, y) \;;\; (y, z) \stackrel{\text{def}}{=} (x, z)}$$

This example shows that arrows need not be functions.

A category of particular interest to us is **Type**, the category of types where objects are monomorphic types, arrows are functional programs between these types, and arrow composition is function composition. In Chapter 5 we introduce the category of algebras and partial algebras, where arrows are, respectively, algebra homomorphisms and partial homomorphisms.

3.5 Duality

For every categorial notion involving diagrams there is always a dual one in which the direction of the arrows is reversed. If $\mathbf{C}$ is a category, the *dual* or ***opposite category*** $\mathbf{C}^{\mathrm{op}}$ has the same objects and arrows as $\mathbf{C}$ only that the direction of the arrows is reversed:

$$\frac{A \in Obj(\mathbf{C})}{A \in Obj(\mathbf{C}^{\mathrm{op}})} \qquad \frac{f \in Arr(\mathbf{C}) \qquad f : A \to B}{f \in Arr(\mathbf{C}^{\mathrm{op}}) \qquad f : B \to A}$$

3.6 Initial and final objects

Definition 3.6.1 Given a category $\mathbf{C}$, $0 \in Obj(\mathbf{C})$ is an ***initial object*** iff for every object A there is a unique arrow $!_A : 0 \to A$. Accordingly, $!_0 = id_0$. Dually, given a category $\mathbf{C}$, $\mathbf{1} \in Obj(C)$ is a ***terminal object*** iff for every object A there is a unique arrow $!_A : A \to \mathbf{1}$. Accordingly, $!_{\mathbf{1}} = id_{\mathbf{1}}$. □

Arrows $x : \mathbf{1} \to A$ *from* terminal objects are called ***constants*** of A [Pie91, p17]. The motivation is that, for example, in the category of types, functional programs from the terminal type $\mathbf{1}$ (called ***unit type***) to any other type A can be put into *one-to-one* correspondence with the values in A. In other words, there is an *injective* function $i : A \to (\mathbf{1} \to A)$ such that if x is a value of type A then $i(x)$ is a value of type $\mathbf{1} \to A$, *i.e.*, a function. For instance, given the type of natural numbers:

```
data Nat = Zero | Succ Nat
```

the nullary value constructor `Zero` : `Nat` is a constant which can be lifted to a function `Zero` : $\mathbf{1} \to$ `Nat`. We use this device for other categories in Section A.3.1.

3.7 Isomorphisms

DEFINITION 3.7.1 Two objects A and B in a category are isomorphic when there are arrows $f : A \to B$ and $g : B \to A$ whose composition is the identity. In other words, $f; g = id_A$ and $g; f = id_B$. □

3.8 Functors

Categories are themselves mathematical structures. Functors are maps between categories which preserve the categorial structure.

DEFINITION 3.8.1 A ***functor*** $F : \mathbf{C} \to \mathbf{D}$ is an overloaded total map[1] between categories $\mathbf{C}$ and $\mathbf{D}$ mapping objects to objects and arrows to arrows while preserving composition and identities. More precisely,

$$F : Obj(\mathbf{C}) \to Obj(\mathbf{D})$$

$$F : Arr(\mathbf{C}) \to Arr(\mathbf{D})$$

such that:

1. $\forall A \in Obj(\mathbf{C}).\ F(A) \in Obj(\mathbf{D})$

2. If the functor is ***covariant*** in its arrow argument then:

$$\frac{f \in Arr(\mathbf{C}) \qquad f : A \to B}{F(f) \in Arr(\mathbf{D}) \qquad F(f) : F(A) \to F(B)}$$

3. If the functor is ***contravariant*** in its arrow argument then:

$$\frac{f \in Arr(\mathbf{C}) \qquad f : B \to A}{F(f) \in Arr(\mathbf{D}) \qquad F(f) : F(A) \to F(B)}$$

However, a functor is just 'contravariant' when $f : A \to B$ but $F(f) : F(B) \to F(A)$.

[1] Or if the reader prefers, two maps with overloaded name.

4. Finally, and more importantly, the functor preserves the categorial structure:

$$\frac{f \in Arr(\mathbf{C}) \qquad g \in Arr(\mathbf{C})}{F(f;g) \;=\; F(f)\;;\; F(g)} \qquad\qquad \frac{A \in Obj(\mathbf{C})}{F(id_A) \;=\; id_{F(A)}}$$

□

Figure 3.2 characterises a functor diagrammatically. If F is a functor and the diagram in $\mathbf{C}$ commutes, the diagram in $\mathbf{D}$ also commutes.

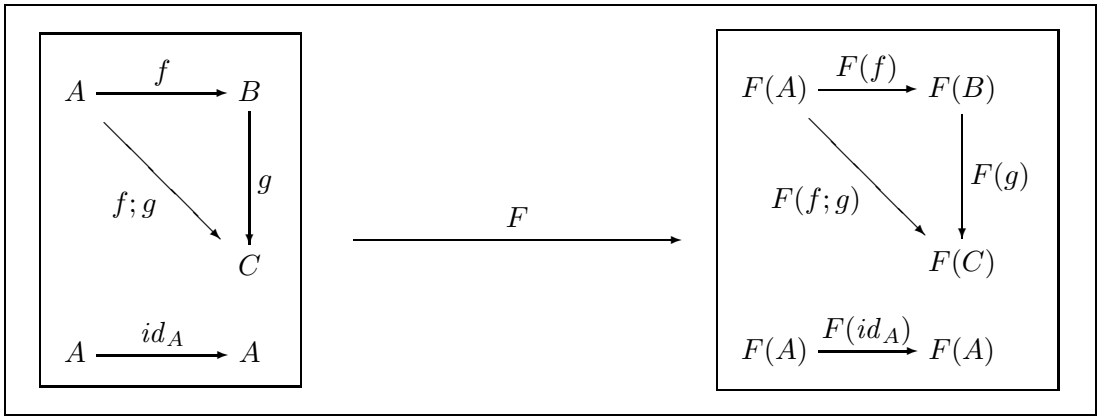


Figure 3.2: A functor F diagrammatically.

For simplicity, Definition 3.8.1 defines a unary functor. Functors of any arity are defined in terms of cartesian products of categories.

DEFINITION 3.8.2 The ***product category*** of two categories $\mathbf{C}$ and $\mathbf{D}$, denoted $\mathbf{C} \times \mathbf{D}$, is a category where:

$$\begin{aligned} Obj(\mathbf{C} \times \mathbf{D}) &\stackrel{\text{def}}{=} Obj(\mathbf{C}) \times Obj(\mathbf{D}) \\ Arr(\mathbf{C} \times \mathbf{D}) &\stackrel{\text{def}}{=} Arr(\mathbf{C}) \times Arr(\mathbf{D}) \end{aligned}$$

such that:

$$\frac{f \in Arr(\mathbf{C}) \qquad f : A \to C \qquad g \in Arr(\mathbf{D}) \qquad g : B \to D}{(f,g) \in Arr(\mathbf{C} \times \mathbf{D}) \qquad (f,g) : (A,B) \to (C,D)}$$

□

The previous definition can be trivially extended to n-tuples; we talk then about n-product categories. For instance, a *binary functor* (or *bifunctor*) is a functor from a product category to another category, *e.g.*, $F : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{E}$, and similarly for n-functors. We will be mostly interested in *endofunctors*, that is, in functors from $\mathbf{C}^n$ to $\mathbf{C}$, where $\mathbf{C}^n$ is the n-product of $\mathbf{C}$. For the sake of clarity, let us illustrate how Definition 3.8.1 is adapted for a binary covariant functor $F : \mathbf{C}^2 \rightarrow \mathbf{C}$:

$$\frac{(A,B) \in Obj(\mathbf{C}^2)}{F(A,B) \in Obj(\mathbf{C})} \qquad \frac{(f,g) \in Arr(\mathbf{C}^2)}{F(f,g) \in Arr(\mathbf{C})} \qquad \frac{(f,g) : (A,B) \rightarrow (C,D)}{F(f,g) : F(A,B) \rightarrow F(C,D)}$$

In the category of types, a functor $F : \mathbf{Type} \rightarrow \mathbf{Type}$ at the object level is a type operator that maps types to types: if F is a type operator and A is a manifest type then $F(A)$ is a manifest type. At the arrow level, *i.e.*, functional programs, F must satisfy the following:

$$\begin{aligned} F(f) &: F(A) \rightarrow F(B) \\ F(f;g) &= F(f);F(g) \\ F(id_A) &= id_{F(A)} \end{aligned}$$

Which means that at the arrow level F is the **`map`** function for the type operator. For example, in the case of type operator `List`:

```
map f :: List a → List b
map (f `;` g)       ==  map f `;` map g
map (id :: a → a)   ==  (id :: List a → List a)
```

where `f ';' g` $= g \circ f$. We have used explicit type annotations to illustrate the types of each **`id`** instance.

Sections 3.9.1 and 3.10 present examples of binary functors.

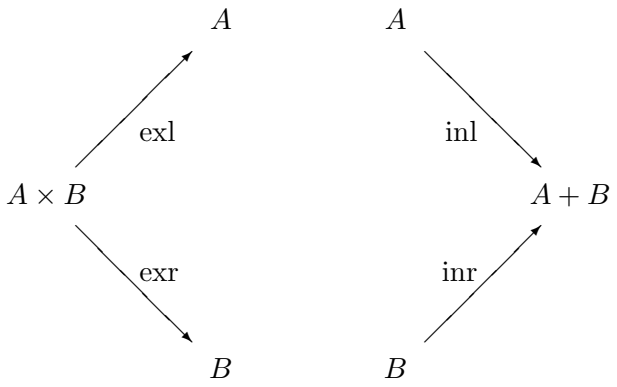
3.9 (Co)Limits

A limit is a solution to a diagram, *i.e.*, another diagram consisting of an object and a collection of arrows that satisfies the universal property that any other solution factors uniquely, *i.e.*, there is a unique arrow from the other solution to the object in the limit diagram which makes the combined diagrams commute. The colimit is the solution in

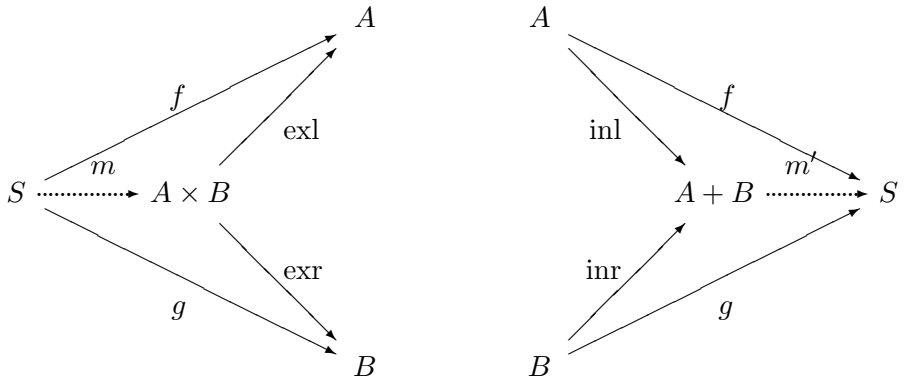
the dual diagram. Limits and colimits can be studied bottom-up from empty diagrams by adding objects and arrows. In the next sections we only present (co)limits for diagrams that are used later on.

3.9.1 (Co)Products

A (co)product is the (co)limit of a diagram involving two objects A and B and no arrows. Following [SS03] we present both notions simultaneously. The product is an object $A \times B$ and two arrows exl and exr. The coproduct is an object $A + B$ and two arrows inl and inr:



such that for any other similar solution (object S with arrows f and g), there is a unique mediating arrow from it to the product (or from the coproduct to it) that makes the following diagrams commute:



That is:

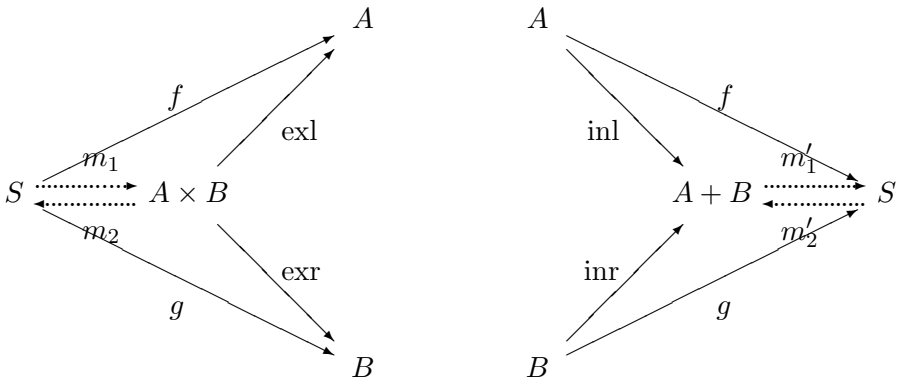
$$m; \mathrm{exl} = f \ \wedge \ m; \mathrm{exr} = g$$

$$\mathrm{inl}; m' = f \ \wedge \ \mathrm{inr}; m' = g$$

Both m and m' are unique for every solution diagram, *i.e.*, they are uniquely determined from the arrows involved. Following [Fok92, MFP91] we make this functional relationship explicit and use functions $\triangle$ and $\triangledown$ such that:

$$\begin{array}{lcl} m & \stackrel{\mathrm{def}}{=} & f \ \triangledown \ g \\ m' & \stackrel{\mathrm{def}}{=} & f \ \triangle \ g \end{array}$$

An important consequence of universality is that any other diagram solution is isomorphic. If S is another product there is a unique arrow $m_2 : A \times B \to S$. If S is another coproduct there is a unique arrow $m'_2 : S \to A + B$. Because mediating arrows are unique and the resulting diagrams commute, the composition of mediating arrows is the identity and their source and targets are isomorphic (Section 3.7). More precisely:



$$\left.\begin{array}{lcl} m_1; \mathrm{exl} & = & f \\ m_2; f & = & \mathrm{exl} \end{array}\right\} \Rightarrow m_1; m_2; f = f \qquad f; m'_2; m'_1 = f \Leftarrow \left\{\begin{array}{lcl} \mathrm{inl}; m'_1 & = & f \\ f; m'_2 & = & \mathrm{inl} \end{array}\right.$$

$$\left.\begin{array}{lcl} m_1; \mathrm{exr} & = & g \\ m_2; g & = & \mathrm{exr} \end{array}\right\} \Rightarrow m_1; m_2; g = g \qquad g; m'_2; m'_1 = g \Leftarrow \left\{\begin{array}{lcl} \mathrm{inr}; m'_1 & = & g \\ g; m'_2 & = & \mathrm{inr} \end{array}\right.$$

Consequently:

$$m_1; m_2 = id_S \wedge m_2; m_1 = id_{A \times B} \qquad m'_2; m'_1 = id_S \wedge m'_1; m'_2 = id_{A+B}$$

3.9.2 (Co)Products and abstraction

The definition of product captures the general notion of an object that is uniquely formed by combining two objects such that we can recover them via arrows exl and exr *irrespective of the internal structure* of that composite object. For example, in the category of sets, the categorial product is the cartesian product, which can be defined internally in many ways:

$$A \times B \stackrel{\text{def}}{=} \{\ \{\{a\},\{a,b\}\} \mid a{\in}A \wedge b{\in}B\ \}$$
$$A \times B \stackrel{\text{def}}{=} \{\ \{\{b\},\{a,b\}\} \mid a{\in}A \wedge b{\in}B\ \}$$
$$A \times B \stackrel{\text{def}}{=} \{\ \{\{a,0\},\{b,1\}\} \mid a{\in}A \wedge b{\in}B\ \}$$

The product object is a generalisation, *i.e.*, an *abstract set* with operations for construction and observation. In the category of types, $A \times B$ is an *abstract type* (a composite of A and B with two selector operators) which abstracts from the internal structure (representation) of the object.

A coproduct is a type into which we can inject two types using the two arrows. The mediating arrow $f \vartriangle g$ provides lifted construction in products and $f \triangledown g$ provides lifted discrimination plus selection in coproducts as shown in Figure 3.3, where lifting refers to the process of turning values into functions.

```
exl  :: Prod a b → a
exr  :: Prod a b → b
△ :: (c → a) → (c → b) → (c → Prod a b)

(f △ g) x = prod (f x) (g x)
prod :: a → b → Prod a b

inl  :: a → CoProd a b
inr  :: b → CoProd a b
▽ :: (a → c) → (b → c) → (CoProd a b → c)

(f ▽ g) x = if isl x then (f ∘ asl) x else (f ∘ asr) x
asl  :: CoProd a b → a
asr  :: CoProd a b → b
isl  :: CoProd a b → Bool
```

Figure 3.3: Type `Prod` stands for a product type and `CoProd` for a coproduct type.

Notice that types `Prod` and `CoProd` are abstract, we have not provided their definition

in terms of concrete types. In a functional language (*i.e.*, Haskell), binary products and coproducts are manipulated through concrete representations introduced in type definitions, *i.e.*, cartesian products and disjoint sums (Figure 3.4).

```
type Prod   a b = (a,b)
data CoProd a b = Inl a | Inr b

exl = fst
exr = snd
inl = Inl
inr = Inr
isl (Inl _) = true
isl (Inr _) = false
asl (Inl x) = x
asl (Inr _) = undefined
asr (Inl _) = undefined
asr (Inr y) = y
```

Figure 3.4: An 'implementation' of Figure 3.3 which describes the internal structure of the objects and arrows.

Binary (co)products are generalised to n-ary (co)products trivially: the (co)product is an object and n arrows [Pie91].

Interestingly, product and coproduct construction, *i.e.*, $\times$ and $+$, are both endofunctors from $\mathbf{C}^2$ to $\mathbf{C}$. It is standard to deviate from the prefix functor application notation and write $A \times B$ instead of $\times(A, B)$ and similarly for $+$. More precisely:

$$\frac{(A,B) \in Obj(\mathbf{C}^2)}{A \times B \in Obj(\mathbf{C})} \qquad \frac{(f,g) \in Arr(\mathbf{C}^2)}{f \times g \in Arr(\mathbf{C})} \qquad \frac{f : A \to C \qquad g : B \to D}{f \times g : A \times B \to C \times D}$$

In the category of types, at the arrow level $\times$ corresponds to the function:

```
map_Prod :: (a → c) → (b → d) → Prod a b → Prod c d
map_Prod f g = (f ∘ exl) △ (g ∘ exr)
```

In the case of concrete tuple types (Figure 3.4), the function can be written more familiarly:

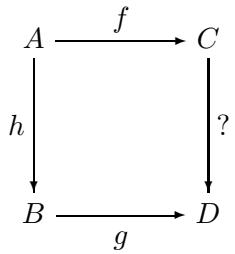
```
map× :: (a → c) → (b → d) → Prod a b → Prod c d
map× f g (x,y) = (f x, g y)
```

Similarly, for coproducts:

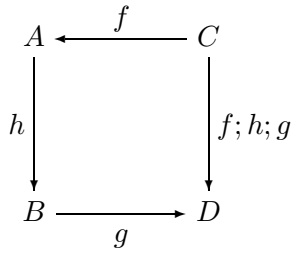
```
map_CoProd :: (a → c) → (b → d) → CoProd a b → CoProd c d
map_CoProd f g = (inl ∘ f) ▽ (inr ∘ g)

map+ :: (a → c) → (b → d) → CoProd a b → CoProd c d
map+ f g (Inl x) = Inl (f x)
map+ f g (Inr y) = Inr (g y)
```

3.10 Arrow functor

In Section 3.3 we introduced the notation $Arr(A, B)$ to express the collection of arrows from A to B in a given category $\mathbf{C}$. Interestingly, Arr can be understood as an endofunctor from $\mathbf{C}^2$ to $\mathbf{C}$. In the category of types, for any $(A, B) \in Obj(\mathbf{Type}^2)$, $Arr(A, B)$ is another type: the type of functions (arrows in $\mathbf{Type}$) from A to B. This functor has the peculiar characteristic that it is contravariant on its first argument. Why this is so becomes apparent by looking at the following diagram:



At the arrow level, we cannot define $Arr(f, g) : Arr(A, B) \to Arr(C, D)$ by composing f and g with arrows $h \in Arr(A, B)$ to yield an arrow in $Arr(C, D)$. We can do it if Arr is contravariant on its first argument:



DEFINITION 3.10.1 Given a category $\mathbf{C}$, the ***arrow functor*** $Arr : \mathbf{C}^2 \to \mathbf{C}$ is defined as follows:

$$\frac{(A, B) \in Obj(\mathbf{C^2})}{Arr(A, B) \in Obj(\mathbf{C})}$$

$$\frac{f : C \to A \qquad g : B \to D \qquad h \in Arr(A, B)}{Arr(f, g) \ : \ Arr(A, B) \to Arr(C, D) \qquad (Arr(f, g))(h) \stackrel{\text{def}}{=} f; h; g}$$

□

In the category of types, $Arr(A, B)$ is the function space $A \to B$. At the arrow level, $f \to g$ can be written using more familiar Haskell notation:

```
map→ :: (c → a) → (b → d) → (a → b) → (c → d)
map→ f g h = g ∘ h ∘ f
```

3.11 Algebra of functors

Just like there is an algebra of manifest types and type operators which can be combined to form type-terms, there is an algebra of objects and functors which can be combined to form object expressions. The following definitions provide the machinery.

Identity functor: The identity functor $Id : \mathbf{C} \to \mathbf{C}$ is defined as follows:

$$\frac{A \in Obj(\mathbf{C})}{Id(A) \in Obj(\mathbf{C}) \qquad Id(A) \stackrel{\text{def}}{=} A} \qquad \frac{f \in Arr(\mathbf{C}) \qquad f : A \to B}{Id(f) \in Arr(\mathbf{C}) \qquad Id(f) \stackrel{\text{def}}{=} f}$$

Constant functor: The constant functor $K_B : \mathbf{C} \to \mathbf{C}$ for *every* object B of $\mathbf{C}$ is defined as follows:

$$\frac{A \in Obj(\mathbf{C})}{K_B(A) \in Obj(\mathbf{C}) \qquad K_B(A) \stackrel{\text{def}}{=} B} \qquad \frac{f \in Arr(\mathbf{C}) \qquad f : C \to D}{K_B(f) \in Arr(\mathbf{C}) \qquad K_B(f) \stackrel{\text{def}}{=} id_B}$$

Polynomial functors and pointwise lifting: Objects described by object expressions can be obtained from applications of the constant functor, the identity functor, and the product and coproduct functors to other objects. For example, if A, B and C

are objects of $\mathbf{C}$, so is $A + (B \times C)$. The object is not named but written in terms of applications of functors to objects.

Functors can also be defined in terms of object expressions. In the previous expression if A stands for a free variable instead of an object, the object turns into a functor $(\cdot + (B \times C)) : \mathbf{C} \to \mathbf{C}$, where we indicate by $\cdot$ the place were the actual parameter would go. More commonly, functors are named in definitions:

$$F(X) \stackrel{\text{def}}{=} X + (B \times C)$$

(Inexplicably, Lambda Calculus notation has never caught on in Category Theory or in maths as a whole.)

It is sometimes convenient to define F only in terms of the functors involved and not in terms of functors and objects. To do that we define a notion of ***pointwise lifting*** for functors.

DEFINITION 3.11.1 Let $F : \mathbf{C}^2 \to \mathbf{C}$ be a functor. The lifting of F, denoted $\dot{F}$ is defined as follows:

$$\frac{A \in Obj(\mathbf{C}) \qquad G : \mathbf{C} \to \mathbf{C} \qquad H : \mathbf{C} \to \mathbf{C}}{\dot{F}(G,H) : \mathbf{C} \to \mathbf{C} \qquad (\dot{F}(G,H))(A) \stackrel{\text{def}}{=} F(G(A), H(A))}$$

□

With this definition at hand it is not difficult to check that $F(X) \stackrel{\text{def}}{=} X + (B \times C)$ can be defined in terms of functors and pointwise-lifted functors:

$$F \stackrel{\text{def}}{=} Id \dot{+} (K_B \dot{\times} K_C)$$

At the object level:

$$\begin{aligned} F(A) &= (Id \dot{+} (K_B \dot{\times} K_C))(A) \\ &= Id(A) + (K_B \dot{\times} K_C)(A) \\ &= A + (K_B(A) \times K_C(A)) \\ &= A + (B \times C) \end{aligned}$$

At the arrow level:

$$\begin{aligned} F(f) &= (Id \dot{+} (K_B \dot{\times} K_C))(f) \\ &= Id(f) + (K_B \dot{\times} K_C)(f) \\ &= f + (K_B(f) \times K_C(f)) \\ &= f + (id_B \times id_C) \end{aligned}$$

The last expression is more commonly written in Haskell as follows for fixed types `C` and `B`:

```
map+ f (map× (id :: B → B) (id :: C → C))
```

Pointwise lifting produces ***higher-order functors***:

$$\frac{F : \mathbf{C}^2 \to \mathbf{C}}{\dot{F} : \mathbf{Func}(\mathbf{C}, \mathbf{C})^2 \to \mathbf{Func}(\mathbf{C}, \mathbf{C})}$$

where $\mathbf{Func}(\mathbf{C}, \mathbf{C})$ is the category of functors from $\mathbf{C}$ to $\mathbf{C}$ (yep, functors make up a category, see Section 3.12). It is common to write $\mathbf{Func}(\mathbf{C}, \mathbf{C})$ as $\mathbf{C} \to \mathbf{C}$, making the 'type signature' of $\dot{F}$ more obvious to a functional programmer:

$$\dot{F} : (\mathbf{C} \to \mathbf{C}) \to (\mathbf{C} \to \mathbf{C})$$

The final touch is provided by lifting objects to functors. An object $A \in Obj(\mathbf{C})$ is lifted to a functor $\dot{A} : \mathbf{1} \to \mathbf{C}$, where $\mathbf{1}$ is the initial category (the initial object in the category of *small* categories, see Section 3.12). If we drop the notational distinction between lifted and regular functors, objects, and lifted products, we end up in a 'language-game' similar to that of values and (higher-order) functions, or manifest types and (higher-order) type operators. This facilitates the treatment of type operators as functors.

The last ingredient in this setting is the introduction of fixed points to account for recursive equations involving functors from $\mathbf{C}^n$ to $\mathbf{C}$. Let us state here the definition for $n = 1$:

Definition 3.11.2 Let $\mathbf{C}$ be a category and $F : \mathbf{C} \to \mathbf{C}$ a functor. A ***fixed point*** of F is a pair (A, α) where $A \in Obj(\mathbf{C})$ and $\alpha : F(A) \to A$ is an isomorphism. □

The technical machinery needed to explain the definition in detail is beyond the scope

of this presentation. Let us just mention that the fixed points of F form a category and the least fixed point is the initial object. Such category is a subcategory of the category of F-algebras, discussed in Chapter 5. The proof of existence of the initial algebra is basically a categorial generalisation of Tarski's fixed-point theorem [Pie91, p61–72].

3.12 Natural transformations

Categories are themselves mathematical structures and can be taken to be objects of another category where functors are the arrows. To avoid circular notions such as the category of all categories (whose objects are all categories) that may lead to paradoxes similar to Russell's in Set Theory [Ham82], categories are classified into ***small*** and ***large***, where in the former objects are not categories.

It is interesting to consider whether *arrows* of a category are *objects* of another category and what would then be the corresponding notion of arrow in this second category, called an ***arrow category***. Functors are maps between categories which preserve the categorial structure. A ***functor category*** is an example of an arrow category: objects are functors between small categories; arrows are called natural transformations.

Definition 3.12.1 Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Let $\mathbf{Func}(\mathbf{C}, \mathbf{D})$ be the category of functors from $\mathbf{C}$ to $\mathbf{D}$ and F and G two functors (objects) of this category that have the same variance. A ***natural transformation*** $\eta : F \dot{\rightarrow} G$ is an arrow in $\mathbf{Func}(\mathbf{C}, \mathbf{D})$. (The notation $\dot{\rightarrow}$ is introduced for arrows between functors.) More precisely, η is a family of $\mathbf{D}$-arrows indexed by objects of $\mathbf{C}$:

$$\eta \;=\; \{\; \eta_X \in Arr(F(X), G(X)) \mid X \in Obj(\mathbf{C}) \;\}$$

In words, a natural transformation $\eta : F \dot{\rightarrow} G$ assigns to each object $A \in Obj(\mathbf{C})$ an arrow $\eta_A \in Arr(F(A), G(A))$. The arrows must satisfy the following coherence (or naturality) property: depending on whether F and G are both covariant or contravariant,

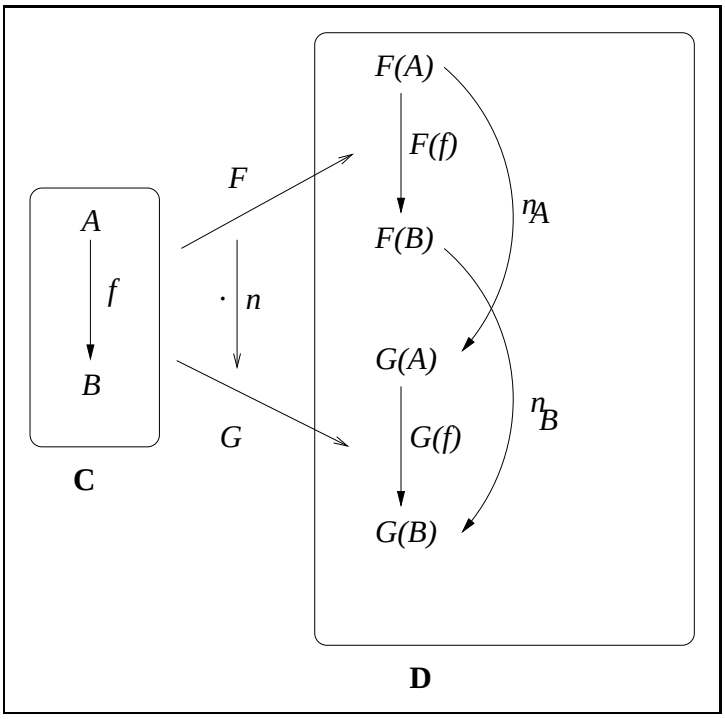


Figure 3.5: η is a natural transformation when $\eta_A; G(f) = F(f); \eta_B$ for every pair of objects A and B in $\mathbf{C}$.

the squared diagrams in $\mathbf{D}$ commute respectively:

$$\begin{array}{c} A \\ \downarrow f \\ B \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

$\mathbf{C}$ $\qquad$ $\mathbf{D}$ $\qquad$ $\mathbf{D}$

□

Figure 3.5 is a more illustrative depiction of the definition when F and G are both covariant. Given $A \in Obj(\mathbf{C})$, we can draw an arrow η_A from $F(A)$ to $G(A)$, *i.e.*, between two objects in $Obj(\mathbf{D})$ arising from the same object A by two different functors. And this can be done for *any* object in $Obj(\mathbf{C})$. Furthermore, for any other object $B \in Obj(\mathbf{C})$, there are two ways of defining an arrow from $F(A)$ to $G(B)$, namely,

$F(f);\eta_B$ and $\eta_A;G(f)$. Both must be equal:

$$F(f)\ ;\ \eta_B\ =\ \eta_A\ ;\ G(f)$$

In the category of types, natural transformations are polymorphic functions between type operators. For example:

```
flatten :: List (List a) → List a
flatten xs = foldr (++) []
```

is a natural transformation:

```
flatten : List;List ·→ List
```

as proven by the following equation:

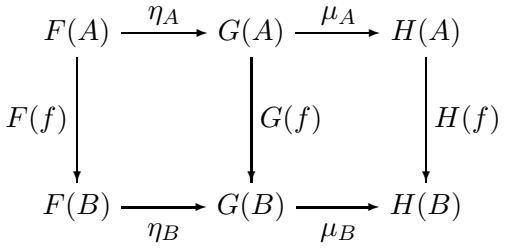
```
(map ';' map) f ';' flatten_b ==  flatten_a ';' map f
```

where `flatten`$_a$ and `flatten`$_b$ are instances of `flatten` at any two types `a` and `b` respectively. The equation can be put in the general form as follows:

$F \stackrel{\text{def}}{=}$ `List;List`, which at the arrow level is **`map ';' map`**.
$G \stackrel{\text{def}}{=}$ `List`, which at the arrow level is **`map`**.
$\eta_A \stackrel{\text{def}}{=}$ `flatten`$_a$
$\eta_B \stackrel{\text{def}}{=}$ `flatten`$_b$

Both sides of the equation are functions of type `List (List a)` → `List b` which are equal. The first function maps `f :: a` → `b` over the list of list of `a`s and then flattens the resulting list of `b`s. The second function flattens the list of lists of `a`s into a list of `a`s and then maps `f` to get a list of `b`s.

That functors as objects and natural transformations as arrows make up a category is illustrated by the following diagram. The composition of natural transformations is associative: given $\eta : F \stackrel{\cdot}{\to} G$ and $\mu : G \stackrel{\cdot}{\to} H$, their composition $\eta;\mu : F \stackrel{\cdot}{\to} H$ defined by $(\eta;\mu)_A = \eta_A;\mu_A$ for every $A{\in}Obj(C)$ is natural (*i.e.*, the diagram commutes):



The identity natural transformation $\iota : F \stackrel{\cdot}{\to} F$ is the collection of identities of the objects in the image of F, *i.e.*, $\iota_A = id_{F(A)}$.

Chapter 4
Generic Programming

> ABSTRACTION PRINCIPLE: Each significant piece of functionality in a program should be implemented in just one place in the source. Where similar functions are carried out by distinct pieces of code, it is in general beneficial to combine them into one by *abstracting out* the varying parts. [Pie02, p339]

The notion of 'genericity' in programming arose independently from within different paradigms that still exert an influence on its meaning, a meaning connoted by technological contingencies and theoretical advances. The term 'generic' is misused as well as overused. One is tempted to say it is overloaded, or to take the pun further, that it is polymorphic. It is certainly not generic.

This chapter provides an introduction to Generic Programming and aims at clarifying some of the confusion. We begin by relating 'generic' to another overused term, namely, 'abstraction', and wind up discussing where the present thesis stands in the described setting.

4.1 Genericity and the two uses of abstraction

Abstraction is the basic mechanism for tackling complexity and excessive detail. It is central to the top-down or bottom-up hierarchical decomposition or imposition of structure (a model) on a highly interconnected domain that cannot be completely understood in isolation. The focus at a particular level of description, highlighting what is relevant and ignoring or hiding what is irrelevant at that level, is more tellingly dubbed ***information hiding***.

Abstraction made its early appearance in Mathematics and Engineering in the guise of ***functional abstraction*** where the focus is directed toward the 'function', purpose, or role of components irrespective of how they work internally. It made its debut in programming in the form of ***control abstraction*** with the advent of assembly macros and subroutines. The first high-level programming languages pushed the notion fur-

ther with the introduction of structured programming constructs (loops, conditionals, etc.), procedures, and synchronisation mechanisms for dealing with concurrency. The idea is summarised by the (Control) Abstraction Principle quoted at the beginning of the chapter. Implicit in it are two important notions, namely, *parametrisation* and *encapsulation*.[1]

Parametrisation is the idea of making something a parameter to something else. For example, an expression with free variables can be abstracted into a function where the free variables are understood to be parameters. Many definitions of Generic Programming focus on parametrisation, *i.e.*, on making programs more flexible, adaptable, and general by "allowing a wider variety of entities as parameters than is available in more traditional programming languages" [Gib03, p1]. Parametrisation enables a *Single Program on Multiple Data* model of computation and is therefore central to code reuse. In the last decade, code reuse has been mostly popularised and achieved through *extension*, inheritance being a conspicuous example.

Encapsulation is information hiding with respect to the behaviour of components: their interaction is fixed but their internals may change. In programming terms, the interaction is set down in the *specification* (the *what*) of some computation or data structure and the internals are described by its *implementation* (the *how*). A consequence of encapsulation is the *interchangeability* of components that satisfy the specification. Thus, if we view our abstracted function as a black box of which we only care about its name, type, and semantics, then we have encapsulated it. Another function of the same type and semantics can be considered equal, irrespective of whether it calculates its outputs in the same fashion.

Genericity is already present in the notion of parametrisation. Programs whose varying domain-specific details have been factored out as parameters[2] could be considered generic in that regard. Nonetheless, genericity has wider connotations than parametrisation. We provide a definition of Generic Programming in a catch-phrase:

> Generic Programming = Parametrisation + Instantiation + Encapsulation

[1] The reader should not confuse parametrisation with *parametricity*. The latter has a specific meaning in relation to parametric polymorphism [Rey74, Wad89].

[2] Pedantic remark: factoring out something that can *vary* justifies the use of the term 'parameter *variable*' even in functional languages where variables are immutable. However, some people prefer the more neutral 'identifier'.

More precisely, forms of genericity can be classified according to the following criteria:

1. Which entities are parameters to which other entities at a level of description, *e.g.*, level of values and types, modules, components, agents, etc. Parameterised entities acquire a new status as a *mapping* or *relation* of sorts but they may live at the same level as other entities, *i.e.*, may be first-class and therefore parameters to other entities. The higher the entity is located in the hierarchy, the more general it is (*e.g.*, compare modules to functions) and the more possibilities for parametrisation and independence on entities in levels below—not to mention the possibility of mutual parametrisation when some of the levels are flattened (*e.g.*, dependent types).

2. How parameters are provided to and manipulated by an entity. Parametrisation is not necessarily a synonym of uniform behaviour (*i.e.*, uniform semantics); the mechanics of ***instantiation*** are of central importance. Among other things, it plays a role in determining whether parameterised entities are first-class or can be compiled separately from their instantiations (see Sections 4.5 and 4.7.1). Furthermore, instantiation itself needs not be uniform: some parameters may be instantiated differently than others. Finally, the number of parameters may be variable, they may depend on each other, or be provided by default, etc.

3. Upholding encapsulation. Some researchers have characterised Generic Programming as the attempt at "finding the abstract representation of efficient algorithms, data structures, and other software concepts" so as to make them generally applicable, interoperable and reusable [CE00, cit. p169]. Key ideas are lifting algorithms to a general level without loosing efficiency and expressing them with *minimal assumptions* about data types. In other words, lifting control abstraction to a new level of generality, usually via parametrisation, but upholding encapsulation in data types, known as ***data abstraction***.

Encapsulation in control abstraction is achieved by hiding the implementation of an algorithm behind an interface (*e.g.*, function name, type, and semantics). Encapsulation in data abstraction is also achieved by hiding data types behind an interface, but unfortunately there is some disagreement about how this should be done (Section 4.2). At any rate, upholding data abstraction is a necessary condition for Generic Programming (Chapter 7). Sometimes, the two uses of 'abstraction', control and data, do not

come hand by hand. Naturally, the lack of sufficient abstraction at a level of description makes our programs dependent on specific detail and therefore less generic.

4.2 Data abstraction

Data abstraction, *i.e.*, functional abstraction for data, appeared later than control abstraction. Originally, the introduction of base data types and their operations provided machine independent representation and manipulation of data. More complex data structures were *simulated* or *represented* in terms of combinations of base types, structured types like arrays and records (also called type operators) and previously defined types. But the impact of the language of types in correctness was soon to be noticed, and part of the outcome has been a split of the world of data types into ***concrete data types*** and ***abstract data types*** (ADTs). The motivation for this split was threefold:

1. In early computer languages, programmer-defined types were not given the same status as base types. For instance, a programmer-defined `Point` represented as a pair of integers could be manipulated by any function on pairs of integers irrespective of whether the function maintained the properties of `Point` (*e.g.*, that its values lie within a certain range).

2. Many correctness problems, such as safety problems, arose, and still continue to arise, because the language of types is not expressive enough to capture the semantics of the programmer's ***intended*** (or ***imagined***) type. Or to replace a negative with a positive, the *representing* concrete type can be too big: the 'values' of the intended type are represented by a subset of the values of the representing type(s), those that fulfil some particular criteria.

3. Software Engineering practice, in particular maintenance and evolution, dictated the conceptual separation between intended type (specification) and representing type (implementation). The need to minimise the repercussions of ever changing implementations led to a solution based on encapsulation where the representing type was hidden behind an interface.

By the mid 1970s, the notion of an abstract data type was engraved on most programmers' minds, but its theoretical formalisation and its embodiment in programming language constructs has taken long to mature, the very idea and variations on the theme

(*e.g.*, objects) still a subject of current research [AC96, Cas97, DT88, GWM+93, Mit96, Mar98, Pie02].

Data abstraction has been one of the contributions of Computing Science to Type Theory in what is popularly known as the 'formulas-as-types' correspondence or Curry-Howard isomorphism [CF68, How80] which identifies formulas of constructive logics with types, and proofs with functional programs. Programmers write much larger 'proofs' than mathematicians. Advances in program structuring and abstract data types took place in programming without a foundational theory; in fact, many advances in programming have taken place independently or without any intervention of Type Theory or otherwise. Other epitomising examples are object-oriented programming and the serendipitous re-discovery by John Reynolds in programming of Jean-Yves Girard's type-theoretical System F (Section 2.7.3). This cycle of invention followed by formalisation is not uncommon and to bring it into question is to deny reality:

> type checking rules or language ideas are put forth without an underlying model [formal type theory]. Ad hoc rules are not necessarily less desirable for the lack of a model; it seems to us that if such rules lead to a consistent and useful programming methodology, there probably *is* a satisfactory model. [DT88, p.61]

4.3 Generic Programming and Software Engineering

According to conventional wisdom, software suffers from an endemic crisis of unreliability, unmanageability, and unprovability. Because our imagination outstrips our abilities [Bud02, p2], one way of tackling the crisis is, amongst others, to change the way we channel our imaginations, *i.e.*, to change the programming model. In this setting, it is natural to expect Generic Programming to help reduce software development and maintenance costs, as fewer programs are developed, maintained and, so much the better, they can be re-used for varieties of data.

However, there are hardly any standardised or widely accepted software life cycles, or development processes, or specification and design formalisms, or program construction methodologies centred around Generic Programming or taking it into account. Exceptions are the minor notational extensions in UML,[3] the informal ontology of concepts and methodological aspects brought forth by the design of the C++ STL [MS96, MS94,

[3] `www.uml.org`

VJ03], and the work of the *squiggol* community, *e.g.*, [BJJM99, Hoo97], which includes polytypism and theories of *concrete* data types.

It is hard to determine the impact that a discipline of Generic Programming will have on software development, and whether it will dispense with maintainability. Some authors are skeptical about the latter:

> Because each new problem typically requires a slightly different set of behaviours, it is often difficult to design a truly useful and general-purpose software component the first time. Rather, useful reusable software components evolve slowly over many projects until they finally reach a stable state. [Bud02, p284]

In terms of programming languages and implementations, there is available technology, experimental prototypes, and idioms at the level of functions and types, with a bit less at the level of modules and libraries. So far, the most widespread success story in Generic Programming is polymorphism in all its varieties, popular examples being the parametric polymorphism of functional languages and the C++ STL. Hopefully, the rest of the field will soon expand and encompass the full extent of the software development process.

Understandability. One aspect of genericity that has an impact on development and is usually taken for granted is understandability. The more generic a function is, the more things it can do, supposedly the more effort required to construct it, to understand what parameters are, in which mode they are instantiated, how they are manipulated, etc.

Ideally, understandability must not compromise control abstraction, *e.g.*, require knowledge of the generic function's implementation or of the internal mechanics of instantiation, although sometimes this is not entirely possible (Section 4.7.2).

When it comes to program proving, already in a first-order setting proofs require heavy mathematical machinery and get complicated and lengthy for small programs, let alone in a higher-order setting where there is the danger that some properties may be undecidable.

An example of the tension between genericity and understandability is found in functional languages, where programmers can capture recursion patterns as higher-order

functions which factor out specific behaviour as function parameters. However, an excessive use of the latter can make programs difficult to read, especially if these function parameters have to circumvent the higher-order function's behaviour for specific situations. More concretely, it is well-known that many programs can be expressed using only the higher-order function `fold` [Hut99, GHA01], but programmers rarely try to write all they can in terms of it. They had better rely on generic traversals written in terms of combinator libraries [LV02a] or on Generic Programming libraries or language extensions (Chapter 6).

A careful study on the balance between genericity and specificity at various levels of description (design, component, module, function, etc.) is yet to be produced. Its study in ***design patterns*** [GHJV95] could perhaps be a useful starting point. Design patters are high-level specifications in which there is a mixture of genericity and specificity. Issues of abstraction, parametrisation, and encapsulation appear at various levels. There is already some work in this direction in the functional paradigm [OG05, LV02b]

Efficiency. There is also a tension between genericity and efficiency. Initially, a generic function would seem to be less efficient than a specialised function for a given set of parameters, if only because of the extra cost of passing and instantiating parameters and the risk of code bloating in generative approaches.

However, the same can be said about compiled code being less efficient than handwritten machine code or about C being 'faster' than Java. We don't want to write our programs in assembly language unless in critical situations, nor do object-oriented programming in C. Every time there is a leap in abstraction some degree of efficiency is lost.

Indeed, programming languages are better at their job when designed to provide good automatic support for tasks otherwise performed by the programmer. Opposition to this trend in defense of technologically contingent notions of efficiency is striking. Had the stigma of inefficiency not been ignored in the past we would not have procedures, functional languages, or automatic garbage collection today. The compromise between genericity and efficiency is, like the hoary space versus time, one of the inevitable tradeoffs that arise in Computing Science (Box 4.1).

Standard repartees are: (1) it is the job of research in program transformation and optimising compilers to obtain *reasonable* efficiency and, (2) mechanisms for supplying

BOX 4.1: Genericity vs Efficiency

"Correctness must come first. Clarity must usually come second, and efficiency third. Any sacrifice of clarity makes the program harder to maintain, and must be justified by a significant efficiency gain." [Pau96, p10]

"More computing sins are committed in the name of efficiency (without necessarily achieving it) than for any other single reason—including blind stupidity." [Wul72]

efficient specialised code for critical situations can always be provided. An example of the former can be found in [AS05] where a program transformation technique (*fusion*) is enhanced for the optimisation of code generated by the Generic Haskell compiler. An example of the latter is *partial template specialisation* in C++ [VJ03] and polytypic extension in *Scrap your Boilerplate* (Section 6.2).

Notice the use of the expression 'reasonable efficiency'. Admittedly, efficiency improvements usually entail a loss of clarity, for the improvements on an originally inefficient algorithm are obtained after exploiting properties of the problem that are not immediate. Syntactic and semantic program transformation techniques are theoretically limited and cannot produce fully optimal results:

> Program transformations can indeed improve efficiency, but we should regard executable specifications with caution... The ideal of ***declarative programming*** is to free us from writing programs—just state the requirements and the computer will do the rest. Hoare [Hoa87] has explored this ideal in the case of the Greatest Common Divisor, demonstrating that it is still a dream. A more realistic claim for declarative programming is to make programs easier to understand. [Pau96, p10]

The moral of the story is that machines cannot replace programmers.

4.4 Generic Programming and Generative Programming

Generic Programming represents a key implementation technique for ***Generative Programming***, a Software Engineering paradigm that aims at manufacturing software

components in the same 'automated assembly line' fashion as most goods are manufactured in other industries. More precisely:

> Generative Programming focuses on software system families rather than one-of-a-kind systems. Instead of building single family members from scratch, they can all be generated based on a common *generative domain model* ... that has three components: a means of specifying family members, the *implementation components* from which each member can be assembled, and the *configuration knowledge* mapping between a specification of a member and a finished member. [CE00, p5]

The implementation components must be highly orthogonal, combinable, reusable, and non-redundant. Not surprisingly, here is where Generic Programming comes into the picture. However, the view of generic programs as implementation components for Generative Programming does not alleviate the criticisms of Section 4.3, for even if the system family specification is lifted to the generative domain model, the specification and development of generic components still needs to be carried out.

In the other direction, Generative Programming techniques also provide implementation solutions to Generic Programming. Examples are program generation from generic specifications, generative compilation, and meta-programming techniques (we have already touched upon this in Section 2.5). Generative Programming also includes the possibility of *generic program generation*, where "the parameters to the [generic] program generation process remove unnecessary options of the general model and fill in some application-specific detail" [CE00, p209].

4.5 Types and Generic Programming

Types are influential in programming and inexorably too in Generic Programming. The need to type check generic entities is part of the quest for richer type languages and systems.

For instance, most of the available technology in statically type-checked languages is based on type parametrisation. Also, in some typed languages, generic function definitions are at most type-checked by the compiler; code is only generated for their applications to actual arguments. Examples are C++ Templates and Generic Haskell. In C++, template functions are not type-checked, their instantiations are. In Generic Haskell,

so-called polytypic functions are only type-checked by the compiler. Code is generated for their applications. Both language extensions support separate compilation: template instantiation is type-checked as late as link time, and applications of polytypic functions may appear in different modules. However, the compilation of applications requires reading the module with the definitions.

Generic Programming in untyped or dynamically type-checked languages is not entirely a non-issue. Types are implicitly present in strongly and dynamically type-checked languages. At run-time one gets 'type' errors even if types are not explicitly included in the language. Even in untyped languages, generalising certain constructions is not trivial and types somehow arise naturally [CW85, p3].

The definition and instantiation of generic entities must be well-typed and strong static type-checking should not be sacrificed for the sake of genericity. For example, in many languages arrays consist only of pointers to a chunk of memory of a fixed size. Functions on these arrays have to be given their size as a separate parameter—the source of a correctness problem—but they can work on arbitrary arrays. Bundling memory and size in an abstract data type means, for most languages, that functions can only work on arrays of particular sizes. The solution is not to sacrifice safety for genericity but to design a language of types where genericity can be safely accommodated.

4.6 Types and program generators

Sometimes the language of types is expressive enough to allow typed generic programs and data to be directly or indirectly defined within the language itself, or via libraries or reflection. When the existing language of types is not powerful enough, type and term language extensions are proposed which are either incorporated as part of the language, with occasional unexpected interactions with existing features (*e.g.*, C++ Templates [Str92]), or implemented using ***program generators*** (*e.g.*, Generic Haskell [HJ02]):

> [A generator] produces 'code' in some language which is already implemented [and is] used to extend the power of the base language ... This process should be compared with that of functional abstraction. The difference lies in [that] manipulation is performed before the final compiling [where the semantic checks of the whole program take place]. Macrogeneration seems to be particularly

> valuable when a semantic extension of the language is required [and the] only alternative to trickery with macros is to rewrite the compiler—in effect, to design a new language... In a more sophisticated language the need for macrogenerator diminishes. [Str00]

Another important aspect of generic entities that may cause confusion is the relationship between being first-class, typed, part of language extensions, or implemented in terms of program generators. All these possibilities are orthogonal:

- An entity at a particular level (*e.g.*, agents, modules, types, terms) is first-class if it plays the role of a "value" at that level. Amongst other things, it can be a part of other entities and a parameter or result of other entities. In a typed language, a first-class value has a type; but having a type does not grant first-class status. (*e.g.*, functions in C). Haskell modules are examples of entities that are not first-class and don't have types.
- Typed generic entities implemented using generators may not be first-class because of particular design decisions regarding generator or compiler implementation, not because of theoretical limitations. Some Draconian restrictions on C++ Templates, like forbidding virtual member template functions, are examples of this. On the contrary, generic functions are not first-class in Generic Haskell because their types are parametric on attributes of actual type-operator arguments, which are not known until instantiation (Section 6.1).

4.7 The Generic Programming zoo

Generic Programming is better characterised by a classification of its particular manifestations which account for the differences in the usage of the term. As discussed in Section 4.1, genericity qualifies parametrisation by describing what are parameters, how and when they are instantiated, and whether abstraction is upheld. The inhabitants of the Generic Programming zoo can be classified according to this criteria.

Most general-purpose typed programming languages usually have three explicit levels of values, types, and modules (which may contain types and values) plus two more possible implicit or explicit levels, namely, one of kinds (Chapter 2) and another one for checking the correct use of abstractions when they are not checked at the type

level (*e.g.*, Standard ML modules). In some languages abstractions are checked at the type level (*e.g.*, existential types, higher-order records, classes, etc). Special-purpose languages may have other levels.

Most mainstream languages keep values and types apart, *i.e.*, values can only be parameters to values or functions and similarly for types and type operators. It is natural to explore what happens when an entity in a higher level is a parameter to an entity in a lower level (*e.g.*, values parameterised by types) or when the levels are interconnected (Box 4.2). It is also natural to explore the different possibilities of instantiation.

BOX 4.2: Varieties of Parametrisation

The following table lists some possibilities of parametrisation at value, type, and kind level (adapted from [HJ02, p17]).

		Instantiation	
Entity	Parameter	Based on substitution	Based on structure
Value	Value	Ordinary functions	
Value	Type	Polymorphic functions	Polytypic functions
Value	Kind		
Type	Value	Dependent types	
Type	Type	Ordinary type operators	Polytypic types
Type	Kind		Polykinded types
Kind	Value	Dependent kinds	
Kind	Type	Dependent kinds	
Kind	Kind	Kind operators	Kind-indexed kind

This table does not consider binding time (Section 4.7.1). Ordinary functions have types. Ordinary type operators have kinds. Polytypic functions have polykinded types. Polytypic types have kind-indexed kinds.

4.7.1 Varieties of instantiation

Different varieties of instantiation can be entertained depending on how and when actual arguments are provided.

Concerning the *how*, the typical method of instantiation is the ***substitution*** of the ac-

tual argument for the formal parameter. This kind of 'syntactical replacement' is not trivial to implement and is qualified by evaluation order; typically call-by-X, where X is *value*, *push-value*, *reference*, *need*, *name*, etc. All these modes of evaluation assume that function definitions are not evaluated but just optimally compiled. It is the typical method of instantiation in function and type application in classical varieties of polymorphism and dependent types.

Instantiation could also involve some sort of *examination* of the argument that guides the instantiation. This is the case in structural polymorphism, where types and functions are dependent on the *structure* of type arguments and therefore the latter determine the former's semantics. In statically-typed languages, what is manipulated at run-time is a value representing the type argument, if at all.

Non-uniform behaviour is possible in all these instantiation schemes (see Section 4.7.2 and Box 4.2).

Concerning the *when* or the ***binding time***, instantiation can take place at compile time (***static parametrisation***), at run time (***dynamic parametrisation***), or at both. Static parametrisation rules out run-time variability and may complicate separate compilation—*e.g.*, template instantiation in C++ is sometimes deferred until link time. At the value level, the correctness of instantiations that take place dynamically may usually be checked statically (*e.g.*, type checking function applications).

The notions of ***open world*** and ***closed world*** are typically used in relation to extension. Generic entities are open world if they can be extended incrementally without recompiling already written code. However, the notions can also be used with regards to instantiation and separate compilation: generic entities are open world if their definitions can be compiled (or at least, type-checked) without their instantiations. In other words, separate compilation is supported. 'Openness' refers here to the fact that definitions are universal and not determined by how they are used.

4.7.2 Varieties of polymorphism

C++ Templates and the STL are considered examples of Generic Programming. Ada's instantiatable modules are actually called 'generics'. Many of the current uses of genericity conform to what is technically known as ***polymorphism***, a vocable derived

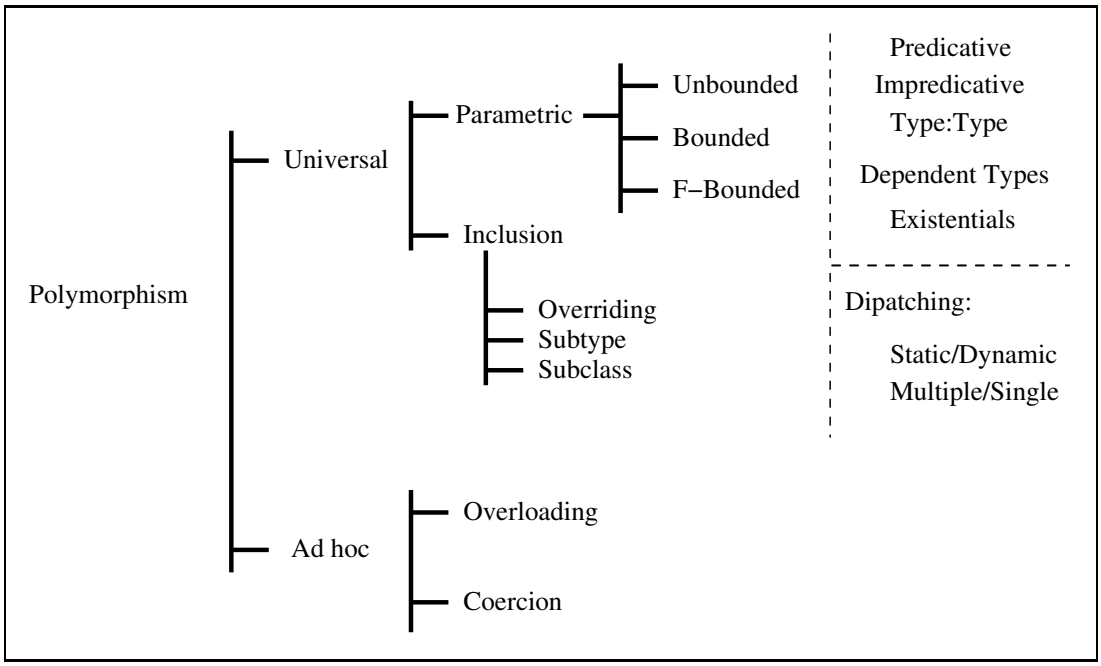


Figure 4.1: Classic polymorphism with instantiation based on substitution [CW85, p4] [CE00, chp. 6] [Pie02, part v] [Mit96, chp. 9].

from the Greek *poly morphos*, literally meaning 'many forms'.[4] Typically, an entity (usually a value or a type) is polymorphic if it has many semantic meanings. Examples of meaning are the value denoted or the type, the latter an approximation of the semantics.

An early but crude classification of polymorphism was proposed by Strachey [Str00] and later improved by Cardelli and Wegner [CW85] from a types perspective; or more precisely, from the perspective of the types of functions, for the type of a polymorphic function specifies the nature of its polymorphism. Figure 4.7.2 depict and extends their classification. The remainder of the section elaborates on it.

The original definition of ***ad-hoc polymorphism*** was given by Strachey [Str00, p37]:

> In ad hoc polymorphism there is no systematic way of determining the type of the result from the type of the arguments. There may be several rules of limited extent which reduce the number of cases, but these are themselves ad hoc both

[4] Morphos is a Greek god who can take any form he wishes, even that of a giant carrot.

in scope and content.

Coercion is an example of ad hoc polymorphism. Coercion is automatic or implicit ***conversion***. A conversion is a transformation of a value from one type to another type. In the case of coercion, on the surface the value seemingly retains its identity and is seen as having many types. Behind the surface the compiler inserts calls to conversions functions and creates new temporary values. Coercions not involving changes of representation are called ***castings***.

Overloading is the foremost and most powerful instance of ad hoc polymorphism. It is basically a mechanism for symbol reuse. A function name, literal value, etc, is used with different meanings (*e.g.*, types) but what is actually being defined is typically a *fixed* set of monomorphic values or functions that are only nominally related.[5] The functions work on different types with unrelated structure and thus have unrelated semantics. Their use in the program is determined by a process of 'best-match' resolution, typically performed at compile-time, based on the context of instantiation, which in the case of functions is determined by the types of the actual arguments and the lexical scope.

Overloading is a very powerful feature, especially if combined with other features like inheritance, redefinition, polyadicity, etc; but it lacks a generality that is often complemented by extension or generative approaches. C++ Templates are an example (Boxes 4.3 and 4.4).

Whether languages are *explicitly* or *implicitly* typed (*i.e.*, require type annotations) has an impact on the overloading scheme. In explicitly typed object-oriented languages like Java or C++, overloaded methods may have different type signatures and different semantics—*e.g.* `myStack.push()`, `myDoor.push()`, etc. In implicitly typed functional languages of the ML family, overloading is either forbidden or has to be achieved via ***type classes*** [Blo91, WB89, Jon92]. Type classes capture a more uniform form of overloading that relates types by making them belong to the same type class (Box 4.5). However, in Haskell the aforementioned `push` methods cannot be defined as overloaded functions within the same scope because the type of stacks and the type of doors do not fall naturally under the same type class. The functions can be overloaded in terms of scope if the types are defined in different modules and the function names are always

[5] We are talking about stand-alone overloading. Matters change when overloading is combined with other forms of polymorphism.

BOX 4.3: C++ Templates (Description)

C++ Templates are a language extension for static parametrisation. More precisely, templates endow C++ with a restricted form of polymorphism in its dependent, unbounded parametric, bounded parametric, and F-bounded parametric variants.

The static parametrisation, the mechanics of instantiation, and the fact that templates were designed to be backward-compatible with C++'s legacy type system is what make templates peculiar and restricted. More precisely, templated functions and classes are compile-time macros for what after instantiation are a set of *overloaded* functions and classes, with 'late' overloading resolution performed at link time if necessary [VJ03]. This *ad hoc modus operandi* requires clever compilation tricks and imposes restrictions on what can be programmed. Also, features that could be present in a uniform treatment have to be added *a posteriori*—*e.g.*, member template functions and typedef templates. Box 4.4 continues with examples.

qualified by module name. In Standard ML overloading is rather Spartan because type reconstruction cannot disambiguate the context if type annotations are missing.[6]

Universal polymorphism contrasts with ad hoc polymorphism in the *uniformity* of type structure and "behaviour". Universally polymorphic entities have many types/semantics, even potentially infinite ones, but can be characterised uniformly by a fixed set of rules, and the range of variability can be expressed syntactically using the $\forall$ quantifier, whence the 'universality'.

Parametric polymorphism is the kind of universal polymorphism that is often identified with Generic Programming because it relies on ***generic parameters*** to achieve uniformity. Generic parameters are type or value parameters to other types. They are instantiated using substitution and are reflected at the type level by the introduction of type variables. Entities that "have many types" in reality have one *unique*, universally quantified type (*e.g.*, Section 2.7.3). More power is added when type operators

[6] For many, type reconstruction is more a hurdle than an aid: it is rather strange to let compilers infer specifications from implementations.

BOX 4.4: C++ Templates (Example)

The following is a *template function* that swaps the values of two variables of the same type:

```
template<typename T>
void swap(T& a, T& b) { T temp = a; a = b; b = temp; }
```

The **template** keyword instructs the compiler to treat the entity that follows as a parameterised entity. Within the angle brackets the programmer specifies the kind of parametrisation. In this example, the parameter is a type variable `T`, so '**template<typename** `T`>' is similar to '$\lambda\tau : *.$' in System F (Section 2.7.3). A *type name* is either a base type or a class. Unfortunately, there is no type annotation specifying which methods are to be implemented by `T`. One has to look at the code to find out possible constraints or rely on the compiler, which might issue the error messages at link time. In our example, `T` must provide a copy constructor and **operator=**. There follows an example of usage:

```
int x=1, y=2;
Car car1("Ford Fiesta");
Car car2("Nissan Micra");
swap(x,y);
swap(car1,car2);
```

Behind the scenes, the compiler generates two *overloaded* instances of `swap` for the types to which the function is *implicitly* applied:[†]

```
void swap (int a, int b)   { int temp = a;  ... }
void swap (Car& a, Car& b) { Car temp = a;  ... }
```

At each function call, the compiler figures out the implicit type parameter by a process of type reconstruction based on the types of actual *value* arguments. In addition to type parametrisation, template functions and classes can be parametric on values and parameterised classes (so-called *template-templates*) [VJ03].

[†]Generating code for each type produces ***code bloating***.

BOX 4.5: Type Classes

Type classes were introduced in Haskell to support overloading while preserving type reconstruction. With type annotations, given the application `foo 4.5`, a compiler can tell which version of `foo` to use:

```
foo :: Int → Int          foo :: Float → Int
foo x = x                 foo x = intPart x
```

Without type annotations, it cannot. Type classes are a way to circumvent this. Programmers can specify classes of types `a` for which `foo :: a → Int` is defined:

```
class Foo a where foo :: a → Int
```

Given this class declaration, the compiler infers that all occurrences of `foo` in the program have type `Foo a ⇒ a → Int`. In order to type-check applications involving `foo`, programmers must provide witnesses:

```
instance Foo Int   where foo x = x
instance Foo Float where foo x = intPart x
```

The application `foo 4.5` type-checks because `4.5` has type **`Float`** and the **`Float`** type is an instance of (is in the class) `Foo`. Code generation is also efficient: a type class introduces a type of record dictionary and `foo` is defined to take an extra dictionary parameter:

```
data FooDict a = FD (a → Int)    -- dictionary type
getf (FD f) = f

foo :: FooDict a → a → Int
foo d x = (getf d) x

dInt :: FooDict Int              -- dictionary values
dInt = FD (λx → x)

dFloat :: FooDict Float
dFloat = FD (λx → intPart x)
```

With help from the type checker, the application `foo 4.5` is replaced by the application `foo dFloat 4.5`.

are included as primitives or via type definitions, with type-level execution (reduction) taking place during type checking in the form of type application (*e.g.*, Section 2.7.4).

One common feature of parametric polymorphism is that type information plays no computational role,*i.e.*, no computational decision is made in terms of types and they can be erased by the compiler after type checking. This is something to expect: monomorphic instances of polymorphic programs differ only in type annotations; recall System F (Section 2.7.3). Consequently, unlike overloading, one definition suffices. Moreover, parametrically polymorphic functions are insensitive to type arguments. They are either too general, *e.g.*:

```
∀ a. a → a
```

or constant, *e.g.*:

```
∀ a. a → Int
```

or combinators, *e.g.*:

```
∀ ab. (a → b) → a → b
```

or involve type operators, *e.g.*:

```
∀ a. List a → b
```

These last functions are insensitive to the ***payload*** type of the type operator, or to the ***shape*** type if the type operator is higher order (Section 6.1.1).

Types with value parameters are called ***dependent types***. In languages supporting dependent types, type-level reduction includes a restricted (*i.e.*, terminating) form of value-level reduction if type-checking is to terminate.

Different manifestations of parametric polymorphism are obtained by twiddling various knobs like the range of the ∀, the instantiation mode, or the interplay with other features like, say, dispatching. The range of the ∀ can be defined in terms of ***type universes***. The possibilities are rightly exemplified by the families of typed lambda calculi classified under the names of predicative, impredicative, and Type:Type. By permitting values to be types in each family we obtain new families of predicative, impredicative, or Type:Type *dependently*-typed polymorphism. Most of the interesting combinations are depicted in the so-called Lambda Cube [Pie02].

In ***predicative*** polymorphism, universally quantified types "live" in a separate universe from other types, a distinction captured syntactically by a separation of types into ***type schemes*** and type-terms, the former including type-terms and universally quantified type-terms. The foremost example of predicative polymorphism is the Hindley-Damas-Milner or ML-style of polymorphism (also called ***let-polymorphism***) of functional languages where type inference is efficiently decided by Milner and Damas' W algorithm—a straightforward exposition of W can be found in [FH88] and [PVV93].

In ***impredicative*** polymorphism, there is no distinction between type-terms and type schemes, and parameters themselves may have universally quantified types. The foremost example is System F (Section 2.7.3).

In ***Type:Type*** polymorphism, the universe of types is also a type, hence the colon notation. Type checking may not be decidable and types cannot be understood naively as sets of values (a set of all sets leads to Russell's paradox [Ham82]). The formalisation of Type:Type polymorphism requires the machinery of dependent types [Car86]. It has applications in the implementation of *typed* module systems, for modules contain values but also types.

In ***bounded parametric polymorphism*** the range of the $\forall$ can be restricted to sub-universes within the universe of types [CW85]. It subsumes parametric polymorphism in the sense that $\forall a.\tau$ is $\forall a{\in}\text{Type}.\tau$, where Type is the whole universe of types. The subtype relation is an example of bound:

```
registrationNumber :: ∀ a≤Vehicle. a → Int
```

Type classes are a form of bounded parametric polymorphism where the bound is given by a class-membership predicate instead of a subtype predicate, *i.e.*:

```
sort :: ∀ a. Ord a ⇒ [a] → [a]
```

is identical to:

```
sort :: ∀ a∈Ord. [a] →[a]
```

F-bounded polymorphism is a form of bounded polymorphism in which sub-universes are parametric and, possibly, recursively defined [Mit96].

In ***inclusion polymorphism***, there is an inclusion relation between sub-universes. It is especially used in combination with bounded polymorphism. A well-known inclusion

relation is ***subtyping***, where the members of a type (values) are also members of a supertype. Type classes provide a different kind of inclusion: members of a type class (types) are also members of its superclass.

Object-oriented languages offer different combinations of polymorphism. Inheritance (subclassing) is wrongly identified with subtyping [AC96, CW85, Cas97]. The former is about reusing implementations whereas the latter is about reusing specifications. Replacement, refinement, dynamic dispatching, and overriding are powerful but orthogonal features [Bud02].

Universal polymorphism is 'too generic': functions are insensitive to the possible values of universally-quantified type variables. ***Structural polymorphism*** or ***polytypism*** is a form of polymorphism in which the *definitional structure* of a type is a parameter to the function, which thereby does not have unique semantics. Instantiation is not based on substitution. Polytypic functions can be typed, but their types are parametric on attributes of type-operator arguments. Polytypic functions are really *meta-functions* that produce code from actual type arguments. Consequently, the mechanics of instantiation must be known in order to understand how polytypic functions work. Polytypism captures and generalises overloading and parametric polymorphism.

Polytypism is a form of compile-time reflection. At run time the type of an object can be inspected using some library extension. At compile time that functionality must be provided by the types themselves and, therefore, a language extension is needed. We examine two popular polytypic language extensions to the Haskell language in Chapter 6

4.8 Where does this work fall?

With structural polymorphism, functions are parametric on the definitional structure of types. When types are abstract, the definitional structure is hidden. Structural polymorphism is thus at odds with data abstraction. But in the classic varieties of polymorphism data abstraction is an orthogonal feature. The present work investigates what is needed to restore the order.

Chapter 5
Data Abstraction

> [The] key problem in the design and implementation of large software systems is reducing the amount of complexity or detail that must be considered at any one time. One way to do this is via the process of abstraction. [Gut77, p397]

The history of programming languages is a history towards higher control abstraction and higher data abstraction. Data abstraction appeared late in Mathematics and Computing (around the 1970s). Key ideas in control abstraction were already present in the seminal papers on the theory of computable functions (1930s). The tardiness can be explained by the fact that aspects which are in theory irrelevant can be in practice of the utmost importance: real programs reach a size of millions of lines and each may bring about a failure of the whole program.

When applied to data, 'abstraction' corresponds with the principle of representation-independence. Abstract types are defined by a specification, not by an implementation. Abstract types are represented or simulated in terms of concrete types using the data definition mechanisms of programming languages. Abstract types define sets of values but the interest shifts towards operators that 'hide' them; that is, abstract values are manipulated only through operators. Constant or literal values are understood as nullary operators. Thus, the separation between abstract types and built-in types is artificial: the latter are abstract with respect to their machine representation; the former are as real as a built-in type.

In the early 1970s, the notions of program module and information hiding garnered from Software Engineering practice converged with the formalisation and understanding of abstract types in terms of Universal Algebra and its categorial rendition. Examples of pioneering work are [LG86, Gut77, Mor73, BG82, GTW79].

The specification of an abstract type consists of two parts: (1) A syntactic part that defines the type's name and operator symbols. The latter implicitly define the set of well-formed terms that can be constructed with them. (2) A semantic part or just

'semantics', which inexorably has to be expressed syntactically, that prescribes operator usage and, consequently, the observable behaviour of the values of the type. Notions of *visibility*, *scope*, and *module* follow as implementation mechanisms.

5.1 Benefits of data abstraction

Abstract data types (ADTs) enforce representation independence which facilitates software evolution and maintainability. Changes to an ADT's implementation do not affect client code as long as the specification and the semantics of the operators remain unchanged. The ADT's specification is a *contract* or ***interface*** between the designer, client, and implementor of the ADT. The separation of interface and implementation allows ADTs to be provided as reusable, portable, and pre-compiled (binary) components.

ADTs are ideal for *modular design* and *division of work*. They befit the requirements of high cohesion, low coupling, and reasonable size. ADT interfaces gather collections of operators and types and provide the only interconnection with client code. For lack of standardised terminology, let us call the coupling between an ADT's interface and its client code ***interface coupling***.

ADTs harmonise with the design methodology of *stepwise refinement* of programs *and* the data they manipulate. Program actions are decomposed into smaller, yet unspecified actions, and the existence of ADTs is postulated. Decisions about representation, set of operators and their implementation are postponed. Details are discovered as the requirements and the design are refined.

ADTs have proven their worth in software development. Extensions to the idea (*e.g.*, objects) are deemed *sine qua non* in modern programming. Object-oriented programming itself is about programming with first-class, extensible abstract types where the notion of operator is replaced by the notion of message. Features like overriding, late dispatching, etc, are orthogonal.

5.2 Pitfalls of data abstraction

ADTs are *less flexible* and make programs *more verbose*. For instance, C/C++ programmers revel in their cryptic pointer fiddlings and the confusion between strings and null-terminated arrays of characters. With Java's **`String`**, only string operators are

applicable to string values; programmers have to use conversion operators and be more *explicit* about what they want. Some consider this to be a pitfall; we do not.

Usually, the choice of ADT is driven by a set of desired efficient operators. However, there is often an efficiency trade-off among them caused by the choice of representation. A simple example is the ADT of complex numbers. There are two typical representations: cartesian and polar. Addition in the cartesian representation is more efficient than in the polar representation whereas for multiplication it is the opposite case. Unfortunately, both representations cannot be used at once without losing efficiency in the translation.

The separation between interface and implementation improves but does not resolve completely in practice the coupling between client code and ADT implementations. An implementation change may entail an interface change that in turn may affect already written client code.

A typical example occurs during maintenance when operators are added, deleted, or modified. In many cases this situation can be anticipated and cared for during the design stage; but often it cannot, especially if the changes are elicited during the usage of the abstract type in a live software system. This problem is intrinsic to the nature of software: it evolves.

A thornier example of implementation decisions affecting interfaces occurs in the case of *parameterised* ADTs when an implementation decision may affect the parametricity, *i.e.*, may impose or modify parameter constraints which in turn affect client code. As a realistic scenario, take the case of a data-analysis application for managing numerical data and producing statistical reports, in particular frequency analysis [Mar98, p163-164]. A natural choice of abstract type would be a table that provides an abstract view of the data base. Among the operators, there is an insertion operator that adds elements to the table (increasing their frequency) and selector operators for returning the ith element on the table or its frequency (elements can be sorted by frequency). The efficient implementation of insertion compromises that of selection: hash tables are more adequate for quickly storing and retrieving large volumes of data by key whereas ordered structures such as heaps or balanced search trees are more suitable for storing and retrieving ordered data. The constraints expressing whether the data parameter is 'hashable' or ordered differ but anyhow have to be stated in the type's interface. The

choice of implementation could conceivably change as the circumstances of the software system's usage change. Going for both constraints from the start may be limiting or impossible.

There is research on automatic selection of efficient representations based on compile-time analysis of operator usage guided by programmer annotations (see, [CH96] for a probabilistic approach to the problem). To the author's knowledge, there is no universally accepted way of dealing with the problem, and most specification or programming languages either ignore it or offer their own custom-made solutions. We come back to this problem in Section 5.6, which illustrates it with an algebraic specification example, and in Sections 5.8.1 and 5.8.2, which discuss it in the context of the Haskell language. In Section 6.1.11 we make Generic Haskell cope with constraints.

5.3 Algebraic specification of data types

Algebraic specifications are axiomatic formal systems for specifying ADTs precisely, unambiguously, and independently of their implementation. They have several advantages beyond the mere specification of a formal object; in particular, they provide an interface for client code, they can be used in the formal construction and verification of client code, there is a formal relation between the specification and the implementation [Mar98, p221-224], and prototype implementations can be obtained automatically (*e.g.*, [GWM$^+$93]).

Algebraic specifications are subject to all the issues that arise in the description of any axiomatic system: the distinction between syntax, semantics, and pragmatics; the notions of syntactic as well as semantic consistency, soundness, and completeness, etc.

Because we cannot communicate ethereal ideas in our heads without syntax, and certainly do not want to talk about meanings too informally, endowing an axiomatic system with meaning amounts to providing a translation or interpretation of that formal system in terms of another formal system that is 'hopefully' better understood—*i.e.*, *squiggles to familiar squiggles.* One should not get the mistaken impression that semantic universes are already there, waiting to be matched syntactically. As specifications get more and more sophisticated, so are semantic universes discovered (or one should say, contrived) in order to endow the former with meaning. The reason for bothering with a translation is non-injectivity: different syntactic entities may have

the same semantic meaning.

The meaning of algebraic specifications is provided by the formal system of Universal Algebra. This is why algebraic specifications are 'algebraic'. Concretely, in this thesis the meaning of an algebraic specification is a ***partial many-sorted algebra***, in particular, the *least* or *initial* one. The reasons for choosing this semantic formalism are fourfold:

1. Algebraic specifications are collections of texts whose meaning is considered *simultaneously*: the meaning of a specification depends on the meaning of the specifications it imports and is therefore convenient to consider all specifications at once. Many-sorted algebras are algebras with many carriers, *i.e.*, sets of values. (They are more precisely called *many-valued* but the terminology has not caught on.)

2. Partial algebras allow us to deal with ***partial operators*** in a natural way. Partial operators are operators that may produce stuck terms, *i.e.*, run-time errors. They are common in strongly-typed languages that separate values from types. For example, the function that returns the head of a list is a partial operator which fails when the list is empty.

3. Classic algebraic specification formalisms have limitations on expressibility. Firstly, the behaviour of each type is described by means of *equational axioms* to be satisfied by any meaning-providing algebra. Equations form a simple specification language but many properties cannot be expressed with equations alone. For example, the axiom of functional extensionality requires a conditional equation:

   $$f\ x\ =\ g\ x\ \Rightarrow\ f = g$$

   Secondly, operators are *first-order* functions.[1] Equations involving higher-order operators are difficult to wield; in particular higher-order unification is in general undecidable (not surprising, for function equality is undecidable). However, there is no reason to worry. On the one hand, Parameterised Programming makes the case for lifting higher-order programming from operators to specifications [Gog88]. On the other hand, there are algebraic specifications with higher-order partial operators and conditional equations that are complete with respect to the class of all extensional models [Mei95].

[1] For some authors, this is what 'algebraic' means, *e.g.* [Mit96, p145].

Our specification formalism will permit ***conditional equations***, which are essential in specifications involving partial operators. As to the order of operators, we remain in a *first-order world*. Firstly, our aim is to explore Generic Programming on classic abstract types which can be described perfectly well in a first-order setting. Higher-order functions such as maps or folds will be written as *generic* programs outside the type using the latter's first-order operators. Secondly, conditional equations are important to us for purposes of specification, not deduction.

4. We want our specifications to have a unique meaning (up to isomorphism). Initiality captures the idea of the least algebra that satisfies the specification, *i.e.*, it does not satisfy equations that are not syntactically provable from the axioms, and the carriers contain values that are symbolised by at least one term (*i.e.*, expression involving operators). There is also another model, the ***final algebra***, which is unique up to isomorphism. The central notion here is *bisimulation*, the dual notion of congruence in initial algebras. A bisimulation establishes an equivalence relation between terms whose external observable behaviour is equivalent. Such information-hiding overtone might render the impression of being a better notion of meaning for abstract types, for the programmer is free to implement a simulation of the type so long as the terms' observable behaviour is the same. However, one thing is the semantics of a specification and another its implementation, of which we care less in this thesis. Finally, initial algebras have been studied in depth and endow specifications with an interesting form of induction.

The following sections describe our algebraic specification formalism. For readability, the technical details about its semantics and set-theoretic formalisation are given in Appendix A. The formalism is presented in two stages. A basic formalism is presented first, namely, signatures, theories with equations, and initial algebra semantics. Section 5.4 presents its syntax and semantics discursively through a few examples. Section A.1 details its set-theoretic formalisation and algebraic semantics. Section 5.5 describes the extension of the basic formalism with partial operators and conditional equations. Section A.2 details the semantics of the extended formalism. Section A.3 glides over the categorial rendition of algebra; in particular, Section A.3.1 introduces the notion of F-Algebra which is essential for a full understanding of Chapter 9.

Partial algebras were first studied in [BW82] and the existence of initial models in

[AC89]. *Ordered-sorted algebras* are a generalisation that includes a notion of inclusion (subtyping) among specifications [GM89]. The books [LEW96, Mar98] are excellent introductions to algebraic specifications.

5.4 The basic specification formalism, by example

Our algebraic specification formalism has a syntactic part or ***signature*** and a semantic part or ***theory***. The syntactic part declares the name of the specification and the name of the types defined, called ***sorts*** in the jargon. A sort is a piece of uninterpreted syntax, a name, but it is meant to stand for something, *i.e.*, a set of values. A signature also lists the set of operator names and their ***sort-signatures*** which specify the arity and sort of operator arguments and return values. Nullary operators are called constants. Figure 5.1 shows a simple example.

```
signature NAT
sorts Nat
ops
  zero : → Nat
  succ : Nat → Nat
end
```

Figure 5.1: Signature `NAT`.

Functional programmers express the same idea with different notation, where only sorts, called 'type names' in this case, and operator names, called 'value constructors', are defined explicitly:

```
data Nat = Zero | Succ Nat
```

As shown above, a signature starts with the keyword **`signature`** followed by the signature's name (in uppercase), the keyword **`sorts`** followed by the list of (capitalised) sort-names, and the keyword **`ops`** with the list of (lowercase) operator names and their sort-signatures. We assume operators are all prefix and written in functional style. An optional **`use`** clause can be included to import other specifications as shown in Figure 5.2.

Whitespace in sort-signatures separates argument sorts. This notation is consistent with most algebraic specification languages and has the advantage that it can be interpreted either as cartesian product or as function space (currying). This last inter-

```
signature STRING                          signature CHAR
sorts String                              sorts Char
use CHAR                                  ops
ops                                         ch0   : → Char
  empty : → String                          ...
  pre   : Char String → String              ch255 : → Char
end                                       end
```

Figure 5.2: Signatures `STRING` and `CHAR`.

pretation conflicts with our previous assertion that operators are first-order. We make an exception for operators that only return functions because there is an isomorphism between the types $A \times B \to C$ and $A \to B \to C$. In some functional languages currying seems the preference and we want to keep in mind the possibility of working with algebraic specifications in such languages, even if not currently supported.

There is clearly a shift of emphasis from values to operators: programmers specify the permitted operators, and terms can be formed out of repeated applications of proper operators to constants. These terms must satisfy the well-sortedness property that every proper operator is applied to arguments of the sorts specified by its sort-signature. The functional style is important as it conveys the notion of *referential transparency*: terms are meant to stand for values of the type, and one term represents a *unique* value.

For example, signature `NAT` defines the sort `Nat`, which is meant to stand for the set of natural numbers. The set of operators generates terms that can be put in correspondence with natural numbers: `zero` for 0, `succ zero` for 1, etc. A natural number is represented by only one `NAT` term.

In general specifications may contain ***laws***, that is, *relations* between terms—*e.g.*, equations or conditional equations—that specify the behaviour of operators. More precisely, relations are *semantic constraints*: an algebra must satisfy them in order to be a model. Figure 5.3 shows an example of an algebraic specification with laws.

The signature name has changed because we have added one operator. The semantic part or theory embeds a signature part by preceding it with the keyword **`theory`** and the name of the theory (in uppercase). A theory adds (1) a list of free-variable declarations for variables appearing in the laws which are assumed universally quantified, and (2) a list of laws (equations) which, in this example, specify the behaviour of `plus`.

```
theory NAT2
signature NAT2
sorts Nat
ops
  zero : → Nat
  succ : Nat → Nat
  plus : Nat Nat → Nat
vars
  x,y : Nat
laws
  plus x zero     = x
  plus x (succ y) = succ (plus x y)
end
```

Figure 5.3: Theory $\mathtt{NAT}_2$.

Equality is an equivalence relation and therefore equations introduce equivalences among terms. Consequently, countably infinite terms may denote the same value; for instance:

```
succ zero
plus zero (succ zero)
plus (succ zero) zero
plus (plus zero zero) (succ zero)
...
```

all represent the natural number 1.

The meaning of an algebraic specification is a ***many-sorted algebra***: broadly, sets of values, called ***carriers***, and functions on those values, called ***algebraic operators***, that satisfy the laws. Many different algebras can be ***models*** of the specification, *i.e.*, we can make a correspondence between terms and values, and algebraic operators satisfy all the axiomatic equations and those derivable (syntactically provable) from them. The ***initial algebra*** approach provides a definition of the most natural model. The carriers of an initial algebra contain values that are represented by at least one term, *i.e.*, there are ***no junk*** values in the algebra. And the algebraic operators only satisfy the equations of the specification, *i.e.*, there is ***no confusion*** between values; in other words, the model does not satisfy extra equations between values that are not reflected as equational axioms between terms, or as equations that can be derived from the latter.

Take signature `NAT`, for example. We assume that it specifies the set of natural numbers. However, we can also make a correspondence between the set of terms and the set of integers. In this case, negative integers are not represented by any term (junk). We can also make a correspondence with the set $\{0, 1, 2\}$, where `succ` is interpreted as 'addition modulo 3', that is, $2 + 1 = 0$, but there are no laws in the specification reflecting this property (confusion).

The following EBNF grammar describes the syntax of the specification formalism. Non-terminals *Uname*, *Cname*, and *Lname* stand for upper-case identifier, capitalised identifier, and lowercase identifier respectively:

Theory `::= theory` *Uname Signature Vars*`?` *Laws*
Signature `::= signature` *Uname UseList*`?` *Param*`?` *Body*
UseList `::= use` *Uname*$^+$
Body `::= sorts` *Cname*$^+$ `(ops` *OpList*`)?`
OpList `::= (`*Lname* `:` *SortSig*`)`$^+$
SortSig `::=` *Cname** $\rightarrow$ *Cname*
Vars `::= vars (`*Lname* `:` *Cname*`)`$^+$
Laws `::= (`*Term* `=` *Term*`)`$^+$
Param `::= param` *Cname* `(`*OpList Vars*`? Laws`$)^*$

We will often drop the **`vars`** clause, for free variables are those symbols that are not declared by other clauses and their sorts can be inferred from the context of use without effort.

As indicated by non-terminal *Param*, an algebraic specification can be parametric on another algebraic specification. A formal parameter is declared using the keyword **`param`** followed by the list of capitalised sort names and the set of constraints (laws) that must be satisfied by the parameter specification. Without constraints, the parameterised specification is (unbounded-)parametrically polymorphic, and with constraints, bounded-parametrically polymorphic (Chapter 4).

Parameterised specifications are *functions* on specifications and they could be higher order, *i.e.*, parametric specifications could take parametric specifications as parameters. The meaning of parametric specifications can be studied directly in terms of signature and theory morphisms [BG82, LEW96], which also describe other composition mechanisms such as inclusion, derivation, etc. Another possibility is to study the meaning of actual instantiations, that is, of the specifications resulting from the instantiation of

the parameters to non-parametric (or manifest, if you will) specifications. This is the approach we follow in Appendix A, for it is the simplest.

5.5 Partial specifications with conditional equations.

In strongly and statically type-checked languages that separate types from values, ADT operators may be partial. Partial operators produce stuck or non-reducible terms (when equations are directed and turned into reduction rules). Examples of partial operators are `tos` and `pop`, which return or remove respectively the top of a stack. The following are stuck terms:

```
tos emptyStack
pop emptyStack
```

Many people think of stuck terms as being *undefined*. Specification languages of yore ignored them for that reason. However, the term 'undefined' is also a synonym of *non-termination*, this meaning stemming from Partial Recursion Theory. A stuck term is not the same as a non-terminating term. We can always test whether the argument of a partial function meets a condition (*i.e.*, whether the stack is empty) at run-time whereas testing for non-termination is in general undecidable even at run-time.[2]

Partial functions introduce countably infinite stuck terms if the grammar of terms is recursive. For example:

```
empty? (pop emptyStack)
push 1 (pop emptyStack)
push 2 (pop emptyStack)
...
```

Stuck terms do not stand for values in any carrier. Also, they cannot be proven equal to any other term. They constitute junk and algebraic specifications must somehow deal with them. A possible solution is to introduce the notion of ***error terms*** and axioms for them which are reminiscent of the use of $\perp$ in Complete Partial Orders [Sto77, Ten76]. More precisely, we add to every specification:

1. One constant operator `error_`s of sort s, for every s.

[2] This fact is what supports the idea of replacing run-time tests by static ones in richer type systems; we would not expect richer type systems to decide the halting problem.

2. An error-testing operator `error_`s`?` of sort $s \rightarrow$ `Bool`, for every sort s.

3. Equations for the above.

4. `BOOL` is imported and it has an `if-then-else` conditional operator.

For example, Figure 5.4 shows part of a specification of stacks.

```
theory STACK
signature STACK
sorts Stack
param Elem
use BOOL
ops
  emptyStack   : Stack
  push         : Elem Stack → Stack
  pop          : Stack → Stack
  tos          : Stack → Elem
  empty?       : Stack → Bool
  error_Elem   : Elem
  error_Stack  : Stack
  error_Elem?  : Elem → Bool
  error_Stack? : Stack → Bool
laws
  tos error_Stack = error_Elem
  tos emptyStack  = error_Elem
  tos (push x s)  = if error_Elem? x then error_Elem
                    else if error_Stack? s then error_Elem
                         else x
  pop empty_Stack = error_Stack
  ...
```

Figure 5.4: `STACK` with error terms.

Conditionals are used for testing *at the object level* whether variables stand for error terms. Specifications of this kind can be proven consistent, where each carrier of the initial algebra has one error *value* [Mit96, p200]. However, they are low level and difficult to read and write.

Partial specifications with conditional equations provide a higher-level approach that makes specifications more readable. Partial operators are prefixed by the keyword **`partial`**. An object-level definedness predicate DEF is introduced such that $DEF(t)$ abbreviates $DEF(t) =$ `true` and asserts that t is defined, *i.e.*, it is sugar for $t = t$.

Laws now contain conditional equations of the form:

$$t_1 = t'_1 \ \wedge \ \ldots \ \wedge \ t_n = t'_n \quad \Rightarrow \ E$$

where $n \geq 0$ and E is either an equation $t = t'$ or a ***partial equation*** $t \simeq t'$ which is syntactic sugar for $DEF(t) \wedge DEF(t') \Rightarrow t = t'$. Premises in conditional equations represent *preconditions*; definedness is only an example. An empty premise ($n = 0$) gives rise to an equation E. Figure 5.5 shows the partial specification of stacks.

```
theory STACK
signature STACK
sorts Stack
param Elem
use BOOL
ops
  partial tos : Stack → Elem
  partial pop : Stack → Stack
  emptyS      : Stack
  push        : Elem Stack → Stack
  empty?      : Stack → Bool
laws
  DEF( tos (push x s) )
  DEF( pop (push x s) )
  tos (push x s)    = x
  pop (push x s)    = s
  empty? emptyS     = true
  empty? (push x s) = false
```

Figure 5.5: Partial specification of stacks.

We will use syntactic sugar for boolean terms and write `t` instead of `t=true` and ¬ `t` instead of `t=false` *only in the premises* of conditional equations. This sugar is used in Figure 5.6 which shows the specification of FIFO queues.

5.6 Constraints on parametricity, reloaded

ADT implementations are driven by pragmatic concerns. In the case of parameterised ADTs, some implementations impose constraints on payload types. We have already touched upon this in Section 5.2. This section illustrates the problem with an algebraic specification example.

Consider the parametric ADT of sets shown in Figure 5.7. Sets are unordered collections without repeated elements. Typical expected operators are construction, membership

```
theory FIFO
signature FIFO
sorts Fifo
param Elem
use BOOL
ops
   emptyQ  : Fifo
   enq     : Elem Fifo → Fifo
   emptyQ? : Fifo → Bool
   partial front : Fifo → Elem
   partial deq   : Fifo → Fifo
laws
   DEF( front (enq x q) )
   DEF( deq   (enq x q) )
   emptyQ? emptyQ    = true
   emptyQ? (enq x q) = false
     (emptyQ? q) ⇒ front (enq x q) = x
   ¬ (emptyQ? q) ⇒ front (enq x q) = front q
     (emptyQ? q) ⇒ deq   (enq x q) = x
   ¬ (emptyQ? q) ⇒ deq   (enq x q) = enq x (deq q)
end
```

Figure 5.6: Specification of FIFO queues.

test, and cardinality. Other possible operators not included in the figure are union, intersection, complement, etc.

Unconstrained sets are rather Spartan: payload elements can be put in the set and the cardinality can be calculated. Set membership, and operators derived from it such as union and intersection, imposes an equality constraint on payload types. The implementation of set complement may add another constraint.

Sets can be implemented in various ways. For example, as lists without repeated elements where `insert` leaves only one copy of each element in the list and `card` returns the length of the list. Sets can also be implemented as lists with repeated elements where `insert` inserts unrestrictedly and `card` skips repeated elements from the total count. Another possible implementation is in terms of hash tables where either `insert` or `card` skip repeated elements when dealing with collision lists. This implementation forces a 'hashable' constraint on the payload type. Sets can be implemented in terms of boolean dynamic vectors. In this case payload elements must be indexable, *i.e.*, there must exist an injective function `indexOf` : `Elem` → `Nat`. The application `insert x` sets the position `indexOf x` in the vector to `True`.

```
theory SET
signature SET
sorts Set
param Elem
  use BOOL
  ops
    equal : Elem Elem → Bool
  laws
    equal x x = true
    equal x y = true ⇒ equal y x = true
    equal x y = true ∧ equal y z = true ⇒ equal x z = true
use BOOL NAT
ops
  emptySet  : Set
  emptySet? : Set → Bool
  insert    : Elem Set → Set
  member?   : Elem Set → Bool
  card      : Set → Nat
laws
  emptySet? emptySet     = true
  emptySet? (insert x s) = False
  insert x (insert x s)  = insert x s
  insert x (insert y s)  = insert y (insert x s)
  member? x emptySet     = False
  member? x (insert y s) = equal x y or member? x s
  card emptySet          = Zero
  card (insert x s)
    = plus (card s) (if member? x s then Zero else Succ Zero)
```

Figure 5.7: A possible specification of Sets.

5.7 Concrete types are bigger

Concrete types are *bigger* than abstract types. The reason for this lies partly in the context-free nature of the language of concrete types. Context-dependent properties such as repetition or equality of payload elements are not captured by type definitions. ADTs impose context-dependent constraints on concrete types indirectly by means of equational laws and payload constraints.

Let's look at the problem from another angle. Consider the concrete type:

```
data Ord a ⇒ Tree a = Empty | Node a (Tree a) (Tree a)
```

This type can be used in the implementation of different abstract types: binary search trees, priority queues, ordered sets, ordered bags, etc. In each case, the operators

available and their laws are different: the maximum element in the left subtree of a non-empty binary search tree is at most the root, whereas in a priority queue the ordering in the tree could be different, with smaller values to the right of the root. Binary search trees, priority queues and bags may contain repeated elements; that is not the case for ordered sets. In a priority queue there is the choice of inserting elements with same priority in FIFO fashion or via some collision-resolution function, etc. Elements are removed from a fixed position (*e.g.* the front) in a priority queue. For sets, bags, and binary search trees, the element to remove must be indicated explicitly, *e.g.*:

```
remove  : Elem → Set → Set
```

It is a mistake to think that `Tree` is the 'natural' representation type of any ADT. We have already touched upon this in Chapter 1. This is the reason why ADTs are encapsulated behind an interface of operators that maintain the implementation invariants of the representation type.

5.8 Embodiments in functional languages

Mainstream functional languages, notably Haskell and SML, are not concerned with algebraic specifications of data types in any fashion. The following two sections describe the mechanisms available in Haskell and SML for defining ADTs.

5.8.1 ADTs in Haskell

Haskell supports ADTs poorly, relying on a module concept not dissimilar to C++'s name-spaces [Str92]. A module is a logical entity that does not necessarily correspond to a program file, but this is usually the case. Modules are linguistic constructs for controlling name scope and visibility. They are not first-class entities. ADTs are implemented using modules by means of *exporting* (making public) type and operator names while hiding value constructors and operator implementations. There are mechanisms for controlling the way data and operators are *imported* such as local or qualified imports and transitivity—*e.g.*, if `M` imports `M'` and `M'` imports type `A`, `M` cannot see `A` unless explicitly imported.

The Set ADT shown in Figure 5.7 can be conveyed to Haskell as shown below. In this example there is no separation between specification (only syntax) and implementa-

tion:

```
module Set (Set,emptySet,insert,member,card) where
  data Eq a ⇒ Set a = MkSet [a]
  emptySet :: Eq a ⇒ Set a
  emptySet = MkSet []
  insert :: Eq a ⇒ a → Set a → Set a
  insert x (s@(MkSet l)) = if elem x l then s else MkSet (x:l)
  ...
```

The module name `Set` is followed by a parenthesised list in the heading which declares the exported names. The type name `Set` is exported but not its value constructor `MkSet`. (The overloading of module and type-operator name is legal.) The constraint on the payload type is not shown in the export clause but must be figured out by looking at the type definition. It must be included also in the type signatures of operators. (Changing the implementation to boolean vectors would impose a different type-class constraint which would affect client code—Section 5.8.2).

Type classes can be employed to separate specification from implementation as shown in Figure 5.8.

```
module Set (Set(..)) where
class Set s where
 emptySet :: Eq a ⇒ s a
 insert   :: Eq a ⇒ a → s a → s a
 ...

module Set' (Set') where
import Set
data Eq a ⇒ Set' a = MkSet [a]

instance Set Set' where
  emptySet               = MkSet []
  insert x (s@(MkSet l)) = if elem x l then s else MkSet (x:l)
  ...
```

Figure 5.8: ADTs in Haskell using type classes.

In the first module, the names `Set` and `Set'` are overloaded: `Set` names a *type class* and the module where it is defined. A type `s` is a member of type class `Set` if it implements the required set operators. The type-class constraint **`Eq`** `a` has to be written in the type signature of every operator. In the second module, `Set'` is also overloaded: it names a

type operator and the module where it is defined. The `Set'` type is made an instance of `Set` by providing an implementation for every operator. The module exports only the type, not the value constructor `MkSet`.

We conclude with an example of usage. Function `addList` adds the elements of a list into a set:

```
addList :: (Eq a, Set s) ⇒ [a] → s a → s a
addList [] s     = s
addList (x:xs) s = insert x (insert xs s)
```

Notice the `Set` constraint in the type signature: `s` must satisfy the `Set` interface.

5.8.2 On constrained algebraic types

In Haskell, it seems natural to implement constrained ADTs in terms of constrained type operators. A first-order constrained type operator is type-class constrained on some or all of its payload.

However, constrained type operators are contentious. Paraphrasing Section 4.2.1 of the online Haskell Report,[3] a declaration such as:

```
data Eq a ⇒ Set a = MkSet [a]
```

is equivalent to the following:

```
data Set a where
  MkSet :: Eq a ⇒ [a] → Set a
```

That is, constrained type operators do not carry constraints, their value constructors do. Hence, construction or pattern-matching with `MkSet` gives rise to an **`Eq`** constraint. '[T]the context in the data declaration has no other effect whatsoever'. The last sentence from the Haskell report is proven by the following snippet:

```
foo :: Set a → Int
foo _ = 3

foo (undefined :: Set (Int → Int))
> 3
```

[3] `http://www.haskell.org/onlinereport/decls.html#sect4.2.1`

The code type-checks and runs despite that integer functions are not instances of class **Eq**. This is because `MkSet` is not involved. On the other hand, function `bar` below has a constraint in its type signature because it pattern-matches against `MkSet`:

```
bar :: Eq a ⇒ Set a → a
bar (MkSet xs) = ...
```

The legality of a type such as `Set (Int → Int)` can be explained more accurately using System F_ω notation (Section 2.7.4). The constrained definition of `Set` would give the impression that *at the type level* the type application of `Set` to an argument is legal only when the latter is in class **Eq**. However, as already explained in Section 2.7.4, the kind system accounts for arity and order, not for class membership. Suppose we can annotate kinds with classes such that, for instance, $*^{Eq}$ is the collection of types of kind $*$ in class **Eq**. The definition of the `Set` type and its value constructor `MkSet` would be expressed in this setting thus:

$$
\begin{array}{lcl}
Set & : & *^{Eq} \rightarrow * \\
Set & \stackrel{\text{def}}{=} & \lambda\alpha{:}*^{Eq} \,.\, [\alpha] \\
MkSet & : & \forall\alpha : *^{Eq} .\ [\alpha] \rightarrow Set\ \alpha \\
MkSet & \stackrel{\text{def}}{=} & \Lambda\alpha{:}*^{Eq} \,.\, \lambda x{:}[\alpha] \,.\, x
\end{array}
$$

In Haskell, however, type classes are orthogonal to the kind system. `Set` retains its $* \rightarrow *$ kind and `MkSet` carries the constraint. If constraints were associated with type operators they would provide more security: writing expressions with illegal types such as `Set (Int → Int)` would be impossible.

Many Haskell programmers avoid constrained type operators and prefer to constrain functions. This is practical from the point of view of type-operator reusability. For example, there is only one type of lists. Lists with constrained payload are not defined by a new type, but manipulated by constrained functions such as **sort**. In other words, there is no such thing as a constrained list type in the program.

However, there are reasons why constrained types are useful. For once, because they are constrained, the idea is that they are to be used for specific applications. More importantly, constraints are useful for documentation purposes. Logically, it makes more sense to specify constraints in a single place (a type definition) instead of in all places where the type is used (functions). The fact that constraints in type signatures of functions can be inferred makes this point stronger. However, explicit type signatures

should be written down for documentation purposes (they are part of a function's specification) and with higher-rank polymorphism they are unavoidable (Chapter 4). This brings us to the point about the impact of constraints in maintainability.

When it comes to ADTs, constraints are formally associated with the type operator: in the case of sets, for example, membership test imposes an **`Eq`** constraint on the payload type. We can also argue that ADT values are manipulated via exported operators and that it is perfectly possible to remove constraints from implementation type operators as long as exported operators carry them. For example, in the module-based definition of sets in Section 5.8.1, `MkSet` is not exported and we could remove the constraint from `Set`'s data declaration. Similarly for the class-based definition.

But putting constraints on operators hinders maintenance [Hug99]. We have touched upon this in Sections 5.2 and 5.6: payload constraints are fragile with respect to implementation changes. They can be affected by, and therefore disclose, implementation decisions.

For instance, changing the `Set` implementation from lists to dynamic boolean vectors entails changing **`Eq`** to **`Ix`** (indexable) in the type signatures of set operators. (The constraint is changed, not added, because **`Ix`** is a sub-type-class of **`Eq`**.) Client functions using set operators are affected by constraint propagation if their type signatures are given explicitly and contain an **`Eq`** constraint on set payload. Grappling with this problem leads to solutions based on constraint or class parametrisation (Section 6.2 and Section 6.1.10).

5.8.3 The SML module system

ADTs are also implemented in Standard ML with the help of modules. SML has a more sophisticated module system with a sound theoretical basis (which employs the machinery of dependent types) [Tof96, Pau96, DT88]. An SML module is an abstract concept in terms of which SML programs can be structured. This section overviews the linguistic constructs whose manipulation and combination make up the notion of SML module.

An ***SML signature*** bundles a set of type names, type signatures, exceptions, and SML structure names under the same scope. Signatures play the role of specifications or interfaces, *i.e.*, a signature is the module-level equivalent of the 'type' of an SML

structure.

An ***SML structure*** bundles a set of type and value *definitions* under the same scope. In short, a signature declares a module's interface whereas the structure declares its implementation. This model is similar to Ada's and Modula/2's.

Figure 5.9 shows the signature `Stack` and a possible structure `S1` implementing `Stack`. In SML, whether a function raises an exception is not reflected on its type signature, hence the comments.

```
signature Stack = sig
  type α t
  exception Empty
  val empty : α t
  val push  : α * α t → α t
  val tos   : α t → α    (* raises Empty *)
  val pop   : α t → α t  (* raises Empty *)
end

structure S1 :> Stack = struct
  type α t = α list
  exception Empty
  val empty       = []
  fun push (x,s)  = x::s
  fun tos []      = raise Empty
    | tos (x::xs) = x
  fun pop []      = raise Empty
    | pop (x::xs) = xs
end
```

Figure 5.9: Signature `Stack` and structure `S1` implementing `Stack`.

Signatures and structures lack laws. A structure *conforms* to a signature when it provides definitions for all the signature's types, values, and structure names. SML is an implicitly typed language and the compiler is expected to infer the signature of a structure. By default, the inferred signature contains the types of all declared items in the structure. Programmers may impose structure-to-signature conformance explicitly. This is what `:>` denotes in Figure 5.9.

Structures do not hide information: 'declaring a structure hardly differs from declaring its items separately, except that a structure declaration is taken as a unit and introduces compound names' [Pau96]. Information hiding is achieved by omitting private items from the signature; hence, a structure may contain more items than specified by a

signature.

Signatures are not type-terms and structures are not values: in SML, type-terms and terms are separated; therefore, an entity that encompasses both kinds of terms lives at a different linguistic level. Structure values are created and manipulated at compile/link time, where 'type-level computation' amounts to enforcing scope and visibility plus type checking of constituent items and signature conformance.

Signatures can be combined in various ways to form new signatures and similarly for structures. SML's module system goes a bit further and permits the definition of parameterised modules, surprisingly called 'functors'. An ***SML functor*** takes a structure that conforms to a signature and creates another structure as a result, which conforms to another signature. Functors can be type-checked and compiled to machine code before applied to their arguments. Figure 5.10 shows a functor example. The `SET` signature declares the interface of a set. The `EQ` signature declares the interface of a type with equality. Functor `LIST_SET` takes a structure conforming to `EQ` and returns a structure conforming to `SET` where sets are implemented as lists.

5.9 Classification of operators

Section 5.7 has already introduced some operator terminology, and we have been using some of it when calling an operator a 'constructor' or a 'selector'. We now provide a more detailed classification according to the role operators play in specifications. This classification is used by subsequent chapters.

There are two major operator groups [Mar98, p189-190]: ***constructors*** and ***observers***. Constructors generate values of the type. They can be free or non-free (*i.e.*, there are or there are not equations among them). We have already seen some examples to which we add some more:

```
emptyS : → Stack
enq    : Elem → Fifo  → Fifo
insert : Elem → Set → Set
mkTreeNode : Elem → Tree → Tree → Tree
```

There can be ***multiple constructors***. A typical example is the type of double-ended queues where we can queue values at the front and at the rear of the queue. Lists can be constructed using `nil` and `cons`, `nil` and `snoc`, `nil` and `singleton` and `concat`,

```
signature EQ = sig
 type t
 val eq : t * t → bool
end

signature SET = sig
 type Set
 type Elem
 val empty   : Set
 val isEmpty : Set → bool
 val insert  : Elem * Set → Set
 val member  : Elem * Set → bool
 val card    : Set → int
end

functor LIST_SET (Element : EQ) : SET = struct
 type Elem = Element.t
 datatype ListSet = Nil | Cons of Elem * ListSet
 type Set  = ListSet
 val empty = Nil
 fun isEmpty Nil = true
   | isEmpty _   = false
 ...
end
```

Figure 5.10: SML Functor example.

etc.

The second group of operators is that of ***observers***. Two important subgroups are (boolean) ***discriminators***, which enquire about which constructor has created the value, and (partial) ***selectors*** which enable the selection (extraction, if you will) of data components. Examples of discriminators are:

```
isEmptyS : Stack → Bool
isEmptyQ : Fifo → Bool
isLeaf   : Tree → Bool
```

Examples of selectors are:

```
head : List → Elem
tail : List → List
tos  : Stack → Elem
pop  : Stack → Stack
leaf : Tree → Elem
```

```
leftTree : Tree → Tree
lookup   : Key → Table → Elem
```

Removal operators are selectors. There are two kinds of removal: ***implicit*** and ***explicit***. For example:

```
implicit : ADT → (Elem, ADT)
explicit : Elem → ADT → ADT
```

Implicit removals are usually decomposed into as many selector operators as necessary. For example, one selector returns an element and another returns a new ADT value with that element removed. The element is not an argument in either case, for its 'location' is fixed, *e.g.*:

```
front : Fifo → Elem
deq   : Fifo → Fifo
```

Explicit removal consists typically of a single selector that takes the element to remove as an argument. For example:

```
remove : Elem → Set → Set
```

Selectors are also called ***modifiers*** by some authors. Others use the term ***destructor***. We deprecate the latter for it has deallocation connotations that are irrelevant in the garbage-collected functional world.

Interrogators enquire about properties of the type. Examples are:

```
member      : Elem → Set → Bool
cardinality : Set → Int
```

Finally, ***enumerators*** carry payload contents to a different type, usually a concrete one. For example:

```
enumerate : Set → List
```

5.10 Classification of ADTs

This section provides a classification of ADTs which is used by subsequent chapters.

An ADT is ***unbounded*** when the internal arrangement of payload elements is independent of any payload property. In other words, the structure of an unbounded ADT

is described by the *number* and *position* of payload elements, where by position we mean the location of an element in the *abstract* type, not its implementation type.

Examples of unbounded ADTs are stacks, FIFO queues, double-ended queues, lists, arrays, matrices, etc. Notice that unbounded ADTs may have constrained payload. A FIFO queue can be constrained to payload with equality but only to make queue equality decidable. Arrays can be constrained to indexable payload, but in a referentially-transparent world the position of an element is given by an indexing function and is therefore constant; new array values are created by providing new indexing functions.

An ADT is ***bounded*** when the internal arrangement of payload depends on a property of the payload. More precisely, there are context-dependent laws such as ordering, lack of ordering, repetition, etc, which affect the position of payload elements and whose conformance may impose a constraint on the payload. What is more, the position of an element may be irrelevant.

Examples of bounded ADTs are sets, bags, ordered sets, ordered bags, binary search trees, heaps, priority queues, random queues, hash tables, dictionaries, etc. Sets are constrained on payload with equality not only to decide set equality but to be able to *define* the ADT: sets do not have repeated elements and set membership has to be implemented. (Sets are typical examples of insensitive types where the position of elements, being irrelevant, cannot be used in their characterisation). Ordered sets require an order relation on their payload. The payload of hash tables and dictionaries must be, respectively, 'hashable' and 'keyable'.

Finally, an ADT is ***mixed*** when it is both bounded and unbounded. Examples of mixed ADTs are composites of two ADTs where one is bounded and the other is unbounded. Such ADTs may have multiple payload types.

In practical programming, unbounded ADTs are general and bounded ADTs are specific with respect to applications. A very appealing quality of bounded ADTs is that constructors are 'smart': they take care of inserting payload within the abstract structure.

There are other possible ways of classifying ADTs. For instance, based on their *implementation* (which is usually the standard classification), on their *purpose* [Bud02, p387], on the *efficiency* of foremost operators (*e.g.*, searching a list takes linear time,

searching a random access list takes constant amortised time, searching a vector takes constant time...), etc.

Part II

Functional Polytypic Programming and Data Abstraction

Chapter 6
Structural Polymorphism in Haskell

> Domain Structure Principle: *The structure of a function is determined by the structure of its domain.* [Mac90, p202]

Structural polymorphism, or polytypism if you like, pays justice to the dictum "the structure of the problem immediately suggests the structure of the solution and the structure of the data type immediately suggests the structure of each function" [FH88, p41]. In this chapter we examine two popular language extensions for doing polytypic programming in Haskell: *Generic Haskell* [Hin00, Löh04, HJ02] and *Scrap your Boilerplate* [LP03, LP04, LP05]. The latter can be understood as a blend of polytypic and strategic programming techniques [VS04, LVV02], which are also explained.

It is surprising that the Domain Structure Principle quoted above has been around and uttered *ad nauseam*, yet nobody has tried to capture the general pattern until recently.

6.1 Generic Haskell

Generic Haskell is a polytypic language extension of Haskell. It comes in two flavours, ***classic*** and ***dependency-style***. In terms of expressibility, dependency-style supersedes classic style. The latter's principles can be summarised in a few paragraphs. Subsequent sections spell out what these paragraphs mean in more detail:

> A polytypic function is structurally polymorphic on *one*[1] type operator argument of any kind. It captures a family of polymorphic or overloaded functions on that type operator.
>
> The type of each member of the family (or instance) depends on the *kind* of the type-operator argument. All the instance's types can be captured in a single inductive definition which provides a type for the polytypic function. This type is a ***polykinded*** or ***kind-indexed type*** because it is parametric on the type operator's kind [Hin02].

[1] Multiple type-operator arguments are a difficult extension [Löh04, p207].

> The body of each member of the family depends on the definitional structure of the type-operator argument, *i.e.*, its structure in terms of sums of products of base types, type variables, and applications of type operators. All the instance's bodies can be captured in a single inductive definition and, what is more, only the base case of the induction has to be provided.

A polytypic function is somewhat ambiguously called a ***type-indexed value***. The name is ambiguous because polymorphic functions are also type-indexed, *i.e.*, parametric on a type as made explicit in System F (Section 2.7.3). Moreover, polytypic functions are not first-class *values* in Generic Haskell (Section 6.1.9).

Dependency-style Generic Haskell has a type system for keeping track of dependencies between polytypic functions—other polytypic functions called by a polytypic function appear on the latter's type signature (notice the potential impact on maintainability). However, polykinded types are still used internally by the compiler [Löh04, p104]. Dependency-style also provides more flexible notation and supports *polytypic extension*, which endows polytypic functions with non-monotonic behaviour for particular types, and *polytypic types* [HJL04], *i.e.*, types that are parametric on the definitional structure of other types.

Generic Haskell is a language extension with a generative implementation. More precisely, the Generic Haskell compiler is a pre-processor that generates polymorphic instances of polytypic functions for actual type-operator arguments. In order to generate the instances, type-operator arguments must be known at compile time (Section 4.7.1).

6.1.1 Algebraic data types in Haskell

In most functional languages, user-defined data types are ***algebraic***. An algebraic data type definition simultaneously introduces entities at the type and value level. In Haskell this is done in a **`data`** declaration of the form:

$$\texttt{data}\ Q \Rightarrow T\ a_1 \ldots a_n\ =\ C_1\ \tau_{11} \ldots \tau_{1k_1}\ \mid\ \ldots\ \mid C_m\ \tau_{m1} \ldots \tau_{mk_m}$$

where n and k may be 0, $m > 0$, subscripted a symbols stand for type variables, C symbols for value constructors, T is a type name, subscripted τ symbols are type-terms, and Q stands for a qualification or context, *i.e.*, a list of type-class constraints on some of T's type-variable arguments.

At the type level, a **data** definition introduces a new type-operator name T with global (*i.e.*, module) scope that becomes part of the lexicon of type-terms. Types can be monomorphic (manifest) or parametrically polymorphic (type-level functions mapping types to types). Type operators can be higher-order (or higher-kinded). The context Q restricts the range of type-variable arguments to types belonging to the specified type classes. Type definitions can be recursive or mutually recursive. Type-terms are kind-checked and evaluated to manifest types at compile time. Types and values are separated.

The syntax of a **data** definition restricts types to be constructed in terms of *disjoint sums*[2] of *cartesian products* of primitive types like **Int** or **Bool**, type variables, and other manifest types, *i.e.*, monomorphic or fully applied type operators including the predefined 'function space' type operator '$\rightarrow$'. Records can also be defined in a **data** construct but, at the time of writing, Haskell lacks a universally accepted record system. In Haskell 98, records are syntactic sugar for products with labelled fields where labels have global scope. At any rate, records are ignored as data-definition mechanisms by most polytypic language extensions which assume a world of algebraic data types (*i.e.*, sums of products).

At the value level, a **data** definition introduces a set of value constructors which are special value-level functions that become part of the lexicon of terms. Value constructors are differentiated from ordinary functions syntactically: they have capitalised names. Value-constructor names within the same module scope must differ. A value constructor is introduced for each of the sum's alternatives. Value constructors are special in the sense that they provide the means for both *constructing* and *representing* values of the type. The application of value constructor C to arguments $t_{11} \ldots t_{1k_1}$ of types $\tau_{11} \ldots \tau_{1k_1}$ respectively does not involve a function call but at most the evaluation of the arguments themselves. In languages with *lazy constructors* such as Haskell, the term $C\ t_{11} \ldots t_{1k_1}$ is a value of the algebraic type; in languages with *eager constructors*, the term $C\ v_{11} \ldots v_{1k_1}$ is a value of the algebraic type, where v_{ij} is the value of t_{ij}.

Because of this representation role and because of the lack of relations (*e.g.*, equations) between value constructors, the latter can be used in definitions of functions by ***pattern matching*** (Chapter 8). A function defined by pattern matching is defined over the

[2] 'Disjoint' is sometimes replaced by 'discriminated' and 'sum' by 'union', but the concept is the same.

definitional structure of a value of an algebraic type: value constructor names act as tags that are the means of discriminating (doing case analysis)—*i.e.*, "has this algebraic type been constructed using value constructor C_1 or C_2 or ...?"—and their arguments are matched against *pattern expressions* that select the appropriate product components. Definitions by pattern matching are translated by the compiler into case terms of the core language (*e.g.* Figure 2.3). The following is an example of a function defined by pattern matching:

```
length :: List a → Int
length Nil         = 0
length (Cons x xs) = 1 + length xs
```

In type-theoretic jargon, construction, representation without equations, and pattern matching are technically called *introduction*, *freeness*, and *elimination* [Pie02, Sch94].

Nullary value constructors (with empty product) of polymorphic type operators are ***polymorphic values*** whose type depends on the context (term) where they occur. A typical example is `Nil` which has type ∀`a`:∗.`List a`.

Algebraic data types take the adjective 'algebraic' from the fact that they constitute a *free algebra* (Chapter 5): the set of values is defined entirely by means of operators (value constructors) applied to arguments and nothing more.

Figure 6.1 shows a few examples of type definitions and their kinds. We use the notation `t:k` to state that type `t` has kind `k` and retain Haskell's notation `v::t` to state that value `v` has type `t`. The type operator `GTree` is higher-order; it is not only parametric on the elements it can store—its ***payload***—, but also on other type operators that provide the ***shape*** for its recursive substructure. For instance, in a `GTreeList` the children of a `GNode` come in a `List` whereas in a `GTreeFork` they come in a `Fork`. We could have defined their instantiations directly:

```
data GTreeList a = GEmpty | GLeaf a | GNode (List (GTreeList a))
data GTreeFork a = GEmpty | GLeaf a | GNode (Fork (GTreeFork a))
```

The type operator `GTree` is a generalisation that abstracts on (*i.e.*, is parametric on) the type operator applied to the recursive application:

```
data GTree f a = GEmpty | GLeaf a | GNode (f (GTree f a))
GTree  : (∗ → ∗) → ∗ → ∗
```

```
data List a       = Nil | Cons a (List a)
data BTree a b    = Empty | Leaf a | Node b (BTree a b) (BTree a b)
data GTree f a    = GEmpty | GLeaf a | GNode (f (GTree f a))
data BGTree f a b = BGEmpty | BGLeaf a | BGNode b (f (BGTree f a b))
data Fork a       = Fork a a
data BList a      = BNil | BCons a (BList (Fork a)) -- irregular

-- Instantiations
type TreeCharInt = BTree Char Int
type ArithExp    = BTree Int (Int → Int → Int)
type GTreeList   = GTree List
type GTreeFork   = GTree Fork

t1 :: GTreeList Int -- GTree List Int
t1  = GNode [GLeaf 3, GNode [GLeaf 2, GEmpty]]
t2 :: GTreeFork Int -- GTree Tree Int
t2  = GNode (Fork (GLeaf 3) (GNode (Fork (GLeaf 2) GEmpty)))
```

```
Int            : ∗
List           : ∗ → ∗
List Int       : ∗
List Char      : ∗
BTree          : ∗ → ∗ → ∗
BTree Char     : ∗ → ∗
BTree Char Int : ∗
GTree          : (∗ → ∗) → ∗ → ∗
GTree List     : ∗ → ∗
GTree List Int : ∗
BGTree         : (∗ → ∗) → ∗ → ∗ → ∗
BList          : ∗ → ∗
```

Figure 6.1: Some data types and their kinds in Haskell.

```
GEmpty :: ∀ f:∗→∗. ∀ a:∗. GTree f a
GLeaf  :: ∀ f:∗→∗. ∀ a:∗. a → GTree f a
GNode  :: ∀ f:∗→∗. ∀ a:∗. f (GTree f a) → GTree f a
```

The type operator `BList` is an example of an ***irregular*** type. A recursive type operator is irregular when it is recursively applied in its definition to arbitrary well-kinded type-terms, not just type variables. Some authors call these types ***nested*** [BM98] and others ***non-uniform*** [Oka98a]. Irregular types are arrived at by a process of *data-structural bootstrapping* [Oka98a, Chp10]. They capture structural invariants within the data type itself that otherwise would have to be maintained by external operations. These invariants are enforced by the type system when it checks for term well-formedness.

Also, irregular types may provide more efficient representations. For example, the type `BList` is recursively applied to `Fork a` instead of `a`, which means that every recursive substructure is a pair of `BList`, thus obtaining the type of balanced trees as shown in Figure 6.2.

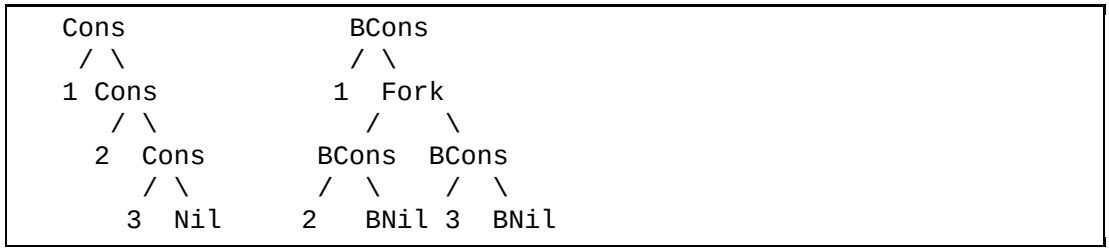


Figure 6.2: `List` vs `BList`.

Functions on irregular types must be ***polymorphically recursive*** [Myc84, Hen93], *i.e.*, each recursive call may have a different type than that of the function. For instance, the length function for `BList` has type `BList a` → **`Int`** whereas its recursive call must have type `BList (Fork a)` → **`Int`**.

6.1.2 From parametric to structural polymorphism

A polytypic function captures a whole family of polymorphic or overloaded functions in a single definition. Let us recall the function that returns the length of a list:

```
length :: List a → Int
length Nil = 0
length (Cons x xs) = 1 + length xs
```

We can think of similar functions for other type operators such as `BTree` and `GTree`. All these functions are instances of a more general, polytypic `gsize` function that returns the 'size' of an arbitrary type operator. What is the type of `gsize`?

```
gsize :: ? → Int
```

One of the insights behind Generic Haskell is that the type of `gsize` depends on the kind of the type operator it works on. Let us write the type signatures and bodies of `gsize` instances starting from a kind-$*$ type operator and moving all the way up the kind hierarchy. For a manifest type `t`, `gsize` must be a constant function:

```
∀ t:∗. gsize_t :: t → Int
```

Possible examples are:

```
gsize_Int  :: Int → Int
gsize_Int   = const 0
gsize_Char :: Char → Int
gsize_Char  = const 0
```

Giving base types a zero size is a design decision whose rationale will be understood shortly.

For a type operator of kind $* \to *$ like `List`, instead of hardwiring the computation of the size attributed to the payload in the function definition we should generalise and pass a function to do the job. Compare **`length`**'s definition to `gsize_List`'s below. (We depart slightly from Haskell's standard syntax and show $\forall$ and kind annotations in type signatures of members of the `gsize` family.)

```
length :: List a → Int
length Nil         = 0
length (Cons x xs) = 1 + length xs

gsize_List :: ∀ a:∗. (a → Int) → (List a → Int)
gsize_List gsa Nil          = gsize_Unit Nil
gsize_List gsa (Cons x xs)  = gsa x + gsize_List gsa xs
gsize_Unit = const 0   -- fixed!
```

Notice that the size for values of `Unit` type is fixed, like those for kind-$*$ types. We can define the original length function as a particular case of `gsize_List`:

```
length  = gsize_List (const 1)
```

We can compute other notions of length by playing with function arguments:

```
ords :: List Char → Int
ords  = gsize_List ord
```

Let us repeat the same process for type operator `BTree` of kind $* \to * \to *$:

```
size_Tree :: BTree a b → Int
size_Tree Empty        = 0
size_Tree (Leaf x)     = 1
size_Tree (Node x l r) = 1 + size_Tree l + size_Tree r
```

```
gsize_Tree :: ∀ a:∗. (a → Int) →
              (∀ b:∗. (b → Int) → (BTree a b → Int))

gsize_Tree gsa gsb Empty        = gsize_Unit Empty
gsize_Tree gsa gsb (Leaf x)     = gsa x
gsize_Tree gsa gsb (Node x l r) = gsb x + gsize_Tree gsa gsb l
                                        + gsize_Tree gsa gsb r
```

Again, the `gsize` version takes an argument function for every type variable and the definition for units is fixed. Let us now write the `gsize` instance for higher-order type operator `GTree`. The argument function `gsf` associated with type variable `f` is a parametrically polymorphic function whose type is that of a `gsize` for a type operator of kind $* \to *$:

```
gsize_GTree
 :: ∀ f:∗→∗. (∀ a:∗. (a → Int) → (f a → Int)) →
              (∀ a:∗. (a → Int) → (GTree f a → Int))

gsize_GTree gsf gsa GEmpty    = gsize_Unit GEmpty
gsize_GTree gsf gsa (GLeaf x) = gsa x
gsize_GTree gsf gsa (GNode y) = gsf (gsize_GTree gsf gsa) y
```

There is an inductive pattern here which can be gleaned by looking carefully at the type signatures:

```
∀ t:∗. gsize :: t → Int

∀ t:∗→∗. gsize :: ∀ a:∗. (a → Int) → (t a → Int)

∀ t:∗→∗→∗.
 gsize :: ∀ a:∗. (a → Int) → (∀ b:∗. (b → Int) → (t a b → Int))

∀ t:(∗→∗)→∗→∗.
   gsize :: ∀ t':∗→∗. (∀ a:∗. (a → Int) → (t' a → Int)) →
                      (∀ b:∗. (b → Int) → (t t' b → Int))
```

Each version of `gsize` takes a function for each of the type operator's type variables. The *number* of function arguments is determined by the arity of the type operator and the *type signature* of each function argument depends on the kind of the type variable.

That is, the type of each `gsize` is determined by the type operator's kind inductively from the type of the kind-$*$ case. This is captured by a so-called polykinded type:

```
type Size⟨∗⟩ t = t → Int
type Size⟨k→v⟩ t = ∀ a. Size⟨k⟩ a → Size⟨v⟩ (t a)
```

The notation `Size⟨k⟩ t` indicates that `Size` is a polykinded type defined by induction on the kind `k` of type operator `t`. The polykinded type is specialised for every actual type-operator argument to get the types of `gsize` instances. Figure 6.3 illustrates this process for `t` = `GTree`. (We have renamed type variables in the figure to avoid variable shadowing.)

```
Size⟨(∗→∗)→∗→∗⟩ GTree
 = ∀ f. Size⟨∗→∗⟩ f → Size⟨∗→∗⟩ (GTree f)
 = ∀ f. (∀ a. Size⟨∗⟩ a → Size⟨∗⟩ (f a)) →
        (∀ a. Size⟨∗⟩ a → Size⟨∗⟩ ((GTree f) a))
 = ∀ f. (∀ a. (a → Int) → f a → Int) →
        (∀ a. (a → Int) → GTree f a → Int)
```

Figure 6.3: Specialisation of polykinded type `Size⟨k⟩ t` where `t` is `GTree` and therefore `k` is `(∗→∗)→∗→∗`.

Let us move on to the function bodies. In each case, the particular instance of `gsize` is defined by pattern matching on the type operator's definitional structure in terms of sums of products. There is one line for each sum, *i.e.*, for each value constructor. Nullary products are considered equivalent to values of `Unit` type. The total size of a proper product is calculated by adding the sizes of each product component: the size of elements whose type is given by a type variable is computed by the function arguments; the size of recursive substructures is computed by recursive calls. Although not shown in the examples, components of other types such as base types or user-defined types would have their particular instances of `gsize` applied to them.

Consequently, the body of each `gsize` instance is determined by the type operator's definitional structure. (This motivates a pun in 'poly-typical': a family of *many typical* or expected functions.) There remains to collect all the bodies into a polytypic definition of a single `gsize` function. This function would compute the size of an arbitrary type operator argument of arbitrary kind. For a particular type operator, say `List`, a call to polytypic `gsize`:

```
gsize⟨List⟩ (const 1) [1,2,4]
```

where angle brackets denote ***polytypic application***, "should amount" to a call to its particular instantiation:

```
gsize_List (const 1) [1,2,4]
```

Another insight behind Generic Haskell is that polytypic function bodies need not be defined by a fully-fledged inductive definition; it suffices to define the base case of the induction, *i.e.*, the behaviour of the function for kind-$*$ types, *binary* sums, and *binary* products. The Generic Haskell compiler "automatically takes care of type abstraction, type application, and type recursion ... in a type-safe manner" [HJ02, p13]. We should also add that it automatically takes care of translating n-ary sums and products into compositions of right-associative binary ones. This translation is a *design choice* that plays an important role in understanding the instantiation process and the behaviour of polytypic functions. Other representations are possible—*e.g.*, left associativity—and have an effect on the semantics of the function: "most [polytypic] functions are insensitive to the translation of sums and products. Two notable exceptions are [encoding and decoding] for which [another representation] is preferable" [HJ02, p47].

Generic Haskell forces programmers to know the mechanics of the instantiation process, in particular, the internal way of encoding products and sums, for that determines the structure and therefore the body and semantics of the instance. At the time of writing, there are some attempts at letting programmers specify associativity explicitly and to allow types to be treated (*viewed*) as other types [HJL05].

In order to define polytypic function bodies, a canonical representation of a type operator's definitional structure is needed. It is at this point that ***representation types***, called ***structure types*** in the Generic Haskell literature, are introduced. A representation type provides a canonical encoding of a type in terms of compositions of binary sums of binary products of base types and type-operator applications. Representation types are built on the following ***base representation types***:

```
data Unit = Unit
data Sum a b = Inl a | Inl b
type Pro a b = (a,b)
type Fun a b = (→) a b
```

We ignore function space (`Fun`) in the following examples and defer its discussion until Section 6.1.4. For readability we will write `a`+`b` for `Sum a b` and `a`×`b` for `Pro a b`. As

previously explained, the Generic Haskell compiler follows the convention that + and × are right associative.

Figure 6.4 shows examples of type operators represented by structurally *isomorphic* representation type operators. Let us ignore for the moment the fact that type-synonym declarations (keyword **type**) cannot be recursive. We will provide a better definition of representation types shortly.

```
data List   a   = Nil | Cons a (List a)
type List'  a   = Unit + (a × (List' a))

data BTree  a b = Empty | Leaf a | Node b (BTree a b) (BTree a b)
type BTree' a b = Unit + (a + (b × ((BTree' a b) × (BTree' a b))))

data GTree  f a = GEmpty | GLeaf a | GNode (f (GTree f a))
type GTree' f a = Unit + (a + (f (GTree' f a)))

data BList  a   = BNil | BCons a (BList (Fork a)) -- irregular
type BList' a   = Unit + (a × (BList' (Fork a)))
```

Figure 6.4: Some type operators and their respective *representation* type operators. These definitions are not legal Haskell 98: type synonyms cannot be recursive.

Value-constructor names are dropped but the actual representation type used by the Generic Haskell compiler includes information about value constructors (names, fixity, arity, etc.), which is essential for programming polytypic serialisers such as pretty-printers.

Polytypic functions are defined on base types and base representation types as shown in the first box of Figure 6.5. The interesting cases are the sum and product ones. The size of a sum `a+b`, whatever `a` and `b`, within a representation type is computed by pattern matching on whether the value of the sum type is an `Inl` or `Inr` and then by applying the appropriate argument function to it. The size of a binary product `a×b`, whatever `a` and `b`, within a representation type is computed by pattern matching on product components and adding their sizes, which are computed by the argument functions. The second box in Figure 6.5 is syntactic sugar for the box above where the recursion is made explicit: the argument functions `gsa` and `gsb` are instances of `gsize` for the types `a` and `b`.

```
gsize⟨t:k⟩ :: Size⟨k⟩ t
gsize⟨Char⟩ = const 0
gsize⟨Int⟩  = const 0
gsize⟨Bool⟩ = const 0
gsize⟨Unit⟩ = const 0
gsize⟨a+b⟩ gsa gsb (Inl x) = gsa x
gsize⟨a+b⟩ gsa gsb (Inr y) = gsb y
gsize⟨a×b⟩ gsa gsb (x,y)   = gsa x + gsb y
```

```
gsize⟨t:k⟩ :: Size⟨k⟩ t
gsize⟨Char⟩ = const 0
gsize⟨Int⟩  = const 0
gsize⟨Bool⟩ = const 0
gsize⟨Unit⟩ = const 0
gsize⟨a+b⟩ (Inl x) = gsize⟨a⟩ x
gsize⟨a+b⟩ (Inr y) = gsize⟨b⟩ y
gsize⟨a×b⟩ (x,y)   = gsize⟨a⟩ x + gsize⟨b⟩ y
```

Figure 6.5: Polytypic `gsize` with implicit and explicit recursion.

Figure 6.6 illustrates the instantiation process. The first box shows instances of `gsize` for base types, units, and binary products and sums. All are generated directly from the polytypic definition. Their type signatures are obtained by instantiating the polykinded type `Size`.

The remaining boxes show the instances of `gsize` for representation type operators `List'`, `BTree'`, and `GTree'`. These instances follow the type-operator's definitional structure to the letter: an argument function is passed for each of the type-operator's type variables; the occurrence of a base type in the representation type translates to a call to its `gsize` instance in the body of the function; the occurrence of a sum translates to a call to `gsize_Sum`; the occurrence of a product to a call to `gsize_Pro`; type-operator application translates to function application.

As an improvement, the type for kind-$*$ type operators can be expressed by a type synonym and the type signatures of all the instances can be written in terms of it as shown in Figure 6.7.

There are two problems with the scheme presented. First, `gsize` has been defined on `List'` not `List`. The generated instantiation is `gsize_List'`, not `gsize_List`. And similarly for the other type operators. Second, the type synonyms in Figure 6.4 are invalid.

```
gsize_Char :: Char → Int
gsize_Char  = const 0

gsize_Bool :: Bool → Int
gsize_Bool  = const 0

gsize_Int :: Int → Int
gsize_Int  = const 0

gsize_Unit :: Unit → Int
gsize_Unit  = const 0

gsize_Sum :: ∀ a. (a → Int) → (∀ b. (b → Int) → Sum a b → Int)
gsize_Sum gsa gsb (Inl x) = gsa x
gsize_Sum gsa gsb (Inr y) = gsb y

gsize_Pro :: ∀ a. (a → Int) → (∀ b. (b → Int) → Pro a b → Int)
gsize_Pro gsa gsb (x,y)   = gsa x + gsb y
```

```
type List' a = Unit + a × (List' a)

gsize_List' :: ∀ a. (a → Int) → List' a → Int

gsize_List' gsa
  = gsize_Sum gsize_Unit (gsize_Pro gsa (gsize_List' gsa))
```

```
type BTree' a b = Unit + a + b × (BTree' a b) × (BTree' a b)

gsize_BTree'
  :: ∀ a. (a → Int) → (∀ b. (b → Int) → BTree' a b → Int)

gsize_BTree' gsa gsb =
  gsize_Sum gsize_Unit
    (gsize_Sum gsa (gsize_Pro gsb
        (gsize_Pro (gsize_BTree' gsa gsb) (gsize_BTree' gsa gsb))))
```

```
type GTree' f a = Unit + (a + (f (GTree' f a)))

gsize_GTree'
  :: ∀ f. (∀ a. (a → Int) → (f a → Int)) →
          (∀ a. (a → Int) → (GTree f a → Int))

gsize_GTree' gsf gsa
  = gsize_Sum gsize_Unit
              (gsize_Sum gsa (gsf (gsize_GTree' gsf gsa)))
```

Figure 6.6: Instantiations of `gsize`.

```
type Size t = t → Int

gsize_Int    :: Size Int
gsize_Unit   :: Size Unit
gsize_Sum    :: ∀ a. Size a → (∀ b. Size b → Size (Sum a b))
gsize_Pro    :: ∀ a. Size a → (∀ b. Size b → Size (Pro a b))

gsize_List'  :: ∀ a. Size a → Size (List' a)

gsize_BTree' :: ∀ a. Size a → (∀ b. Size b → Size (BTree a b))

gsize_GTree' :: ∀ f. (∀ a. Size a → Size (f a)) →
                     (∀ a. Size a → Size (GTree' f a))
```

Figure 6.7: Type signatures of the functions in Figure 6.6 written in terms of a type synonym.

The type synonym of Figure 6.7 will prove useful in understanding how the Generic Haskell compiler sorts out this problem. Before delving into that, let us (1) show some examples of usage, (2) two more examples of polytypic function definitions and (3) discuss some theoretical aspects of the approach in more detail (Section 6.1.3).

Let us suppose that recursive type synonyms were legal and assume that a call to `gsize` on a type operator T is somehow translated to a call on its representation type T'. Figure 6.8 shows examples of usage that illustrate the design objectives. Desired results are shown below each application.

In the figure, type `BTree Int Char` has kind $*$ and therefore there is no argument function passed to `gsize`. The type `BTree Int` has kind $* \to *$ and therefore there is an argument function passed to `gsize`. More precisely, the Generic Haskell compiler replaces the polytypic applications:

```
gsize⟨BTree Int Char⟩
gsize⟨BTree Int⟩
```

by calls to the instances `gsize_BTree_Int_Char` and `gsize_BTree_Int` whose definition has been generated as follows:

```
gsize_BTree_Int_Char :: BTree Int Char → Int
gsize_BTree_Int_Char = gsize_BTree gsize_Int gsize_Char
gsize_BTree_Int :: ∀ a. (a → Int) → BTree Int a → Int
gsize_BTree_Int gsa = gsize_BTree gsize_Int gsa
```

```
aTree :: BTree Int Char
aTree  = Node 'A' (Leaf 2) (Leaf 3)

gsize⟨BTree Int Char⟩ aTree
> 0

gsize⟨BTree Int⟩ (const 1) aTree
> 1

gsize⟨BTree⟩ (const 0) (const 1) aTree
> 1

gsize⟨BTree⟩ (const 1) (const 1) aTree
> 3

gsize⟨List⟩ (const 1) "hello world"
> 11

gsize⟨List⟩ ord "hello world"
> 1116
```

Figure 6.8: Examples of usage of polytypic `gsize`.

The size computed is zero because that is the size given to integers and characters in Figure 6.5, which make up the payload of the type. Had we defined `gsize` differently for base types then the computed size would differ. We come back to this in Section 6.1.12. Other examples in Figure 6.8 show how to obtain different sizes (*e.g.*, counting nodes or counting leaves in `BTree`s) by playing with function arguments.

Polytypic map and polytypic equality. Figure 6.9 shows the definitions of polytypic map and polytypic equality. The latter function shows that overloaded functions (implemented using type classes in Haskell) are also subject to generalisation in a polytypic definition. The polykinded type of polytypic map is at first sight confusing. According to what has been said so far it might appear to be:

```
type Map⟨∗⟩ t = t → t
type Map⟨k→v⟩ t = ∀ a. Map⟨k⟩ a → Map⟨v⟩ (t a)
```

However, this polykinded type is not general enough as shown by the following counter-example:

```
Map⟨∗→∗⟩ List = ∀ a. Map⟨∗⟩ a → Map⟨∗⟩ (List a)
              = ∀ a. (a → a) → List a → List a
```

```
type Map⟨∗⟩ t1 t2  = t1 → t2
type Map⟨k→v⟩ t1 t2  =
 ∀ a1 a2. Map⟨k⟩ a1 a2  → Map⟨v⟩ (t1 a1) (t2 a2)

gmap⟨t:k⟩ :: Map⟨k⟩ t t
gmap⟨Int⟩   =  id
gmap⟨Unit⟩  =  id
gmap⟨a+b⟩ (Inl x) = Inl (gmap⟨a⟩ x)
gmap⟨a+b⟩ (Inr y) = Inr (gmap⟨b⟩ y)
gmap⟨a×b⟩ (x,y)   = (gmap⟨a⟩ x, gmap⟨b⟩ y)

gmap⟨List⟩ chr [65, 66, 67]
> "ABC"

gmap⟨BTree⟩ ord chr (Node 'A' (Leaf 66) (Leaf 67))
> Node 65 (Leaf 'B') (Leaf 'C')
```

```
type Eq⟨∗⟩ t =  t → t → Bool
type Eq⟨k→v⟩ t = ∀ a. Eq⟨k⟩ a → Eq⟨v⟩ (t a)

geq⟨t:k⟩ :: Eq⟨k⟩ t
geq⟨Int⟩   =  (==)
geq⟨Unit⟩ Unit    Unit     =  True
geq⟨a+b⟩  (Inl x) (Inl x') =  geq⟨a⟩ x x'
geq⟨a+b⟩  (Inl x) (Inr y') =  False
geq⟨a+b⟩  (Inr y) (Inl x') =  False
geq⟨a+b⟩  (Inr y) (Inr y') =  geq⟨b⟩ y y'
geq⟨a×b⟩  (x,y)   (x',y')  =  geq⟨a⟩ x x' && geq⟨b⟩ y y'

geq⟨List⟩ (==) [1,2,4] [1,2,4]
> True

geq⟨List⟩ (==) [1,2,4] [1,2,3]
> False
```

Figure 6.9: Polytypic map, polytypic equality, and examples of usage.

For type operators of higher kinds we need two different universally-quantified variables, one for the source type and another for the mapped target type:

```
Map⟨∗→∗⟩ List List = ∀ a b. Map⟨∗⟩ a b → Map⟨∗⟩ (List a) (List b)
                   = ∀ a b. (a → b) → List a → List b
```

The polykinded `Map` type uses two type variables from the start, but the polykinded type of `gmap` is restricted to take the same type operator.

Two remarks are in order. First, polytypic map is not defined for function types: function space is contravariant on its first argument (Section 3.10) and its type signature therefore breaks the pattern captured by the polykinded type of polytypic map. Second, as its polykinded type indicates, polytypic map is slightly more general than a map. For example, the type signature of `gmap_GTree` is:

```
map_GTree :: ∀ f g. (∀ a b. (a → b) → (f a → g b)) →
             (∀ a b. (a → b) → GTree f a → GTree g b)
```

Categorially, `GTree` is a functor if it is functorial in the two arguments. More precisely, we can define two functions:

```
hmap :: (Functor f, Functor g) ⇒ Nat f g → Nat (GTree f) (GTree g)
fmap :: Functor f ⇒ (a → b) → GTree f a → GTree f b
```

where `Nat f g` is the type of natural transformations from functor `f` to functor `g`:

```
type Nat f g = ∀ a. f a → g a
```

(The naturality conditions follow from parametricity, *i.e.*, polymorphic functions are natural transformations in **Type**.) Functions `hmap` and `fmap` must satisfy the functorial laws:

```
hmap (μ ∘ η)          == hmap μ ∘ hmap η
hmap (id :: Nat f f) == (id :: Nat (GTree f) (GTree f))

fmap (i ∘ j)          == fmap i ∘ fmap j
fmap (id :: a → a)   == (id :: (GTree f) a → (GTree f) a)
```

where:

```
μ :: Nat g h          i :: b → c
η :: Nat f g          j :: a → b
```

Because it is possible to manufacture a function of type $f\ a \to g\ a$ from a function k of type $(a \to b) \to f\ a \to g\ b$, namely, $k \circ \mathrm{id}$, `map_GTree` must satisfy the following equation:

```
map_GTree k r == hmap (k ∘ id) ∘ fmap r
```

The body of `gmap` is straightforward. For base types (we only show integers and units) it is the identity. For sums and products it maps over the components. Examples of usage are also shown in the figure.

The polykinded type of `geq` is also straightforward: for kind $*$ it is a binary predicate on the type `t`. Regarding the body, equality for base types is standard equality; two sums are equal if they are both left or right components and their contents are equal; and two products are equal if their components are equal. Examples of usage are also shown.

Notice that we could have defined a more general polykinded type for equality following the spirit of polykinded type `Map` [HJ02]:

```
type Eq⟨∗⟩ t1 t2 = t1 → t2 → Bool
type Eq⟨k→v⟩ t1 t2 = ∀ a1 a2. Eq⟨k⟩ a1 a2 → Eq⟨v⟩ (t1 a1) (t2 a2)
```

where now `geq` would have to have polykinded type `Eq⟨k⟩ t t`. The type of its instance for `List'` would be:

```
geq_List' :: ∀ a b. (a → b → Bool) → List' a → List' b → Bool
```

Surprisingly, the body of polytypic `geq` need not change, so one wonders about what is its *most general* polykinded type and about the expressibility of polykinded type definitions. We come back to this in Section 6.1.5.

6.1.3 Generic Haskell and System F_ω

If instead of Haskell we were programming in System F_ω (Section 2.7.4), our overview of polytypic programming would have come to a conclusion. In System F_ω there is a structural equivalence relation between types. A type operator and its representation type are structurally equivalent and we would program with representation types directly. A polytypic extension of System F_ω would translate polykinded types and polytypic functions following the scheme so far, treating named definitions as sugar for

$$
\begin{array}{lcl}
\mathit{gsize\_Unit} & \stackrel{\text{def}}{=} & \lambda x{:}\texttt{Unit}\,.\,0 \\
\\
\mathit{gsize\_Int} & \stackrel{\text{def}}{=} & \lambda x{:}\texttt{Int}\,.\,0 \\
\\
\mathit{gsize\_Sum} & \stackrel{\text{def}}{=} & \Lambda\alpha{:}*\;.\;\lambda \mathit{gsa} : \alpha \to \texttt{Int}. \\
 & & \Lambda\beta{:}*\;.\;\lambda \mathit{gsb} : \beta \to \texttt{Int}. \\
 & & \lambda s{:}\alpha{+}\beta\;.\;\texttt{case}\;s\;\texttt{of}\;\texttt{Inl}\;x\;\texttt{then}\;\mathit{gsa}\;[\alpha]\;x\;;\;\texttt{Inr}\;y\;\texttt{then}\;\mathit{gsb}\;[\beta]\;y \\
\\
\mathit{gsize\_Pro} & \stackrel{\text{def}}{=} & \Lambda\alpha{:}*\;.\;\lambda \mathit{gsa} : \alpha \to \texttt{Int}. \\
 & & \Lambda\beta{:}*\;.\;\lambda \mathit{gsb} : \beta \to \texttt{Int}. \\
 & & \lambda p{:}\alpha{\times}\beta\;.\;\texttt{plus}\;(\mathit{gsa}\;[\alpha]\;(\texttt{fst}\;[\alpha]\;[\beta]\;p))\;(\mathit{gsb}\;[\beta]\;(\texttt{snd}\;[\alpha]\;[\beta]\;p)) \\
\\
\mathit{List} & \stackrel{\text{def}}{=} & \lambda\alpha{:}*\,.\,\mathbf{1} + \alpha \times (\mathit{List}\;\alpha) \\
\\
\mathit{gsize\_List} & \stackrel{\text{def}}{=} & \Lambda\alpha{:}*\;.\;\lambda \mathit{gsa}{:}\alpha \to \texttt{Int}\;. \\
 & & \quad \mathit{gsize\_Sum}\;[\texttt{Unit}]\;\mathit{gsize\_Unit} \\
 & & \quad\quad [\alpha \times (\mathit{List}\;\alpha)]\;(\mathit{gsize\_Pro}\;[\alpha]\;\mathit{gsa}\;[\mathit{List}\;\alpha]\;(\mathit{gsize\_List}\;[\alpha]\;\mathit{gsa}))
\end{array}
$$

Figure 6.10: Writing and using instances of `gsize` for lists in System F_ω. Type-terms in universal type applications are shown in square brackets.

lambda abstractions, with fixed points when there is recursion.

For example, the code in Figure 6.10 shows the instances of `gsize` for units, integers, binary sums, products, and lists. For readability, we have used meta-level recursive names and thus obviated the use of fixed-point operators. Also, to distinguish universal type applications from term applications more easily, each type argument in a universal application is written between square brackets.

The following code is an example of usage where *xs* abbreviates a value of type *List* `Int` which corresponds to the Haskell list value `[1,2]`:

$$
\begin{array}{l}
\mathit{xs} \stackrel{\text{def}}{=} \texttt{Inr}\;(1, \texttt{Inr}\;(2, \texttt{Inl}\;\texttt{unit})) \\
\mathit{gsize\_List}\;[\texttt{Int}]\;(\lambda x{:}\texttt{Int}\,.\,1)\;\mathit{xs} \\
> 2
\end{array}
$$

The generation of instances like *gsize_List* amounts to producing terms from type-terms, where the latter are described by a type-level STLC (Chapter 2). The programmer defines the translation for base types, units, sums, and products. The rest can be

taken care of automatically. A polytypic System F_ω compiler or interpreter assigns a System F_ω term (an instance of a polytypic function, *e.g.*, *gsize_List*) to a STLC term (a type-operator, *e.g.*, *List*) [HJ02, p42ff].

6.1.4 Nominal type equivalence and embedding-projection pairs

Among other things, Haskell differs from System F_ω in that type equivalence is nominal, not structural. In type systems with ***nominal type equivalence*** naming is not a derived form but an explicit and essential language construct. In such systems two types are equivalent if and only if they have the same name. For several reasons [Pie02, p251–254] nominal type equivalence is the norm in mainstream programming languages. It facilitates the treatment of recursion and plays an important role in enforcing data abstraction. For instance, the types:

```
data Debit  = Debit  Int
data Credit = Credit Int
```

are structurally equivalent but mistaking one for the other can have painful consequences. In Haskell the two types are different, a point underlined by the two different value constructors. In System F_ω, which lacks names, we would only make use of integers.

A representation type operator is structurally equivalent to many structurally isomorphic type operators, but in a nominal type system, programming with, say, a `List'` is not programming with a `List`.

Generic Haskell solves this problem by defining representation types in a different fashion where they only capture the top-level structure of type operators. There is one *embedding* function that translates a type operator to its representation type operator and a *projection* function that performs the converse. Figure 6.11 shows their definitions for the case of lists and binary trees. Embedding-projection functions also follow the structure of the data and their definitions are generated automatically by the Generic Haskell compiler for arbitrary type operators [HJ02, p46–47].

The instances of `gsize` have been defined for the illegal representation types of Figure 6.4. However, for kind-$*$ types and binary sums and products these instances need not change. They will work for the new representation types of Figure 6.11 in the

```
data List a  = Nil | Cons a (List a)
type List' a = Unit + (a × (List a))

from_List :: ∀ a. List a → List' a
from_List Nil         = Inl Unit
from_List (Cons x xs) = Inr (x,xs)

to_List :: ∀ a. List' a → List a
to_List (Inl Unit)   = Nil
to_List (Inr (x,xs)) = Cons x xs

data BTree  a b = Empty | Leaf a | Node b (BTree a b) (BTree a b)
type BTree' a b = Unit + (a + (b × ((BTree a b) × (BTree a b))))

from_BTree :: ∀ a b. Tree a b → Tree' a b
from_BTree Empty        = Inl (Inl Unit)
from_BTree (Leaf x)     = Inr (Inl x)
from_BTree (Node x l r) = Inr (Inr (x,l,r))

to_BTree :: ∀ a b. Tree' a b → Tree a b
to_BTree (Inl (Inl Unit))    = Empty
to_BTree (Inr (Inl x))       = Leaf x
to_BTree (Inr (Inr (x,l,r))) = Node x l r
```

Figure 6.11: Generic Haskell's representation types for lists and binary trees, together with their embedding and projection functions.

forthcoming scheme.

The idea is to find out what needs to be modified from the definition of `gsize_List'` in order to define `gsize_List`. First, a new `gsize_List'` must be generated for the new representation type of Figure 6.11 from the polytypic definition. The occurrence of type operator `List` in the representation type signals the place in the body where its `gsize` instance is to be called:

```
gsize_List' :: ∀ a. Size a → Size (List' a)
gsize_List' gsa = gsize_Sum gsize_Unit
                            (gsize_Pro gsa (gsize_List gsa))
```

Second, looking closely at the types of the functions:

```
gsize_List' :: ∀ a. Size a → Size (List' a)
gsize_List  :: ∀ a. Size a → Size (List a)
```

in order to define `gsize_List` the remaining ingredient would be a function:

```
foo :: ∀ a. Size (List' a) → Size (List a)
```

which would afford to tie the knot:

```
gsize_List gsa = foo (gsize_List' gsa)
```

That is:

```
gsize_List gsa = foo (gsize_Sum gsize_Unit
                                (gsize_Pro gsa (gsize_List gsa)))
```

Fortunately, the embedding-projection functions can help attain this goal:

```
from_List :: ∀ a. List a  → List' a
to_List   :: ∀ a. List' a → List a
```

It remains to define lifted versions of types:

```
bar :: ∀ a. Size (List a)  → Size (List' a)
foo :: ∀ a. Size (List' a) → Size (List a)
```

This lifting is performed by a special map function which must be *polytypic on the type synonym*, as the process must be repeated for other polytypic functions and type synonyms like equality:

```
type Eq a = a → a → Bool
geq_List :: ∀ a. Eq a → Eq (List a)

bar :: ∀ a. Eq (List a)  → Eq (List' a)
foo :: ∀ a. Eq (List' a) → Eq (List a)
```

The first box in Figure 6.12 defines a new data type `EP` and two selector functions for manipulating embedding-projection pairs conveniently. The second box defines instances of a special map function `mapEP` which is the map for `EP` values: for units and integers, embedding-projection pairs are identities; for sums and products, embedding-projection pairs can be obtained by mapping embedding-projection pairs associated with their components. Notice that the arrow type operator is contravariant on its first argument (Section 3.10), thus the definition of `mapEP_Fun`.

As usual, the type of the instance `mapEP_Size` is obtained from the polykinded type:

```
MapEP⟨∗→∗⟩ Size =
 ∀ t t'. MapEP⟨∗⟩ t t' → MapEP⟨∗⟩ (Size t) (Size t')
```

```
data EP a b = MkEP (a → b) (b → a)

from :: EP a b → (a → b)
from (MkEP f t) = f

to :: EP a b → (b → a)
to (MkEP f t) = t
```

```
mapEP_Int :: EP Int Int
mapEP_Int  = MkEP id id

mapEP_Unit :: EP Unit Unit
mapEP_Unit  = MkEP id id

mapEP_Pro :: EP a c → EP b d → EP (a × b) (c × d)
mapEP_Pro (MkEP f1 t1) (MkEP f2 t2)
  = MkEP (map× f1 f2) (map× t1 t2)

mapEP_Sum :: EP a c → EP b d → EP (a + b) (c + d)
mapEP_Sum  (MkEP f1 t1) (MkEP f2 t2)
  = MkEP (map+ f1 f2) (map+ t1 t2)

mapEP_Fun :: EP a c → EP b d → EP (a → b) (c → d)
mapEP_Fun (MkEP f1 t1) (MkEP f2 t2)
  = MkEP (map→ t1 f2) (map→ f1 t2)
```

```
type MapEP⟨∗⟩ t1 t2 = EP t1 t2
type MapEP⟨k→v⟩ t1 t2 =
  ∀ a1 a2. MapEP⟨k⟩ a1 a2 → MapEP⟨v⟩ (t1 a1) (t2 a2)
mapEP⟨t:k⟩ :: MapE⟨k⟩ t t
mapEP⟨Int⟩  = mapEP_Int
mapEP⟨Unit⟩ = mapEP_Unit
mapEP⟨a+b⟩  = mapEP_Sum
mapEP⟨a×b⟩  = mapEP_Pro
mapEP⟨a→b⟩ = mapEP_Fun
```

Figure 6.12: Embedding-projection pairs and polytypic `mapEP`.

```
∀ t t'. EP t t' → EP (Size t) (Size t')
```

Letting `t`=(`List a`) and `t'`=(`List' a`) would seal off the process. This is achieved by creating an embedding-projection pair value for lists:

```
ep_List :: ∀ a. EP (List a) (List' a)
ep_List  = MkEP from_List to_List
```

and by getting the body of `mapEP_Size` from its polytypic definition, which follows to the letter the definitional structure of type synonym `Size`:

```
mapEP_Size :: ∀ t t'. EP t t' → EP (Size t) (Size t')
mapEP_Size mapEPa = mapEP_Fun mapEPa mapEP_Int
```

Function `mapEP_Size` applied to `ep_List` gives us a value of type:

```
∀ a. EP (Size (List a)) (Size (List' a))
```

whose `to` component is our sought-after `foo`:

```
foo :: ∀ a. Size (List' a) → Size (List a)
foo = to (mapEP_Size ep_List)
```

We can now provide the definition of `gsize_List`:

```
gsize_List :: ∀ a. Size a → Size (List a)
gsize_List gsa = (to (mapEP_Size ep_List)) (gsize_List' gsa)
```

6.1.5 The expressibility of polykinded type definitions

The polykinded type `MapEP` and the polytypic function `mapEP` are predefined in the Generic Haskell prelude. This function in part determines the expressibility of a polykinded type, *i.e.*, what sort of polykinded types can be written. The Generic Haskell compiler collects all the polykinded type definitions and generates an instance of `mapEP` for the type synonyms generated from their kind-$*$ cases. As it currently stands, these synonyms can only contain applications of type operators to arguments. In [HJ02, p51], the definition of `mapEP` is extended so that, in general, polykinded types can have the form:

```
type T⟨∗⟩     t1 ... tn x ... z = B
type T⟨k→v⟩ t1 ... tn x ... z =
 ∀ a1 ... an. T⟨k⟩ a1 ... an x ... z → T⟨v⟩ (t1 a1) ... (tn an) x ... z
```

where B is a polymorphic type signature where universal quantification is permitted over type variables of kind $*$ and $* \to *$, and where `x...z` are auxiliary type variables that are passed untouched to the base case. These variables are universally quantified later when expressing the polykinded type of particular polytypic functions such as polytypic reduce, shown in Figure 6.13 (notice how a somewhat spurious function has to be passed to `freduce` in order to get a more meaningful type.)

```
type Reduce⟨∗⟩ t x = x → (x → x → x) → t → x
type Reduce⟨k→v⟩ t x = ∀ a. Reduce⟨k⟩ a x → Reduce⟨v⟩ (t a) x

reduce⟨t:k⟩ :: ∀ x. Reduce ⟨k⟩ t x
{- body of reduce not shown here

   ∀ x. Reduce⟨∗→∗⟩ List x
    =
   ∀ x. ∀ a. Reduce⟨∗⟩ a x → Reduce⟨∗⟩ (List a x)
    =
   ∀ x. ∀ a. (x → (x → x → x) → a → x) →
             x → (x → x → x) → List a → x
-}
freduce⟨t:∗→∗⟩ :: ∀ x. x → (x → x → x) → t x → x
freduce⟨t⟩ = reduce⟨t⟩ (λx y z → z)

fsum⟨t⟩    = freduce⟨t⟩ 0 (+)
fand⟨t⟩    = freduce⟨t⟩ True and
```

Figure 6.13: Polytypic reductions.

We should also mention that Generic Haskell supports ***polytypic types***, called ***type-indexed data types*** in the literature, *i.e.*, types whose definitional structure depends on the definitional structure of another type [HJL04]. Polytypic types may appear in the kind-$*$ case of a polykinded type. Instances of polytypic types are generated before generating the instances of the polykinded type and the polytypic function.

Polykinded types also allow type-class constraints in some situations. We defer this discussion until Section 6.1.10.

6.1.6 Polytypic abstraction

Figure 6.13 also shows two examples of ***polytypic abstraction*** (called 'generic abstraction' in the Generic Haskell literature) where instances of a polytypic function are defined for type operators of particular kinds only. Functions `fsum` and `fand` work on

type operators of kind $* \to *$ only. Another example of polytypic abstraction involving `gsize` follows:

```
flength⟨t:∗→∗⟩ :: t a → Int
flength⟨t⟩ = gsize⟨t⟩ (const 1)
length = flength⟨List⟩
```

6.1.7 The expressibility of polytypic definitions

Generic Haskell allows us to express ***catamorphisms*** such as `gsize` (Section 6.1.13). It even allows us to express more generic ones such as `freduce`, or even more generally, `reduce`. Notice that `reduce` is not quite a polytypic `gfoldr`. The type signature of **`foldr`** for a given type operator takes one argument per value constructor. The base case of the polykinded type of a polytypic `gfoldr` would not only depend on a type operator's kind but also on what is to the right of the equals in a data definition. Also, representation types contain binary products, not n-ary products which are the arguments taken by the functions that would be passed to `gfoldr`. For these reasons, `gfoldr` is not definable in Generic Haskell.

Let us illustrate the problems in more detail. The following code defines **`foldr`** in terms of a type `Alg` representing an operator algebra:

```
data Alg a b = Alg { nil :: b, cons :: a → b → b }
foldr :: Alg a b → List a → b
foldr alg Nil = nil alg
foldr alg (Cons x xs) = cons x (foldr alg xs)
```

Particular algebras are values of type `Alg`:

```
algSum = Alg{ nil = 0, cons = (+) }
algLen = Alg{ nil = 0, cons = λx y → 1+y }
```

The following is an example of usage:

```
foldr algSum [1,2,3]
> 6
foldr algLen [1,2,3]
> 3
```

It is common practice to write **`foldr`** `alg` as $(\!|\mathtt{v}\triangledown\mathtt{f}|\!)$ where `v` stands for `nil alg` and `f`

for `cons alg` [MFP91].

The instance of `gfoldr` for `List` should have the same type signature as **`foldr`**. But `Alg` not only depends on the kind-signature of `List`; it also depends on the number and type signature of `List`'s value constructors.

Even if we could define a *polytypic type* `Alg⟨t⟩` of which `Alg` for lists is an instance, the representation of n-ary products into associations of binary products becomes an issue. This is better illustrated in the case of `BTree`. When it comes to dealing with the tertiary product of a `Node`, `reduce` for `BTree`s takes two functions, one for each binary product into which the tertiary product is arranged:

```
data Alg a b c = Alg{ empty :: c,
                      leaf  :: a → c,
                      node  :: b → c → c → c}

gfoldr_BTree alg Empty          = empty alg
gfoldr_BTree alg (Leaf x)       = leaf alg x
gfoldr_BTree alg (Node x tl tr) = node alg x (gfoldr_BTree alg tl)
                                             (gfoldr_BTree alg t2)

reduce v l n1 n2 (Inl Unit)       = v
reduce v l n1 n2 (Inr (Inl x))    = l x
reduce v l n1 n2 (Inr (Inr (x,(tl, tr)))) = n1 x (n2 t1 tr)
```

In other words, the `gfoldr_BTree` generated would not be the one shown above, but one taking this algebra:

```
data Alg a b c = Alg{ v  :: c,
                      l  :: a → c,
                      n1 :: b → c → c,
                      n2 :: c → c → c}
```

This is what differentiates a *fold* from a *crush*. In Figure 6.13, polytypic reduce is a crush.

6.1.8 Summary of instantiation process

The Generic Haskell compiler collects all the polykinded type definitions `P⟨∗⟩ t` and generates type synonyms `P` based on that kind-$*$ case. It also expands `mapEP⟨P⟩`. For

each type operator `T` of kind `k` passed as an argument to a polytypic function, the compiler generates its representation type `T'` and the relevant machinery for dealing with its associated embedding-projection pairs. Finally, the compiler generates the instance of the polytypic function for `T`. The instance's type is obtained by expanding `P⟨k⟩ T` and the body follows to the letter the structure of `T'` with the exception of the call to `mapEP⟨P⟩` on the embedding-projection value. The Haskell compiler does the rest of the job: it type-checks that generated instance bodies have the generated type signatures and compiles and optimises the code.

6.1.9 Polytypic functions are not first-class

In Generic Haskell, polytypic functions are not first-class, polytypic applications (*i.e.*, generated instances) are. To make polytypic functions first class, polykinded types must be incorporated into Haskell's type system. However, this would require an extension of the type system beyond what is needed for type-checking polytypic functions. For instance, polykinded-type reconstruction is an open question: a polytypic function need not be associated with a unique polykinded type. (Recall the discussion regarding the polykinded types of `gmap` and `geq` on page 124 and Section 6.1.5.)

First-class status is seldom necessary: type-terms τ in polytypic applications $g\langle\tau\rangle$ must be known at compile-time unless *static* type checking is given up. Consequently, they can be replaced by instances. Take, for example, the following function:

```
foo :: ∀ t a. ∀ p. p⟨∗→∗⟩ t → (a → c) → t a → c
foo g f = g⟨t⟩ f
```

The value of type variable `t` must be known at compile time and its kind be $* \to *$. Although universally quantified, `p` cannot be instantiated to `Map⟨∗ → ∗⟩ t₁ t₂`, which expects two type variable arguments. Finally, the usage of `g` in the body imposes the requirement that:

```
p⟨∗→∗⟩ t = (a → c) → t a → c
```

Let us fix `p` and `t` respectively to a particular type-operator and polykinded type:

```
foo :: ∀ t a. Size⟨∗→∗⟩ List → (a → c) → List a → c
foo g f x = g⟨List⟩ f x
```

Now `foo` can only be passed `gsize` as an argument and `c` must be **`Int`**. There is really

no need for parametrisation:

```
foo :: (a → Int) → List a → Int
foo f x = gsize⟨List⟩ f x
```

6.1.10 Type-class constraints and constrained algebraic types

We have not defined polykinded types that involve type-class constraints. It is possible to include constraints in polytypic abstractions, *e.g.*:

```
gsum⟨t:∗→∗⟩ :: Num a ⇒ t a → Int
```

It is not legal to write constraints in polykinded types. This makes sense: constraints would appear on nested ∀s due to the recursive nature of the general case. Polytypic functions would only be applicable to type operators that have those constraints in *all* type-variables.

However, type-class constraints introduced *in data definitions* are ignored by the latest versions of the Generic Haskell compiler at the time of writing. For example, the representation type generated for types:

```
data         List a = Nil | Cons a (List a)
data Ord a ⇒ List a = Nil | Cons a (List a)
```

is exactly the same:

```
type List' a = Unit + (a × (List a))
```

Although the embedding-projection pair `ep_List` for the unconstrained list has type:

```
∀ a. EP (List a) (List' a)
```

for the constrained list it must have type:

```
∀ a. Ord a ⇒ EP (List a) (List' a)
```

The Haskell compiler complains about the *generated* code, issuing a type error.

Constraints must also appear in the type signatures of generated instances, *e.g.*, in the type signature of `gsize_List`. The reason: `ep_List` appears in `gsize_List`'s body and certainly if `List` has its type argument constrained so should `gsize_List`:

```
gsize_List :: ∀ a. Ord a ⇒ Size a → Size (List a)
```

```
gsize_List gsa = (to (mapEP_Size ep_List)) (gsize_List' gsa)
```

Notice the propagation of constraints:

```
ep_List :: ∀ a. Ord a ⇒ EP (List a) (List' a)
gsa     :: ∀ a. Ord a ⇒ Size a
(mapEP_Size ep_List) ::
  ∀ a. Ord a ⇒ EP (Size (List a)) (Size (List' a))
```

However, the type signatures of instances are all derived from polykinded types which *cannot* accommodate constraints because they are introduced by type-operator *arguments* in polytypic applications. Indeed, if `gsize` is applied to the type:

```
data Eq a ⇒ Set a = ...
```

the constraint in the type signature of the instance differs:

```
gsize_Set :: ∀ a. Eq a ⇒ Size a → Size (Set a)
```

The polykinded type `Size⟨k⟩ t` cannot capture all possible constraints that may appear in the definition of an arbitrary type operator `t`. Therefore, Generic Haskell fails to work with type-class-constrained type operators. It captures an 'unbounded polymorphism' kind of polytypism.

Interestingly, polytypic function *bodies* need not change. The body of `gsize_List` for an unconstrained `List a` is insensitive to the type `a`. The same polymorphic function *body* can compute with a constrained list as long as the constraint is accommodated in the function's type signature. The implicit dictionary introduced by the constraint is ignored:

```
gsize_List :: ∀ a. Ord a ⇒ Size a → Size (List a)
gsize_List gsa = (to (mapEP_Size ep_List)) (gsize_List' gsa)
```

If `List` had been **`Eq`** constrained:

```
data Eq a ⇒ List a = Nil | Cons a (List a)
```

only the type signature would be affected:

```
gsize_List :: ∀ a. Eq a ⇒ Size a → Size (List a)
gsize_List gsa = (to (mapEP_Size ep_List)) (gsize_List' gsa)
```

We conclude this section mentioning that, in Haskell, parametrically polymorphic functions are often lifted automatically to constrained ones that ignore their dictionary arguments, *e.g.*:

```
silly :: ∀ a. Num a ⇒ (∀ a. Num a ⇒ a → a) → a → a
silly f x = f (x + x)

sillyid :: ∀ a. Num a ⇒ a → a
sillyid  = silly id
```

In this example, function `silly` expects a constrained function but it is passed an unconstrained **`id`** in `sillyid`. Behind the scenes, what is really passed is a wrapped identity that discards its dictionary:

```
id_wrap :: NumD a → a → a
id_wrap dict x = x
```

6.1.11 Polykinded types as context abstractions

There are two possible ways of coping with constrained type operators in Generic Haskell. One way is to deal *away* with them, *i.e.*, never to define constrained type operators and put constraints on the functions that compute with them. This has advantages and disadvantages which have been discussed in Section 5.8.2.

Another way is to extend Generic Haskell. However, the extension is surprisingly simple; in fact, it is possible to leave the language unchanged and to only extend the Generic Haskell compiler.

The idea is that the constraints for a `gsize_T` instance, say, are determined from the constraints of type operator `T`. Therefore, the polykinded type `Size⟨k⟩ t` must *abstract* over `t`'s *context* (*i.e.*, list of constraints). In other words, polykinded types must be *context-parametric*. Fortunately, there is no need to change the Generic Haskell *language*. Context-parametric polykinded types and their expansion rules can be hidden from the programmer and manipulated internally by the Generic Haskell *compiler*.

Constraint expressions and lists. We introduce the syntax of constraint expressions and constraint lists in Figure 6.14.

Constraint expressions and constraint lists are type-terms. A ***constraint list*** can be

```
Type           ::= ConstraintList

ConstraintList ::= ϵ
                 | Constraint # ConstraintList

Constraint     ::= ∅
                 | TypeClassName
                 | λ TypeVar . Body

Body           ::= ( TypeClassName Basic ( , TypeClassName Basic )* )

Basic          ::= TypeVar
                 | Basic Basic
```

Figure 6.14: Grammar of constraints and constraint lists.

empty, denoted by ϵ, or constructed by prefixing a constraint expression (explained shortly) to a constraint list using the # operator. As expected, # associates to the right and we allow the following syntactic sugar:

$$[c_1; \ldots; c_n] \stackrel{\text{def}}{=} c_1 \# \ldots \# c_n \# \epsilon$$

when $n > 0$.

Given a data declaration of the form:

```
data Δ ⇒ T a1 ... an = ...
```

the constraint list $\Delta\mathsf{T}$ associated with a type operator `T` contains *one* constraint expression for every one of `T`'s type variables. The constraint expression appears on the list in the same position from left to right as that of the variable in the data declaration. More precisely, when $n > 0$, $\Delta\mathsf{T} = [c_1; \ldots; c_n]$ and c_i is the constraint expression associated with a_i. When $n = 0$, $\Delta\mathsf{T} = \epsilon$.

A ***constraint expression*** can be an empty constraint $\emptyset$, a type-class name (*e.g.*, **Eq**), or a new form of lambda-abstraction whose body consists of applications of type-class names to (applications of) type variables. Lambda-abstractions collect all the constraints associated with a type variable in a single constraint expression, *e.g.*:

```
λx. Eq x
λx. (Eq x, Show x)
λx. Ord (a x)
```

In the last example type variable `a` is free. We explain the role of free type variables shortly.

Here are some type operators and their associated constraint lists:

```
data Eq a ⇒ T1 a = ...
data (Ix a, Eq b) ⇒ T2 a b = ...
data (Eq a, Show a) ⇒ T3 a = ...
data (Eq a, Show a, Num b) ⇒ T4 a b = ...
data (Functor f, Eq (f a)) ⇒ T5 f a = ...
data Ord b ⇒ T6 a b = ...

ΔT1 = [ Eq ]
ΔT2 = [ Ix ; Eq ]
ΔT3 = [ λx.(Eq x, Show x) ]
ΔT4 = [ λx.(Eq x, Show x) ; Num ]
ΔT5 = [ Functor ; λx.Eq (a x) ]
ΔT6 = [ ∅ ; Ord ]
```

Notice that in `ΔT5`, type variable `a` occurs free. Free variables are chosen in alphabetical order according to positions in constraint lists. For example, in `ΔT5`, `a` refers to the type variable to which the *first* constraint expression in `ΔT5` is to be applied. In contrast, in a constraint expression of the form `λx.Eq (b x)`, free variable `b` would refer to the type variable to which the *second* constraint expression in the constraint list is to be applied. Free variables become bound when expanding polykinded types, as shown shortly.

A constraint list `ΔT` only contains constraints associated with `T`'s type variables, not with the type variables of potential type-operator arguments to `T`, which are unknown. For instance, `T5 List` is a legal type, but `ΔT5` does not mention constraints associated with `List`'s type variables.

Context-parametric polykinded types. A polykinded type $P\langle\kappa\rangle\ \overline{t}$ whose general form is given in the first box of Figure 6.15 can be translated automatically by the Generic Haskell compiler to a ***context-parametric polykinded type*** $P'\langle\kappa\rangle\ \overline{t}$ shown in the second box. The third box explains meta-notation. For simplicity, we ignore auxiliary free type variables (Section 6.1.5) which are just carried around when expanding polykinded types.

$$
\begin{array}{lcl}
P\langle * \rangle\ \overline{t} & = & \tau \\
P\langle \kappa \to \nu \rangle\ \overline{t} & = & \forall \overline{\alpha}.\ P\langle \kappa \rangle\ \overline{\alpha}\ \to\ P\langle \nu \rangle\ \overline{(t\ \alpha)}
\end{array}
$$

$$
\begin{array}{lcl}
P'\langle * \rangle\ \overline{t} & = & \gamma q.\ \tau \\
P'\langle \kappa \to \nu \rangle\ \overline{t} & = & \gamma q.\ (\forall \overline{\alpha}.\ q\ \overline{\alpha} \Rightarrow P'\langle \kappa \rangle\ \overline{\alpha}\ \epsilon\ \to\ P'\langle \nu \rangle\ \overline{(t\ \alpha)}\ q)
\end{array}
$$

$$
\begin{array}{lcl}
\overline{\alpha} & \stackrel{\text{def}}{=} & \alpha_1 \ldots \alpha_n \\
\overline{t} & \stackrel{\text{def}}{=} & t_1 \ldots t_n \\
\overline{(t\ \alpha)} & \stackrel{\text{def}}{=} & (t_1\ \alpha_1) \ldots (t_n\ \alpha_n) \\
q\ \overline{\alpha} & \stackrel{\text{def}}{=} & (q\ \alpha_1, \ldots, q\ \alpha_n) \\
n & > & 0
\end{array}
$$

Figure 6.15: A polykinded type and its context-parametric version.

We have introduced a new form of abstraction using the symbol γ as binder:[3] a ***γ-abstraction*** is a new type-term of the form $\gamma q.\tau$ where q may occur free in τ. (In the $P'\langle * \rangle$ case, however, q must be picked so as not to occur free in τ.)

A ***γ-application*** is the application of a γ-abstraction to a constraint list. This is a type-term of the form $(\gamma q.\tau)\ cs$, where cs is a type-term whose compile-time value is a constraint list. The new application is denoted by whitespace but, as we show shortly, its type-level reduction rules are more complicated than simple substitution. In the second box of Figure 6.15, there are two γ-applications in the $P'\langle \kappa \to \nu \rangle$ case: $P'\langle \kappa \rangle\ \overline{\alpha}$ is γ-applied to ϵ and $P'\langle \nu \rangle\ \overline{(t\ \alpha)}$ is γ-applied to q.

The context-parametric versions of `Size` (Figure 6.5) and `Map` (Figure 6.9) are shown in Figure 6.16.

Figure 6.17 shows the type-reduction (or 'expansion') rules of context-parametric polykinded types (first box), which includes the type-reduction rules of γ-application (second box), and the mechanics of constraint-expression application (third box).

Rule C-START shows how the instance of a polykinded type $P\langle \kappa \rangle\ \overline{T}$ on an actual type-operator argument T is to be obtained by reducing its context-parametric version $P'\langle \kappa \rangle\ \overline{T}\ \Delta T$ which takes T's list of constraints into account.

[3] Capital gamma is often used for denoting contexts, so lowercase gamma makes sense as a binder for constraint lists.

```
type Size'⟨∗⟩ t = γq. t → Int
type Size'⟨k→v⟩ t
  = γq.( ∀a. q a ⇒ (Size'⟨k⟩ a ϵ) → (Size'⟨v⟩ (t a) q ) )

type Map'⟨∗⟩ t₁ t₂ = γq. t₁ → t₂
type Map'⟨k→v⟩ t₁ t₂
  = γq.( ∀a₁ a₂. (q a₁, q a₂) ⇒
           (Map'⟨k⟩ a₁ a₂ ϵ) → (Map'⟨v⟩ (t₁ a₁) (t₂ a₂) q ) )
```

Figure 6.16: Context-parametric polykinded types `Size'` and `Map'`.

$$P\langle\kappa\rangle\ \overline{T} \quad \blacktriangleright \quad P'\langle\kappa\rangle\ \overline{T}\ \Delta T \quad \text{(C-START)}$$

$$\begin{array}{lcll}
(\gamma q.\ (\forall\overline{\alpha}.\ q\ \overline{\alpha} \Rightarrow \sigma))\ \epsilon & \blacktriangleright & \forall\overline{\alpha}.\sigma[q/\epsilon] & \text{(C-NULL)} \\
(\gamma q.\ (\forall\overline{\alpha}.\ q\ \overline{\alpha} \Rightarrow \sigma))\ (\emptyset\#cs) & \blacktriangleright & \forall\overline{\alpha}.\sigma[q/cs] & \text{(C-EMPTY)} \\
(\gamma q.\ (\forall\overline{\alpha}.\ q\ \overline{\alpha} \Rightarrow \sigma))\ (c\#cs) & \blacktriangleright & \forall\overline{\alpha}.\ c\ \overline{\alpha} \Rightarrow \sigma[q/cs] & \text{(C-PUSH)} \\
(\gamma q.\ \tau)\ cs & \blacktriangleright & \tau \quad \text{if } q \notin \mathrm{FV}(\tau) & \text{(C-DROP)}
\end{array}$$

$$c\ \overline{\alpha} \quad \stackrel{\text{def}}{=} \quad (c\ \alpha_1, \ldots, c\ \alpha_n)$$

$$c\ \alpha_i \quad \blacktriangleright \quad \begin{cases} \mathsf{C}\ \alpha_i & \text{if } c = \mathsf{C} \\ (\mathsf{subscript}\ i\ B)[x_i/\alpha_i] & \text{if } c = \lambda x.B \end{cases}$$

Figure 6.17: Type-reduction rules for context-parametric types.

Rule C-NULL shows the reduction when the constraint list is empty: the context is removed from the type-term and ϵ is substituted for q in σ. Rule C-EMPTY shows the reduction of a list with an empty constraint in the head: the context is removed from the type-term and q is replaced by the tail of the list. This way other constraints are pushed down to its associated variables. Rule C-PUSH shows the reduction when the constraint list is not empty: the constraint expression on the head is applied to the universally-quantified type variables, and the tail of the constraint list is substituted for q in σ. Rule C-DROP shows the reduction when $q \notin \mathrm{FV}(\tau)$: the abstraction is dropped.

Some remarks are in order. In the $P'\langle * \rangle$ case, $q \notin \mathrm{FV}(\tau)$ and therefore type reduction will always proceed by Rule C-DROP. In the general case, q is imposed on the universally-quantified type variables and it is γ-applied to the $P'\langle \nu \rangle$ case, whereas an empty constraint list is γ-applied to the $P'\langle \kappa \rangle$ case. This is because $\Delta\mathsf{T}$ only includes constraints associated with T's type variables, as explained above. Finally, notice that expanding a context-parametric polykinded type with $[\emptyset; \ldots; \emptyset]$ produces the same result as expanding it with ϵ.

The last box in Figure 6.17 indicates how the application of a constraint expression to a type variable works. The first line explains meta-notation. The second line is a reduction rule for the application of a constraint expression c to a type variable α_i. The constraint expression c cannot be $\emptyset$ because empty constraints are dealt with by γ-application (C-EMPTY). If c is a type-class name C, constraint application simply juxtaposes the constraint to the variable. If c is a lambda expression $\lambda x.B$, first the auxiliary function $\mathsf{subscript}\ i\ B$ adds subscripts to B's free variables (including x), performing multiple substitution where every free variable μ in B is substituted by μ_i. Then, x_i is substituted by α_i. We omit $\mathsf{subscript}$'s implementation details which are inessential. Here are a few examples that illustrate how it works:

$$(\gamma q.\ (\forall a_1.\ q\ a_1\ \Rightarrow \ldots))\ [\mathsf{C}] \quad = \quad \forall a_1.\ \mathsf{C}\ a_1\ \Rightarrow \ldots$$

$$(\gamma q.\ (\forall a_1.\ q\ a_1\ \Rightarrow \ldots))\ [\lambda x.(\mathsf{C_1}\ x, \mathsf{C_2}\ x)] \quad = \quad \forall a_1.\ (\mathsf{C_1}\ a_1, \mathsf{C_2}\ a_1)\ \Rightarrow \ldots$$

$$(\gamma q.\ (\forall a_1.\ q\ a_1\ \Rightarrow \ldots))\ [\lambda x.\ \mathsf{C}\ (z\ x)] \quad = \quad \forall a_1.\ \mathsf{C}\ (z_1\ a_1)\ \Rightarrow \ldots$$

$$(\gamma q.\ (\forall a_1 a_2.\ (q\ a_1, q\ a_2)\ \Rightarrow \ldots))\ [\mathsf{C}]\ =\ \forall a_1 a_2.\ (\mathsf{C}\ a_1, \mathsf{C}\ a_2)\ \Rightarrow \ldots$$

$$\begin{aligned}&(\gamma q.\ (\forall a_1 a_2.\ (q\ a_1, q\ a_2)\ \Rightarrow \ldots))\ [\lambda x.(\mathsf{C_1}\ x, \mathsf{C_2}\ x)]\\ &\quad =\ \forall a_1 a_2.\ ((\mathsf{C_1}\ a_1, \mathsf{C_2}\ a_1), (\mathsf{C_1}\ a_2, \mathsf{C_2}\ a_2))\ \Rightarrow \ldots\end{aligned}$$

$$\begin{aligned}&(\gamma q.\ (\forall a_1 a_2.\ (q\ a_1, q\ a_2)\ \Rightarrow \ldots))\ [\lambda x.\ \mathsf{C}\ (z\ x)]\\ &\quad =\ \forall a_1 a_2.(\mathsf{C}\ (z_1\ a_1), \mathsf{C}\ (z_2\ a_2))\ \Rightarrow \ldots\end{aligned}$$

Instantiation examples. We conclude this section illustrating the instantiation process for the example type-operators given above and for polykinded types `Size` and `Map`. Universally-quantified variables are chosen in order so as to avoid shadowing when expanding context-parametric polykinded types.

The first example involves `T1` which has an **`Eq`** constraint on its type variable. For reasons of space, we omit the details of constraint-expression applications which take place in rule C-PUSH. We also obviate subscripts when polykinded types have only one universally-quantified type variable:

$$\texttt{Size}\langle *\to *\rangle\ \texttt{T1}$$
$$\blacktriangleright \quad \{\ \text{C-START}\ \}$$
$$\texttt{Size'}\langle *\to *\rangle\ \texttt{T1}\ \texttt{[Eq]}$$
$$\blacktriangleright \quad \{\ \text{def. of } \texttt{Size'}\ \}$$
$$(\gamma q.\ (\forall a.\ q\ a \Rightarrow \texttt{Size'}\langle *\rangle\ a\ \epsilon \to \texttt{Size'}\langle *\rangle\ (\texttt{T1}\ a)\ q))\ \texttt{[Eq]}$$
$$\blacktriangleright \quad \{\ \text{C-PUSH}\ \}$$
$$\forall a.\ \texttt{Eq}\ a \Rightarrow \texttt{Size'}\langle *\rangle\ a\ \epsilon \to \texttt{Size'}\langle *\rangle\ (\texttt{T1}\ a)\ \epsilon$$
$$\blacktriangleright \quad \{\ \text{def. of } \texttt{Size'} \text{ (twice)}\ \}$$
$$\forall a.\ \texttt{Eq}\ a \Rightarrow (\gamma q.\ a \to \texttt{Int})\ \epsilon \to (\gamma q.\ (\texttt{T1}\ a) \to \texttt{Int})\ \epsilon$$
$$\blacktriangleright \quad \{\ \text{C-DROP (twice)}\ \}$$
$$\forall a.\ \texttt{Eq}\ a\ \Rightarrow\ (a \to \texttt{Int}) \to \texttt{T1}\ a \to \texttt{Int}$$

The second example involves `T2` which has two type variables, each with a different constraint:

$$\texttt{Size}\langle *\to *\to *\rangle\ \texttt{T2}$$

▶ { C-START }

$\texttt{Size'}\langle * \to * \to * \rangle\ \texttt{T2}\ \texttt{[Ix;Eq]}$

▶ { def. of Size' }

$(\gamma q.\ (\forall a.\ q\ a \Rightarrow \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T2}\ a)\ q))\ \texttt{[Ix;Eq]}$

▶ { C-PUSH }

$\forall a.\ \texttt{Ix}\ a \Rightarrow \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T2}\ a)\ \texttt{[Eq]}$

▶ { def. of Size' }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (\gamma q.\ a \to \texttt{Int})\ \epsilon \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T2}\ a)\ \texttt{[Eq]}$

▶ { C-DROP }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T2}\ a)\ \texttt{[Eq]}$

▶ { def. of Size' }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\gamma q.\ (\forall b.\ q\ b \Rightarrow \texttt{Size'}\langle * \rangle\ b\ \epsilon \to \texttt{Size'}\langle * \rangle\ (\texttt{T2}\ a\ b)\ q))\ \texttt{[Eq]}$

▶ { C-PUSH }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\forall b.\ \texttt{Eq}\ b \Rightarrow \texttt{Size'}\langle * \rangle\ b\ \epsilon \to \texttt{Size'}\langle * \rangle\ (\texttt{T2}\ a\ b)\ \epsilon)$

▶ { def. of Size' }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\forall b.\ \texttt{Eq}\ b \Rightarrow (\gamma q.\ b \to \texttt{Int})\ \epsilon \to \texttt{Size'}\langle * \rangle\ (\texttt{T2}\ a\ b)\ \epsilon)$

▶ { C-DROP }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\forall b.\ \texttt{Eq}\ b \Rightarrow (b \to \texttt{Int}) \to \texttt{Size'}\langle * \rangle\ (\texttt{T2}\ a\ b)\ \epsilon)$

▶ { def. of Size' }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\forall b.\ \texttt{Eq}\ b \Rightarrow (b \to \texttt{Int}) \to (\gamma q.\ (\texttt{T2}\ a\ b) \to \texttt{Int})\ \epsilon)$

▶ { C-DROP }

$\forall a.\ \texttt{Ix}\ a \Rightarrow (a \to \texttt{Int}) \to (\forall b.\ \texttt{Eq}\ b \Rightarrow (b \to \texttt{Int}) \to \texttt{T2}\ a\ b \to \texttt{Int})$

The following example involves T3 which has multiple constraints on its only type variable:

$\texttt{Size}\langle * \to * \rangle\ \texttt{T3}$

▶ { C-START }

$\texttt{Size'}\langle * \to * \rangle\ \texttt{T3}\ [\lambda x.(\texttt{Eq}\ x, \texttt{Show}\ x)]$

▶ { def. of Size' }

$(\gamma q.\ (\forall a.\ q\ a \Rightarrow \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \rangle\ (\texttt{T3}\ a)\ q))\ [\lambda x.(\texttt{Eq}\ x, \texttt{Show}\ x)]$

▶ { C-PUSH and constraint application }

$\forall a.\ (\texttt{Eq}\ a, \texttt{Show}\ a) \Rightarrow \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \rangle\ (\texttt{T3}\ a)\ \epsilon$

▶ { def. of `Size'` (twice) and C-DROP (twice) }
$\forall a.$ (**Eq** a, **Show** a) $\Rightarrow$ $(a \to \texttt{Int}) \to \texttt{T3}\ a \to \texttt{Int}$

In the following example, there are two constraints on `T4`'s first type variable and one constraint on the second:

`Size`$\langle * \to * \to * \rangle$ `T4`
▶ { C-START }
`Size'`$\langle * \to * \to * \rangle$ `T4` $[\lambda x.$(**Eq** x, **Show** x); **Num**$]$
▶ { def. of `Size'` }
$(\gamma q.\ (\forall a.\ q\ a \Rightarrow$ `Size'`$\langle * \rangle\ a\ \epsilon \to$ `Size'`$\langle * \to * \rangle$ (`T4` a) $q))$ $[\lambda x.$(**Eq** x, **Show** x); **Num**$]$
▶ { C-PUSH and constraint application }
$\forall a.$ (**Eq** a, **Show** a) $\Rightarrow$ `Size'`$\langle * \rangle\ a\ \epsilon \to$ `Size'`$\langle * \to * \rangle$ (`T4` a) [**Num**]
▶ { ... }
$\forall a.$ (**Eq** a, **Show** a) $\Rightarrow$ $(a \to \texttt{Int}) \to (\forall b.$ **Num** $b \Rightarrow$ `Size'`$\langle * \rangle\ b\ \epsilon \to$ `Size'`$\langle * \rangle$ (`T4` $a\ b$) $\epsilon)$
▶ { ... }
$\forall a.$ (**Eq** a, **Show** a) $\Rightarrow$ $(a \to \texttt{Int}) \to (\forall b.$ **Num** $b \Rightarrow (b \to \texttt{Int}) \to \texttt{T4}\ a\ b \to \texttt{Int})$

The following example involves `T5` which has a constraint on both of its type variables, and the first one has kind $(* \to *)$. The example illustrates the use of free type variables: the type variables after the $\forall$ are chosen in the same alphabetical order so that a free variable in position i in a constraint list is bound by the ith universal quantifier introduced by a $P'\langle\nu\rangle$ case. We could have worked up to some context of free variables but we prefer to use this convention for simplicity:

`Size`$\langle$`(`$* \to *$`)`$\to * \to * \to *\rangle$ `T5`
▶ { C-START }
`Size'`$\langle$`(`$* \to *$`)`$\to * \to * \to *\rangle$ `T5` [**Functor**;$\lambda x.$**Eq** $(a\ x)$]
▶ { def. of `Size'` }
$(\gamma q.\ (\forall a.\ q\ a \Rightarrow$ `Size'`$\langle * \to * \rangle\ a\ \epsilon \to$ `Size'`$\langle * \to * \rangle$ (`T5` a) $q))$ [**Functor**;$\lambda x.$**Eq** $(a\ x)$]
▶ { C-PUSH }
$\forall a.$ **Functor** $a \Rightarrow$ `Size'`$\langle * \to * \rangle\ a\ \epsilon \to$ `Size'`$\langle * \to * \rangle$ (`T5` a) $[\lambda x.$**Eq** $(a\ x)]$
▶ { def. of `Size'` }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow \texttt{Size'}\langle * \to * \rangle\ a\ \epsilon \to (\gamma q.\ (\forall b.\ q\ b \Rightarrow \texttt{Size'}\langle * \rangle\ b\ \epsilon \ldots))\ [\lambda x.\mathbf{Eq}\ (a\ x)]$

▶ { C-PUSH }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow \texttt{Size'}\langle * \to * \rangle\ a\ \epsilon \to (\forall b.\ \mathbf{Eq}\ (a\ b) \Rightarrow \texttt{Size'}\langle * \rangle\ b\ \epsilon \ldots)$

▶ { ... }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow \texttt{Size'}\langle * \to * \rangle\ a\ \epsilon \to (\forall b.\ \mathbf{Eq}\ (a\ b) \Rightarrow (b \to \texttt{Int}) \to \ldots)$

▶ { ... }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow (\forall b.\ (b \to \texttt{Int}) \to a\ b \to \texttt{Int}) \to$
$(\forall b.\ \mathbf{Eq}\ (a\ b) \Rightarrow (b \to \texttt{Int}) \to \texttt{T5}\ a\ b \to \texttt{Int})$

We have not shown Rule C-NULL at work. It has been used in the steps from the penultimate equation to the last. For reasons of space, let us use the following abbreviation:

$$etc \stackrel{\text{def}}{=} (\forall b.\ \texttt{Eq}\ (a\ b) \Rightarrow (b \to \texttt{Int}) \to \texttt{T5}\ a\ b \to \texttt{Int})$$

These are the details of the derivation:

$\forall a.\ \mathbf{Functor}\ a \Rightarrow \texttt{Size'}\langle * \to * \rangle\ a\ \epsilon \to etc$

▶ { def. of `Size'` }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow (\gamma q.\ (\forall b.\ q\ b \Rightarrow \texttt{Size'}\langle * \rangle\ b\ \epsilon \to \texttt{Size'}\langle * \rangle\ (a\ b)\ q))\ \epsilon \to etc$

▶ { C-NULL }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow (\forall b.\ \texttt{Size'}\langle * \rangle\ b\ \epsilon \to \texttt{Size'}\langle * \rangle\ (a\ b)\ \epsilon) \to etc$

▶ { def. of `Size'` and C-DROP (twice) }

$\forall a.\ \mathbf{Functor}\ a \Rightarrow (\forall b.\ (b \to \texttt{Int}) \to a\ b \to \texttt{Int}) \to etc$

The following example involves `T6` and shows Rule C-EMPTY at work:

$\texttt{Size}\langle * \to * \to * \rangle\ \texttt{T6}$

▶ { C-START }

$\texttt{Size'}\langle * \to * \to * \rangle\ \texttt{T6}\ [\emptyset;\mathbf{Ord}]$

▶ { def. of `Size'` }

$(\gamma q.\ (\forall a.\ q\ a \Rightarrow \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T6}\ a)\ q))\ [\emptyset;\mathbf{Ord}]$

▶ { C-EMPTY }

$\forall a.\ \texttt{Size'}\langle * \rangle\ a\ \epsilon \to \texttt{Size'}\langle * \to * \rangle\ (\texttt{T4}\ a)\ [\mathbf{Ord}]$

▶ { ... }

$\forall a.\ (a \to \texttt{Int}) \to (\forall b.\ \mathbf{Ord}\ b \Rightarrow (b \to \texttt{Int}) \to \texttt{T4}\ a\ b \to \texttt{Int})$

Lastly, the following example shows the expansion of `Map'` which has two universally-quantified type variables, thus illustrating the use of subscript:

$$\texttt{Map'}\langle(*\to*)\to*\to*\rangle\ \texttt{T5}\ [\textbf{Functor};\ \lambda x.\textbf{Eq}\ (a\ x)]$$
$$\blacktriangleright \quad \{\ \text{def. of } \texttt{Map}\ \}$$
$$(\gamma q.\ (\forall a_1 a_2.\ (q\ a_1, q\ a_2) \Rightarrow$$
$$(\texttt{Map'}\langle*\to*\rangle\ a_1\ a_2\ \epsilon) \to (\texttt{Map'}\langle*\to*\rangle\ (\texttt{T5}\ a_1)\ (\texttt{T5}\ a_2)\ q)))\ [\textbf{Functor};\ \lambda x.\textbf{Eq}\ (a\ x)]$$
$$\blacktriangleright \quad \{\ \text{C-PUSH}\ \}$$
$$\forall a_1 a_2.\ (\textbf{Functor}\ a_1, \textbf{Functor}\ a_2) \Rightarrow$$
$$(\texttt{Map'}\langle*\to*\rangle\ a_1\ a_2\ \epsilon) \to (\texttt{Map'}\langle*\to*\rangle\ (\texttt{T5}\ a_1)\ (\texttt{T5}\ a_2)\ [\lambda x.\textbf{Eq}\ (a\ x)])$$

We abbreviate the result of reducing $\texttt{Map'}\langle * \to * \rangle\ a_2\ a_2\ \epsilon$ as follows:

$$S \stackrel{\text{def}}{=} (\forall b_1 b_2.\ (b_1 \to b_2) \to a_1\ b_1 \to a_2\ b_2)$$

And resume:

$$\forall a_1 a_2.\ (\textbf{Functor}\ a_1, \textbf{Functor}\ a_2) \Rightarrow S \to$$
$$(\texttt{Map'}\langle*\to*\rangle\ (\texttt{T5}\ a_1)\ (\texttt{T5}\ a_2)\ [\lambda x.\textbf{Eq}\ (a\ x)])$$
$$\blacktriangleright \quad \{\ \text{def. of } \texttt{Map'}\ \}$$
$$\forall a_1 a_2.\ (\textbf{Functor}\ a_1, \textbf{Functor}\ a_2) \Rightarrow S \to$$
$$((\gamma q.\ (\forall b_1 b_2.\ (q\ b_1, q\ b_2)\ \Rightarrow$$
$$(\texttt{Map'}\langle*\rangle\ b_1\ b_2\ \epsilon) \to (\texttt{Map'}\langle*\rangle\ (\texttt{T5}\ a_1\ b_1)\ (\texttt{T5}\ a_2\ b_2)\ q)))$$
$$[\lambda x.\textbf{Eq}\ (a\ x)])$$
$$\blacktriangleright \quad \{\ \text{C-PUSH (invoking subscript)}\ \}$$
$$\forall a_1 a_2.\ (\textbf{Functor}\ a_1, \textbf{Functor}\ a_2) \Rightarrow S \to$$
$$(\forall b_1 b_2.\ (\textbf{Eq}\ (a_1\ b_1), \textbf{Eq}\ (a_2\ b_2)) \Rightarrow$$
$$(\texttt{Map'}\langle*\rangle\ b_1\ b_2\ \epsilon) \to (\texttt{Map'}\langle*\rangle\ (\texttt{T5}\ a_1\ b_1)\ (\texttt{T5}\ a_2\ b_2)\ \epsilon))$$
$$\blacktriangleright \quad \{\ \ldots\ \}$$
$$\forall a_1 a_2.\ (\textbf{Functor}\ a_1, \textbf{Functor}\ a_2) \Rightarrow (\forall b_1 b_2.\ (b_1 \to b_2) \to a_1\ b_1 \to a_2\ b_2) \to$$
$$(\forall b_1 b_2.\ (\textbf{Eq}\ (a_1\ b_1), \textbf{Eq}\ (a_2\ b_2)) \Rightarrow (b_1 \to b_2) \to (\texttt{T5}\ a_1\ b_1) \to (\texttt{T5}\ a_2\ b_2))$$

Expansion and instance generation. We conclude the section discussing how the expansion of context-parametric types fits into the overall generation process. For each polytypic application $g\langle T\ \tau_1 \ldots \tau_n\rangle$ where T is a type operator of kind κ and τ_i are well-kinded type applications of kind κ_i, the Generic Haskell compiler must generate

the g_T instance whose type is obtained by expanding $P'\langle\kappa\rangle\ \overline{T}\ \Delta T$, where P' is the context-parametric polykinded type of g. The polytypic applications $g\langle\tau_i\rangle$ must be compiled to instances g_τ_i whose type is obtained by expanding $P'\langle\kappa_i\rangle\ \overline{\tau_i}\ \epsilon$, where the constraint list is empty. Consequently, constrained type operators cannot be applied to other constrained type operators. This restriction can be overcome by modifying, among other things, the definition of constraint lists to account for constraints in τ_i, and by γ-applying the $P'\langle\kappa\rangle\ \overline{\alpha}$ case in a context-parametric polykinded type to the constraint list of τ_i during expansion.

At any rate, we have introduced context-parametric polykinded types in this thesis to be able to write polytypic functions on *first-order* ADTs which may be implemented in terms of constrained types (Chapter 9). First-order type operators cannot be applied to proper type operators, let alone constrained ones. The solution presented here is sufficient for our purposes.

6.1.12 Parameterisation on base types

The values for base types and units are fixed in polytypic function definitions. For example, in the case of `gsize`, the size for integers is 0 and so is the size computed for a value of `List Int` type. It is necessary to abstract over a type-operator's payload to compute the size properly:

```
gsize⟨List Int⟩ [1,2,3]
> 0
gsize⟨List⟩ (const 1) [1,2,3]
> 3
```

Fixing the value for base types can be limiting. For instance, the size for units is 0 and it is therefore impossible to count things such as the number of empty trees hanging from leaves in a `BTree`. Serialisation functions (*e.g.*, encoders) take into account units and base type values; their definition must be changed and recompiled if their encoding needs to be changed. Furthermore, every time a base type is added to the system (for example by linking with a library), the polytypic definition has to be modified and recompiled unless the new case is added by extending the polytypic function (dependency-style supports 'polytypic extension' [Löh04]).

```
type Size⟨∗⟩ t = Int → Int → t → Int
type Size⟨k→v⟩ t = ∀ a. Size⟨k⟩ u → Size⟨l⟩ (t a)

gsize⟨t:k⟩ :: Size⟨k⟩ t
gsize⟨Char⟩ v w _ = v
gsize⟨Int⟩  v w _ = v
gsize⟨Bool⟩ v w _ = v
gsize⟨Unit⟩ v w _ = w
gsize⟨a+b⟩  v w (Inl x) = gsize⟨a⟩ v w x
gsize⟨a+b⟩  v w (Inr y) = gsize⟨b⟩ v w y
gsize⟨a×b⟩  v w (x,y)   = (gsize⟨a⟩ v w x) + (gsize⟨b⟩ v w y)

gsize⟨List⟩ (λv w a → 0) 0 1 [1,2,3]
> 1
```

Figure 6.18: Parameterising `gsize` on the values of base types and units.

Parametrisation allows us to use code once by adapting parameters whereas extension requires us to provide new definitions for specific types. And different (overlapping) definitions for the same type are not possible.

A somewhat ugly solution is to parameterise every polytypic function definition with values for base types and units. For instance, Figure 6.18 shows a redefinition of `Size` and `gsize` where the size for base types (argument `v`) and for units (argument `w`) is passed as an argument to `gsize` and to its argument functions. This definition is coarse-grained: base types are given the same size. Making distinctions would require more arguments, and the addition of new base types would require editing the polykinded type, forcing recompilation and affecting client code.

6.1.13 Generic Haskell, categorially

This section is concerned with expressing polytypic function definitions in a way that abstracts from the concrete representation of binary sums and products. Following the categorial definitions and concepts discussed in Section 3.9.2, in this section we read `a×b` and `a+b` as sugar for `Prod a b` and `CoProd a b` respectively. Figure 6.19 shows the definitions of `gsize`, `gmap` and `geq` in the light of this change. Notice that `gsize` and `gmap` both take one *value* argument and follow the same pattern. Figure 6.20 shows their instantiations for a couple of functor expressions.

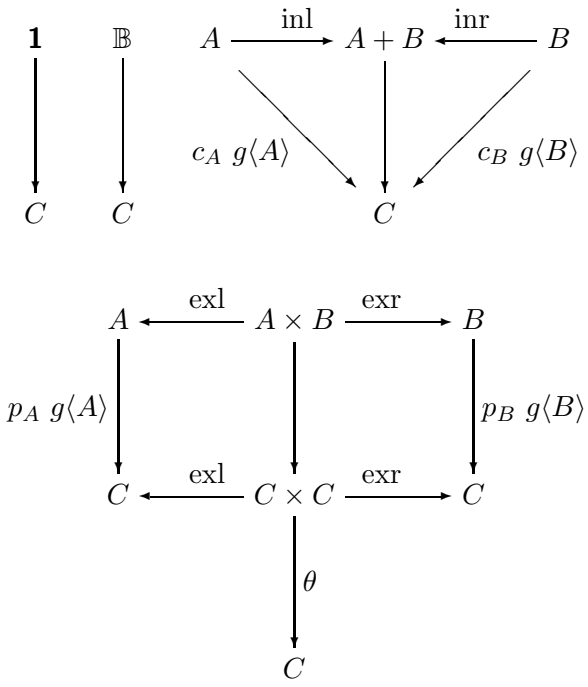
The general pattern of a polytypic function definition that takes one *value* argument is

```
gsize⟨1⟩   = const 0
gsize⟨Int⟩ = const 0
gsize⟨a+b⟩ = gsize⟨a⟩ ▽ gsize⟨b⟩
gsize⟨a×b⟩ = plus ∘ map× gsize⟨a⟩ gsize⟨b⟩
  where plus p = exl p + exr p

gmap⟨1⟩   = id
gmap⟨Int⟩ = id
gmap⟨a+b⟩ = (asLeft ∘ gmap⟨a⟩) ▽ (asRight ∘ gmap⟨b⟩)
gmap⟨a×b⟩ = id ∘ map× gmap⟨a⟩ gmap⟨b⟩

geq⟨1⟩   = const (const true)
geq⟨Int⟩ = (==)
geq⟨a+b⟩ c1 c2 = if (isLeft c1 && isLeft c2) then
                   g⟨a⟩ (asLeft c1) (asLeft c2)
                 else if (isRight c1 && isRight c2) then
                   g⟨b⟩ (asRight c1) (asRight c2)
                 else False

geq⟨a×b⟩ p1 p2 = geq⟨a⟩ (exl p1) (exl p2) &&
                 geq⟨b⟩ (exr p1) (exr p2)
```

Figure 6.19: Polytypic `gsize`, `gmap`, and `geq` in terms of products and coproducts.

```
F a   = 1 + a × (F a)
G f a = 1 + a × f (G f a)

gsize_F gsa     = gsize_Unit ▽ (gsa ‘plus‘ (gsize_F gsa))
gsize_G gsf gsa = gsize_Unit ▽ (gsa ‘plus‘ (gsf (gsize_G gsf gsa)))

gmap_F gma     = (asLeft  ∘ gmap_Unit) ▽
                 (asRight ∘ (gma ‘prod‘ (gmap_F gma)))

gmap_G gmf gma = (asLeft  ∘ gmap_Unit) ▽
                 (asRight ∘ (gma ‘prod‘ gmf (gmap_F gmf gma)))
```

Figure 6.20: Some functors and their polytypic function instances. Notice that **map**$_\times$'s definition has been expanded.



$$
\begin{array}{lcl}
g\langle \mathbf{1} \rangle & = & g_{\mathbf{1}} \\
g\langle \mathbb{B} \rangle & = & g_{\mathbb{B}}
\end{array}
$$

$$
\begin{array}{lcl}
g\langle A{+}B \rangle & = & (c_A\ g\langle A \rangle) \triangledown (c_B\ g\langle B \rangle) \\
\quad \text{where}\ \ c_A\ g & = & \mathit{cafter}_A \circ g \circ \mathit{cbefore}_A \\
\qquad\quad\ \ c_B\ g & = & \mathit{cafter}_B \circ g \circ \mathit{cbefore}_B
\end{array}
$$

$$
\begin{array}{lcl}
g\langle A{\times}B \rangle & = & \theta \circ (p_A\ g\langle A \rangle) \times (p_B\ g\langle B \rangle) \\
\quad \text{where}\ \ l \times r & = & (l \circ \mathrm{exl}) \vartriangle (r \circ \mathrm{exr}) \\
\qquad\quad\ \ p_A\ g & = & \mathit{pafter}_A \circ g \circ \mathit{pbefore}_A \\
\qquad\quad\ \ p_B\ g & = & \mathit{pafter}_B \circ g \circ \mathit{pbefore}_B
\end{array}
$$

Figure 6.21: General pattern of a polytypic function definition expressed categorially. $\mathbb{B}$ ranges over base types.

shown in Figure 6.21. The first box depicts each case as a diagram. The programmer defines c_A, c_B, p_A, p_B and θ. The latter function specifies what is to be done with each product member after the (possibly) recursive application of the polytypic function. The cases for units and base types are fixed.

Notice that `geq` takes two *value* arguments and follows a different pattern. In order to express its definition in point-free fashion, like `gsize`, we would have to define a sort of zipping pairwise extension of $\triangledown$, $\triangle$, and θ.

Figure 6.21 illustrates more clearly the fact that a polytypic function definition has the recursive structure of a ***catamorphism*** [MFP91, BdM97]. We come back to this figure in Chapter 9 when exploring its adaptability in the presence of data abstraction.

6.2 Scrap your Boilerplate

The *Scrap your Boilerplate* (SyB) approach to generic programming [LP03, LP04, LP05] is a blend of polytypic and strategic programming ideas. Section 6.2.1 explains Strategic Programming and Section 6.2.2 provides a paper-by-paper tour of SyB.

6.2.1 Strategic Programming

Data structures are heterogeneous trees of sorts. Computing with them entails traversing them (recursion scheme) and performing actions at every node in the structure. Some actions may involve combining results of other actions into a value (so-called ***type-unifying*** computations such as catamorphisms), or transforming a value (so-called ***type-preserving*** computations such as maps).[4]

Given a set of data types, programming by hand functions on them by pattern matching is non-generic, laborious, fragile with respect to changes in the type's definitions, and rigid with respect to the recursion scheme and traversal control. Functions are not parametric on "the ability to determine which parts of the representation are visited in which order and under which conditions" [VS04, p16]. This problem becomes particularly annoying when functions only work on some parts of the structure, making the rest of the code which traverses it (*i.e.*, which recurses in order to get to those parts) ***boilerplate code***.

[4] The terminology has been borrowed from [LVK00].

The use of higher-order functions capturing recursion schemes is of little help. Examples of such functions are *generalised folds* [Hut99, GHA01, SF93, BP99, MH95]. In particular:

- The fold function for every type has to be programmed explicitly and so is each particular fold algebra (operators that replace value constructors) with which we want to compute. An exception is the programming language Charity,[5] where folds are automatically generated by the compiler for *first-order* type operators. Programmers may assume the existence of these folds when writing their programs.
- Folds are rigid with respect to traversal control: the recursion scheme is fixed and control refinement has to be hardwired in algebraic operators. For example, to 'stop folding' over a binary tree upon a certain condition we need to hardwire the stop condition in algebraic operators (fold would not 'stop' but simply collapse the pruned subtree into a default value) :

```
fold_until p :: (a → Bool) → Alg a c → List a → c
fold_until p alg l = foldr alg' l
where
alg' = Alg {nil  = nil alg,
            cons = λx y → if p x then nil alg else cons alg x y}
```

In [LVK00] a solution is proposed for reusing algebras by extension and overriding which is implemented in terms of extensible records.

Strategic Programming [VS04, LVV02] is a more general, *paradigm independent* solution to generic traversal (computation) with heterogenous structures. It is based on the following key ideas: action at a node and traversal must be separated, and traversals must be decomposed into *one-layer* traversals on the one side and recursion schemes given by an explicit fixed-point definition on the other side. This separation of concerns permits composition and parametrisation: actions can be composed and one-layer traversals and recursion schemes can be provided as action-parametric generic combinators.

In order for an action at a node to be generic, Strategic Programming assumes the possibility of type-based dispatch, usually implemented in terms of a *dynamic type-case*,

[5] `http://www.cpsc.ucalgary.ca/Research/charity/home.html`

i.e., code that enquires about a value's type at run time (run-time type information or RTTI) in order to perform the appropriate computation. Enquiring about type information makes 'generic' code more inefficient than non-generic, purpose-built code (check out [LP03] and Section 4.3).

`id`, `fail`	Primitive actions
`seq`, `choice`	Action composition
`adhoc`	Type-based dispatch
`all`, `one`	One-layer traversal
`topdown`, `bottomup`	Recursion schemes
`f◇t`	Apply strategy `f` to input term `t`

Table 6.1: Examples of Strategic Programming combinators.

Table 6.1 lists a few paradigm-independent combinators [VS04]. The first row lists the actions `id` and `fail` which are applied to an input node. The former returns the node untouched and the latter returns a special value denoting failure or raises an exception. The second row lists some ways of composing actions. Actions can be composed by `seq f g` which applies first `f` and then `g` to the node, or by `choice f g` which applies `g` only if `f` fails. The third row is the dynamic type-case combinator `adhoc f g` which applies `g` to the node if `g`'s source type is the same as that of the node, or otherwise applies `f`, which *must* have the same source type as the node's. The fourth row lists two one-layer traversal combinators: `all f` which applies `f` to all the immediate subnodes of its input node, and `one f` which applies `f` to the leftmost subnode only. These combinators are not recursive. Recursion is achieved by tying the knot:

```
topdown f  = seq f (all (topdown f))
bottomup f = seq (all (bottomup f)) f
```

Notice here the pattern:

$$traversal\_scheme\ f = combinators\ traversal\_scheme\ f$$

which is indeed a fixed-point definition of the form $f\ x = F\ f\ x$ where f is a fixed point of F.

The following example illustrates the power of Strategic Programming: given a heterogeneous data structure `t` which contains, among other things, integer values, the following combinator only increments the values of those integers, traversing the structure in top-down fashion:

```
incInts t = (topdown (adhoc id (+1))) ◇ t
```

Notice the separation between genericity (traversal scheme with identity) and specificity (action at specific nodes only).

Combinators are composable and have nice algebraic properties. This permits the definition of new combinators for particular purposes. Theoretically, it seems that two combinators suffice to express all strategic programs [LV02c].

Strategic Programming combinators are paradigm independent. There are incarnations with running applications for term-rewrite systems and attribute grammars [VS04], for object-oriented systems as improvements of the *Visitor* pattern [Vis01] and, in the functional paradigm, there is the *Strafunski* bundle which includes a functional combinator library and tool support for doing Strategic Programming in Haskell.[6]

6.2.2 SyB tour

This section describes SyB paper by paper.

The first paper [LP03] begins by presenting a *dynamic nominal type-case* operator. It is called *type-safe cast* in the paper for historical reasons and because it is implemented using a cast operator that cannot cause run-time errors.

The implementation of `cast` relies on clever type-class and reflection tricks that enable it to determine the type of a value at run time by means of applying the instance *for that type* of an overloaded function `typeOf` that has been figured out at compile time by the type-checker.

More precisely, the `cast` operator has type:

```
cast :: (Typeable a, Typeable b) ⇒ a → Maybe b
```

where the `Typeable` type class declares the aforementioned `typeOf` function. The definition of `typeOf` is derived automatically by the compiler for a data type using Haskell's **`deriving`** clause. The function returns a value that makes for the representation of a manifest type.

Operationally, the application `cast x` within a context of manifest type **`Maybe`** `T` returns

[6] Strafunski comes from Strategic and Functional programming resembles the music of Igor Stravinski (`http://www.cs.vu.nl/Strafunski`).

Just x if x has type T; otherwise returns **Nothing**:

```
(cast 1) :: Maybe Int
> Just 1
(cast 1) :: Maybe Char
> Nothing
```

In other words, we get a **Just** value when, at run-time, a=b. The name 'cast' is justified in that a value of type **Maybe** b is obtained from a value of type a. However, the behaviour of cast x is that of a type-case that checks at run time whether the value x is of type T, returning **Just** x of type **Maybe** T if the answer is positive or returning **Nothing** of type **Maybe** T' if the answer is negative, where T' is the actual type of x.

The paper goes on to introduce several operators that can be seen as versions of adhoc implemented in terms of cast: mkT for type-preserving actions, mkM for monadic type-preserving actions, and mkQ for type-unifying actions. For example, given a function f of type a → a, mkT f x applies f to x only if x has type a, returning x otherwise. Clearly, mkT f corresponds to adhoc id f. Function mkT (or 'make transformation') lifts a transformation (action in Strategic Programming jargon) on a value of a fixed type into a transformation on a value of type ∀a. Typeable a ⇒ a. It is therefore called a *generic transformation* by the authors.

There are three one-layer traversals called gmapT (type-preserving), gmapM (monadic type-preserving) and gmapQ (type-unifying) defined in type class Data:

```
class Typeable a ⇒ Data a where
 gmapT :: (∀ b. Data b ⇒ b → b) → a → a
 gmapM :: Monad m ⇒ (∀ b. Data b ⇒ b → m b) → a → m a
 gmapQ :: (∀ b. Data b ⇒ b → r) → a → [r]
```

Their behaviour is sketched below for an arbitrary n-ary value constructor C. *One-layer traversals are polytypic*, *i.e.*, defined on the structure of the node:

```
gmapT mt (C t1 ... tn) = C (mt t1) ... (mt tn)

gmapM mt (C t1 ... tn) = do t1' ← mt t1
                            ...
                            tn' ← mt tn
                            return C t1' ... tn'
```

```
gmapQ mt (C t1 ... tn) = [mt t1, ... ,mt tn]
```

In words, `gmapT` applies a generic transformation `mt` (built using `mkT`) to all the immediate subchildren of 'node' `C` whereas `gmapM` applies a *monadic* generic transformation (built using `mkM`). In contrast, `gmapQ` applies a *generic query* (built using `mkQ`, which returns a result) to the immediate subchildren but returns a list of results. Functions `gmap`X can be applied to any data type that has been made an instance of type class `Data`. The compiler can be instructed to generate the instances automatically using the **`deriving`** clause.

Rank-2 polymorphism is required for typing generic transformations and queries. It is also required for typing traversal combinators which take generic transformations or queries as arguments. There are three traversals for the three modes: `everywhere f x` applies a transformation to every node in a data structure in bottom-up fashion:

```
everywhere mt x = mt (gmapT (everywhere mt) x)
```

Similarly, there is a monadic traversal scheme `everywhereM` and a type-unifying traversal scheme `everything`, all declared as members of type class `Data`.

The paper shows that all one-layer traversals are idioms of a one-layer traversal called `gfoldl`:

```
class Typeable a ⇒ Data a where
  gfoldl :: ∀ w a. (∀ a b. Data a ⇒ w (a → b) → a → w b)
                → (∀ g. g → w g) → a → w a
```

Its behaviour is sketched below for an arbitrary n-ary value constructor `C`. *The function is polytypic*, *i.e.*, defined on the structure of a 'node':

```
gfoldl k z (C t1 ... tn) = k ( ... (k (z C) t1) ... ) tn
```

An ordinary fold would replace the value constructor by an n-ary function. In contrast, `gfoldl` passes the value constructor `C` to its second argument and applies its first argument, function `k`, to the result and the first sub-node. The value produced is passed again to `k` which is also passed the second sub-node, and so on. Like function application, `k` associates to the left, hence the name `gfoldl` and not `gfoldr`. For a node `t`:

```
gfoldl ($) id t == t
```

where `($)` is prefix function application.

The one-layer traversal `gmapT` can be defined in terms of `gfoldl` approximately as follows:

```
gmapT mt = gfoldl k id
  where k x y = x (mt y)
```

For instance, given a binary value constructor `C`:

```
gmapT mt (C t1 t2) =  gfoldl k id (C t1 t2)
                   =  k (k (id C) t1) t2
                   =  k (C (mt t1)) t2
                   =  (C (mt t1)) (mt t2)
                   =  C (mt t1) (mt t2)
```

The definition is an approximation because it does not type-check: the value returned by `gmapT` has type `a`, not `w a`. The authors show how to fix the definition of `gfoldl` so that `w` is an identity type, *i.e.*, `w a` = `a`.

The paper also introduces combinators for action composition and other traversal schemes. It uses as a running example a set of heterogeneous kind-$*$ data types but argues that instances of `Data` can be generated automatically for irregular and parametric type-operators. However, Haskell's type-class mechanism imposes limits with respect to the kind of type-operators, and the dynamic nominal type-case is performed on *manifest* types, not *parametric* type-operators.

A `gsize` function can be implemented in SyB as follows:

```
gsize :: Data a ⇒ a → Int
gsize x = 1 + sum (gmapQ gsize x)
```

The type-unifying one-layer traversal `gmapQ` applies `gsize` recursively to the immediate subchildren of `x`, adds up the list of results, and adds one to account for the size of the present node.

We have already seen some examples of `gsize` in Generic Haskell:

```
gsize⟨List⟩ (const 1) (Cons 1 (Cons 2 Nil))
> 2
```

```
gsize⟨List Int⟩ (Cons 1 (Cons 2 Nil))
> 0
```

In SyB, however:

```
xs :: List Int
xs = (Cons 1 (Cons 2 Nil))
gsize xs
> 5
```

First, we have to provide the type of `xs` explicitly in an annotation, the application:

```
gsize (Cons 1 (Cons 2 Nil))
```

will confuse the type checker which cannot decide whether type variable `a` in `gsize`'s type signature is constrained by `Data` or **`Num`**. The culprit is Haskell's monomorphism restriction. Second, `gsize` counts every value constructor into the final size, as illustrated by the evaluation trace:

```
gsize (Cons 1 (Cons 2 Nil))
= 1 + sum (gmapQ gsize (Cons 1 (Cons 2 Nil)))
= 1 + sum ([gsize 1, gsize (Cons 2 Nil)])
= 1 + sum ([1 + sum (gmapQ gsize 1), gsize (Cons 2 Nil)])
= 1 + sum ([1, 1 + sum (gmapQ gsize (Cons 2 Nil))])
= 1 + sum ([1, 1 + sum ([gsize 2, gsize Nil])])
= 1 + sum ([1, 1 + sum ([1,1])])
= 5
```

The trace also illustrates that every recursive call to `gsize` adds one to the sum of the size of the subnodes. In contrast:

```
gsize x = sum (gmapQ gsize x)
gsize xs
> 0
```

No size! Finally, notice that `gsize` is applied to a manifest type of type class `Data` and, so far, we cannot be parametric on the size of its payload.

Another important feature presented in [LP03] is ***polytypic function extension*** or specialisation, *i.e.*, the ability to override the polytypic function's behaviour for specific *monomorphic* types. As expected, there are three extension combinators: `extT`, `extM`,

and `extQ`. An application `extX g f x` applies the specialised *monomorphic* function `f` to `x` if at run time `x`'s type matches `f`'s source type; otherwise it applies the *polytypic* function `g`. Certainly, these operators are another manifestation of `adhoc`, only that the default behaviour is now provided by a polytypic function. An example involving `gsize`:

```
f :: Company → Int  -- provides size for Company values
gsize = gsize_default `extQ` f
  where gsize_default x = 1 + sum (gmapQ gsize x)
```

The original definition of `gsize` is rewritten into a default case, `gsize_default`, and a specialised case, `f`, for values of type `Company`. Notice the fixed-point nature of the definition: `gsize` passes `gsize_default` to `extQ` and `gsize_default` calls `gsize`. Notice also that the type-specific behaviour, `f`, is *fixed* in `gsize`'s definition. Adding new type-specific behaviour entails the recompilation of `gsize`:

```
h :: Client → Int  -- provides size for Client values.
gsize = gsize_default `extQ` f `extQ` h
  where ...
```

Providing the extension in a new function will not avoid recompilation: `gsize_default` has to call the new function:

```
gsize' = gsize `extQ` h
gsize ...
  where gsize_default x = 1 + sum (gmapQ gsize' x)
```

Finally, notice there are no checks for overlapping extensions; the following is possible:

```
gsize = gsize_default `extQ` f `extQ` f `extQ` f
```

The second paper [LP04] extends the original approach endowing the classes `Typeable` and `Data` with reflection operators that allow programmers to dynamically enquire about a value's type, its value constructor names, their fixity, etc. Reflection operators can be derived automatically by the Haskell compiler. Their implementation follows much of the type-class trickery used in the implementation of `cast`.

The paper shows how to program serialisers such as pretty-printers and encoders, de-serialisers that illustrate the benefits of lazy evaluation, and test-data generators. It also adds new one-layer, zip-like combinators for traversing two data structures at the

same time, thus enabling the definition of functions such as `geq`. Finally, new versions of `cast` are provided for performing dynamic nominal type-case on type-operators of up to seven type arguments of kind $*$. Each cast lives in a separate type class `Typeable`i ($i : 1 \ldots 7$) which can again be derived automatically by the compiler. The generalisation of `cast` affords the generalisation of the extension combinators `extT`i, `extQ`i, and `extM`i, which now support *polymorphic* extension of polytypic functions. For example, in `extQ1 g f x`, function `f` is *polymorphic* on a type-operator of kind $* \rightarrow *$; in `extQ2 g f x`, `f` is polymorphic on a type-operator of kind $* \rightarrow * \rightarrow *$, etc.

The implementation is carried out within Haskell, or one should say, within the Haskell compiled by the Glasgow Haskell Compiler which supports the required non-standard extensions and the automatic derivation of classes `Data` and `Typeable`.

The third paper [LP05] refines SyB further with the possibility of extending generic functions in a *modular* fashion. In [LP03], polytypic extension is achieved by means of combinators such as `extQ`, but extending a polytypic function with new type-specific cases entails recompilation—recall the `gsize` and `gsize_default` examples.

In contrast, Haskell's type classes support an open-world approach to function extension: for every newly-defined type, the specific instance of an overloaded function for that type is defined by declaring the type an inhabitant (**`instance`**) of the type class in which the overloaded function name is declared. The instance declaration provides the body of the function for that type. For example:

```
class Size a where
  size :: a → Int

instance Size Int where
  size x = 1

instance (Size a, Size b) ⇒ Size (Pro a b) where
  size (x,y) = size x + size y
```

The problem with overloading is that it is not generic. Each version of the overloaded function has to be programmed explicitly. Providing instances is an incremental process in which there is no need to edit and recompile previously written code.

Modular polytypic extension combines polytypism and the incremental extension

provided by the type-class mechanism. The idea is to get at something like:

```
-- Pseudo-code
class Size a where
  gsize :: a → Int

instance Size a where
  gsize x = 1 + sum (gmapQ gsize x)

instance Size Company where
  gsize = f
```

A polytypic function name is now a type-class operation. Its code and extensions are provided in instances. The first **instance** above specifies `gsize`'s default behaviour for all types `a`. The second specifies the behaviour for `Company` values. Unfortunately, the code does not type-check: type variable `a` is constrained by type class `Data` in `gsize`'s type signature, and this constraint must appear in the **instance** heading:

```
instance Data a ⇒ Size a where
  gsize x = 1 + sum (gmapQ gsize x)
```

Furthermore, `gmapQ` expects a first argument of type:

```
Data a ⇒ a → r
```

but it is passed `gsize`, whose type is less general and has a different type-class constraint:

```
Size a ⇒ a → Int
```

The authors argue that making `Size` a superclass of `Data` would solve the problem, but the type-class mechanism works by extending superclasses with subclasses and not the opposite. The authors propose to extend the type-class mechanism with ***type-class abstraction*** and ***type-class application*** in type-class and instance declarations. They show how to encode the extension directly in Haskell using some of the ideas in [Hug99].

The rest of the paper is devoted to developing the necessary machinery. The most problematic point is the definition of recursive **instance** declarations such as:

```
instance Data Size t ⇒ Size t where
  gsize x = 1 + sum (gmapQ{|Size,t|} gsize x)
```

There is a fixed-point 'equation' in the instance heading which may compromise the decidability of type inference, a problem whose full solution is considered future work (braces with vertical bars denote type-class application).

Now, the RTTI tests of `extX` are avoided and which `gsize` function to call on a value of a type is decided at compile time: either there is an instance of `gsize` for that type or otherwise polytypic `gsize` is applied. Notice that this means there is no way of overlapping or extending a polytypic function in different ways for the same data type unless the compiler supports overlapping **`instance`**s.

Notice that by making polytypic functions members of type classes, these type classes will appear in the type signatures of client functions.[7] For example [LP05, p4]:

```
gdensity :: (Size a, Depth a) ⇒ a → Int
gdensity x = gsize x / gdepth x
```

There is a possible impact on maintainability: changing the implementation of `gdensity` may affect type-class constraints and, hence, affect its client functions. In the original scheme, the type of `gdensity` was the more general:

```
Data a ⇒ a → Int
```

A possible patch that localises the change would consist of hiding classes `Size` and `Depth` behind a `Density` sub-class with no operators:

```
class (Size a, Depth a) ⇒ Density a

gdensity :: Density a ⇒ a → Int
gdensity x = gsize x / gdepth x
```

A change in `gdensity` may entail a change in `Density` but client functions are unaffected.

We conclude this section with an example that illustrates the advantages of polytypic extension. On page 159, we showed how polytypic `gsize` counts value constructors when calculating the size of a list. With polytypic extension, we can customise the `gsize` for lists to count only payload elements:

[7] Compare with Dependency-style Generic Haskell, where the type of a polytypic function includes the *names* of other polytypic functions it calls.

```
instance Size a ⇒ Size (List a) where
  gsize Nil         = 0
  gsize (Cons x xs) = gsize x + gsize xs
```

We cannot be parametric on the size of the payload type—which is computed by `gsize`'s instance for the **`Int`** type—, but at least value constructors are avoided in the total count:

```
xs :: List Int
xs = (Cons 1 (Cons 2 Nil))
gsize xs
> 2
```

6.3 Generic Haskell vs SyB

Generic Haskell and SyB differ in approach and style.

In Generic Haskell, polytypic functions have polykinded types. The former are defined by induction on the structure of representation type operators (where only the behaviour for base types and base representation types is required). The latter are defined by induction on kind signatures. The Generic Haskell compiler replaces polytypic applications by calls to generated instances. It also generates some internal machinery: representation type operators, embedding-projection pairs, etc.

In SyB, polytypic functions are defined in terms of generic one-layer traversals that can be derived automatically by the Haskell compiler for manifest types and type operators whose kind signature is described by the grammar $\kappa ::= * \mid * \to \kappa$ up to seven expansions.

In SyB representation types are unnecessary: type-based dispatch is nominal and relies on `typeOf`. In contrast, Generic Haskell has representation types and type-based dispatch is unnecessary.

SyB employs Strategic Programming ideas to separate specificity (*nominal* action at a node) from genericity (traversal based on *structure*). Generic Haskell is purely structural and specificity can only be achieved via polytypic extension.

SyB is carried out within Haskell with a (supposedly) minimal extension: instructing the compiler to generate type classes `Typeable` and `Data`. In contrast, Generic Haskell

is a language extension requiring a pre-processing compiler. But polytypic functions are not type-checked independently of their instantiations. It is the Haskell compiler the one that type-checks that *generated* instance bodies conform to *generated* type signatures.

Generic Haskell and SyB differ in the model of computation. In SyB there is a set of one-layer traversals that are used in the definition of polytypic functions on *manifest types*. SyB also supports monadic traversals. In contrast, polytypic functions in Generic Haskell work on *unconstrained type operators* of arbitrary kinds. The fact that SyB's implementation is type-class based imposes technical limits: there are only seven `Typeable`i classes.

In SyB, `gfoldl` and one-layer traversal idioms perform computations on value constructors. In Generic Haskell, polytypic functions can perform computations on value constructors but seldom do so (contrast the `gsize` examples in page 159).

Despite the differences, we consider Generic Haskell and SyB polytypic language extensions of Haskell. Abstractly, polytypic programming is characterised thus: in polymorphic languages, functions are 'separated' by the types on which they work. Polytypic functions must work on values of different types. They are a generalisation of families of overloaded or polymorphic functions whose types and bodies can be generated automatically in regular fashion. Polytypic functions are used as if they were a single function. Interestingly, this 'single' function can be given a type. The rest is implementation detail.

We conclude the section with a short description of the paper [HLO06] which is, in part, an attempt to explain the dynamic behaviour of `gfoldl` by means of making the dynamic type information explicit at compile time. More precisely, the paper defines a ***generalised algebraic data type*** [PWW04] for representing type information as values in programs:

```
data Type :: ∗ → ∗ where
  TInt  :: Type Int
  TChar :: Type Char
  TPair :: Type a → Type b → Type (a,b)
  TSum  :: Type a → Type b → Type (Sum a b)
  TList :: Type a → Type (List a)
```

...

For example, the value constructor `TInt` is a value representation of the type **`Int`**; the value constructor `TChar` is a value representation of the type **`Char`**; the expression `TPair TInt TChar` is a value representation of the type **`(Int,Char)`**.

When types are explicitly represented as values at run-time, generic functions can simulate type-based dispatch, for instance:

```
gsize :: Type a → a → Int
gsize TInt  i               = 0
gsize TChar c               = 0
gsize (TPair ta tb) (x,y)   = gsize ta x + gsize tb y
gsize (TSum  ta tb) (Inl x) = gsize ta x
gsize (TSum  ta tb) (Inr y) = gsize tb y
gsize (List ta) xs          = sum (map (gsize ta) xs)
...
```

In the paper, a type `a` is said to be typed if it is represented by a `Type a` value:

```
data Typed a = HasType a (Type a)
```

The authors introduce another generalised algebraic data type, called `Spine` that plays the same role as a LISP S-expression. There is a 'generic' `toSpine` function that translates `Typed` values to ('un-`Typed`') `Spine` values:

```
data Spine :: ∗ → ∗ where
  Constr :: a → Spine a
  (⋄)    :: Spine (a → b) → a → Spine b

toSpine :: Type a → a → Spine a
```

In the definition of `Spine`, `Constr` plays the role of `z` and ⋄ the role of `k` in the definition of `gfoldl`, which is now expressed thus:

```
gfoldl :: Type a → (∀ a b. w (a → b) → Typed a → w b)
                 → (∀ a. a → w a)
                 → a → w a
gfoldl t k z = foldSpine k z ∘ toSpine t
```

However, the generalised algebraic type `Type` is not extensible: the translation from

Haskell to LISP (*i.e.*, from `Typed` values to `Spine` values) explains `gfoldl`'s dynamic behaviour statically within Haskell for one compilation. In SyB, when new types are added the compiler generates their `gfoldl` instances. In contrast, extending `Type` entails its recompilation.

6.4 Lightweight approaches

There are lightweight approaches for doing polytypic programming *within* Haskell. The most notable ones are (1) [CH02] which relies either on existential types[8] or on generalised algebraic data types [PWW04] to enforce the correspondence between a type and its representation type; (2) [Hin04] that extends the previous approach in order to define polytypic functions on type-operators of order 1 within Haskell 98; and (3) [OG05] which generalises and collects the ideas into a programmable design pattern and considers polytypic functions with polytypic (type-indexed) types.

[8] Type-terms where universal quantification occurs in contravariant position; not to be confused with other notions of existential types that model data abstraction [Pie02].

Chapter 7
Polytypism and Data Abstraction

> If you make a small change to a program, it can result in an enormous change in what the program does. If nature worked that way, the universe would crash all the time. (Jaron Lanier)
> Successful software always gets changed. (Fred Brooks)

In Generic Haskell, polytypic functions are defined by induction on the *concrete* definitional structure of a type operator, and their polykinded types by induction on the type operator's kind signature. In SyB, polytypic functions are defined in terms of one-layer traversals whose generated instances are applied to the *concrete* structure of a 'node'.

Access to concrete representations conflicts with the principle of data abstraction. More precisely, data abstraction limits polytypism's genericity.

The present chapter articulates the previous statement. Some readers may deem this unnecessary. For them the step from the fact that "functions that access ADT representations can wreak havoc" to the fact that "*polytypic* functions [or their instances for that matter] that access ADT representations can wreak havoc" requires no arguments nor examples. But it is important to drive home the point for those lured by the 'generic' adjective. There are also conflicts specific to the nature of Generic Haskell and SyB. Calling a function *structurally* polymorphic highlights the fact that the function is dependent on structure and whether structure changes.

The whole issue is bound to spur philosophical disagreement. Access to concrete representations is one of the dearest tools of the functional programmer—or at least of 'non-Lispers'. Think for example of functions defined by pattern matching. Or think about the fact that complex data-structure definitions such as first-class, extensible higher-order records, or first-class modules for that matter, are ignored by polytypic languages which always assume a world of algebraic data types (*i.e.*, sums of products). In Haskell 98, the standard record and module system is even found wanting.

7.1 Polytypism conflicts with data abstraction

First and foremost, concrete representations are logically hidden and often physically unavailable (*e.g.* pre-compiled libraries). Second, if polytypic functions were allowed to sneakily access an ADT's representation, or were tipped off by an oracle, they would not work satisfactorily. Data abstraction brings about a different game. More precisely:

1. A pure function must return the same result when applied to the same argument. This also applies to functions on ADT values. If the function computes its result by accessing the ADT's representation and the representation changes, the value computed may also change despite that the function is applied to the same 'abstract' value. Polytypic functions are subject to this problem just like ordinary, non-generic ones.

   In particular, functions accessing representations can be affected by ***implementation clutter***, *i.e.*, data relevant only to the implementation of the type. More precisely, an ADT may be implemented using various concrete types, parts of which may contain data used for efficiency or structuring purposes. The well-known trade-off between time and space indicates that this is bound to happen often: extra data will be used in order to improve operator speed. It would be rather difficult, if not impossible, for a function to ascertain the *pertinence* of data components *in implementations* and to know what to do with them in a semantics-preserving way. (Notice that clutter is not the same as junk: clutter can be part of a non-junk value of the concrete type that represents a value of the abstract type.)

2. A function accessing ADT representations may violate the ***implementation invariants*** which guarantee that concrete values are valid representations of abstract values. These invariants are maintained by ADT operators and are the *raison d'être* for hiding the representation behind an interface. A violation of the implementation invariants most certainly entails a violation of the type's semantics, *i.e.*, the value computed is not a value of the ADT.

3. Polytypic functions also have problems of their own:

   (a) A manifest (kind-$*$) ADT has the same status as a base type. Polytypic function definitions have to provide cases for them like they do for integers or booleans (Section 6.1.12).

In Generic Haskell, adding a new base type requires editing and recompiling polytypic function definitions (except those defined by polytypic abstraction), unless the new case is added incrementally using polytypic extension. In SyB, the ADT has to be made an instance of `Typeable` and `Data` by the ADT *implementor*, who is the only one entitled to the internals of the type.

In both cases, if the ADT implementation changes the definition of the polytypic function must be changed accordingly. Client code may be affected if the results computed by the new definition differ from its previous version.

From the viewpoint of Generic Programming, it is better to rely on parametrisation than to rely on extension. Parametrisation allows us to write code once by adapting parameters whereas extension requires us to provide new definitions for specific types, *i.e.*, there is no 'generic' programming here other than name reuse and, furthermore, providing different (overlapping) definitions for the same type is currently not possible.

(b) Generic Haskell does not support constrained types (Section 6.1.10) which arise frequently in ADT implementations: order in binary search trees, equality in sets, etc. In contrast, SyB supports constrained types when the payload is also constrained on `Data`, *e.g.*:

```
instance (Data a, Ord a) ⇒ Data (OrdSet a) where
  gfoldl k z s = k (z fromList) (toList s)
  ...
```

With polytypic extension, `gsize` has a `Size` constraint instead of a `Data` constraint, so **`Ord`** must be declared a superclass of `Size`. The type-class parametrisation framework suggested in [LP05] is undergoing research.

4. Polytypic extension is not a satisfactory solution. Let us illustrate this point using `gsize` as a running example.

   With polytypic extension it is possible to write a definition of `gsize` for a given ADT such that it upholds the implementation invariants and ignores implementation clutter. But there are two problems with this:

   (a) *Who writes the definition?* It could be written by the polytypic programmer if granted access to the ADT's implementation. However, it is disturbing to define a function that accesses the implementation *outside* the ADT. The definition

should be provided by the ADT *implementor*, who can update the definition if the implementation changes. However, providing `gsize` as an ADT operator amounts to providing the `size` operator directly. The fact that `gsize` is an overloaded operator name and that there exists a polytypic version of it is an orthogonal issue. The ADT implementor must provide a `size` operator and the polytypic programmer must define the polytypic extension of `gsize` for the ADT in terms of `size`.

Unfortunately, it is not possible for an ADT implementor to foresee all possible operators that can be employed in the polytypic extension of future polytypic functions. Polytypic extension takes place *after* a polytypic function has been defined.

In sum, we end up in a visibility problem: polytypic programmers are ADT *clients* and cannot customise their polytypic functions for those ADTs by accessing their implementation.

(b) *Where is the genericity?* Of course, polytypic programmers can use ADT *interfaces* to define their extensions. However, we would like to have a polytypic `gsize` that works for all types, concrete or abstract, not one whose version for every new ADT has to be explicitly programmed. What is desired is *automatic* polytypic extension.

The following sections elaborate and illustrate these points with a few examples. Please note that this chapter is not meant to be a criticism of Generic Haskell or SyB. That would be unfair, for coping with ADTs is not a design goal of these language extensions. We just aim at exposing polytypism's genericity limitations in order to argue the case for our solution.

7.1.1 Foraging clutter

It is typical of many ADT implementations to use elaborate concrete types with clutter of fixed types or payload type.

As a typical example, consider ordered sets supporting the following operators: `empty`, `isEmpty`, `insert`, `member`, and `remove`. An ordered set can be implemented in terms of Red-Black Trees or in terms of Leftist Heaps which contain, respectively, colour and height components used for re-balancing the tree during insertion and removal [Oka98a,

p197,p203]:

```
data Colour  = Red | Black
data RBT a   = E | N Colour a (RBT a) (RBT a)
data LHeap a = E | N Int a (LHeap a) (LHeap a)
```

Polytypic functions take these clutter components into account. In Generic Haskell, their contribution to the computation depends on the definition for units and integers—recall Figure 6.5 and Section 6.1.12. Polytypic `gsize` calculates the size correctly because the size for integers and units is zero. However, serialisation functions such as pretty-printers or encoders would print or encode the clutter components. In SyB, `gsize` counts all the value constructors in nodes, *e.g.*:

```
t :: LHeap Int
t = N 0 5 E E
gsize t
> 5
```

It is more reasonable to expect the size (cardinality) of the set $\{5\}$ to be 1, counting only the number of payload elements. ADT clients care less about internal value constructors. Polytypic extension comes to the rescue:

```
instance Size a ⇒ Size (LHeap a) where
  gsize E           = 0
  gsize (N i x l r) = gsize x + gsize l + gsize r
```

but, as argued in Section 7.1, from the viewpoint of Generic Programming this is an unsatisfactory solution.

Clutter can be of payload type. Let us present a simple example first. Imagine an ordered *cached* container `CSet a` for which the membership test for the last inserted element takes constant time. It could be implemented in terms of ordered lists or binary search trees, as shown in Figure 7.1. For brevity, only the implementation of insertion is shown.

Value constructor `CE` represents an empty `CSet` and value constructor `C` a non-empty `CSet` with a cached element and a concrete type with all the payload. In the list implementation, a value `C` t `Nil`, where t is an arbitrary term, does not represent a `CSet` value and constitutes *junk* (Chapter 5). Similarly, a term `C` t `BinTree.empty` in

```
module CSet (CSet,empty,isEmpty,insert,member) where
  data Ord a ⇒ CSet a = CE | C a (Payload a)
```

```
-- List implementation
type Payload a = [a]

insert :: Ord a ⇒ a → CSet a → CSet a
insert x CE       = C x [x]
insert x (C y ys) = C x (x:ys)
```

```
-- BinTree implementation
import BinTree
type Payload a = BinTree.BinTree a

insert :: Ord a ⇒ a → CSet a → CSet a
insert x CE       = C x (BinTree.insert x BinTree.empty)
insert x (C y ys) = C x (BinTree.insert x ys)
```

Figure 7.1: `CSet` implemented in terms of ordered lists or binary search trees with respective implementation of insertion.

the binary search tree implementation constitutes junk.

Both representations contain clutter: a unit value and a cached value of payload type. In the list implementation, Generic Haskell's `gsize` counts the latter when computing the size:

```
c = foldr (λx y → CSet.insert x y) CSet.empty [1,2]
gsize⟨CSet⟩ (const 1) c
> 3
```

The result should have been 2. The extra unit is also counted by SyB's `gsize`:

```
c :: CSet Char
c = CSet.insert 'A' CSet.empty
gsize c
>  5
```

The results for the binary search tree implementation depend on the concrete type implementing `BinTree`. If implemented as a Red-Black Tree then SyB's `gsize` will return a different value than if implemented as an ordinary binary search tree.

FIFO-queue implementations afford many examples of clutter. The FIFO-queue interface is shown in Figure 7.2 (top box). There are many possible implementations [Oka98a,

p186–189]: Batched Queues, Physicist's Queues, Banker's Queues, Hood-Melville Queues, etc. Their representation types are shown in Figure 7.2 (bottom box).

```
module Queue(Queue(..)) where
 class Queue q where
   empty   :: q a
   isEmpty :: q a → Bool
   enq     :: a → q a → q a
   front   :: q a → a
   deq     :: q a → q a
```

```
data BatchedQueue a   = BQ [a] [a]
data BankersQueue a   = BnQ Int [a] Int [a]
data PhysicistQueue a = PQ [a] Int [a] Int [a]
data RotationState a = Idle
                     | Reversing Int [a] [a] [a] [a]
                     | Appending Int [a] [a]
                     | Done [a]
data HoodMelvilleQueue a = HMQ Int [a] (RotationState a) Int [a]
data BootStrappedQueue a = E
                         | Q Int [a] (BootStrappedQueue [a]) Int [a]
```

Figure 7.2: Queue interface and some possible implementations.

A Batched Queue uses two lists where the first contains the front elements in correct order and the second the rear elements in reverse order. When the front list is emptied the rear list is rotated and becomes the front list. A Banker's Queue keeps also the length of both lists. Elements are moved from the rear to the front periodically when `length f == length r + 1`, replacing `f` by `f ++ reverse r`, *i.e.*, an expression that in a lazy language like Haskell is only evaluated on demand (a suspension).

A Physicist's Queue also tracks the lengths of the lists but it keeps another list that is a prefix of the front list to avoid the constant evaluation of the suspension.

A Hood-Melville Queue tracks the lengths of the lists and uses an auxiliary data structure that captures the state of the reversal explicitly.

A Bootstrapped Queue is a recursive irregular type (Section 6.1.1) with one unit element representing empty queues. The recursive case has two integers, one counting the length of the front list plus the length of all the suspended lists in the recursive substructure, and another counting the length of the rear list. Irregular types can be converted into regular ones "introducing a new datatype to collapse the different instances into a single

type ... [irregularity] really refers more to how we think about a datatype than to how it is implemented" [Oka98a, p143]. Irregular types are nevertheless preferred for the reasons given in Section 6.1.1. An irregular type can be changed into a regular one introducing more auxiliary types, *i.e.*, more clutter.

The reader is referred to [Oka98a] for details about the implementation of these queues and other functional data structures.

All queue implementations conform to the `Queue` interface. We expect functions operating on queues not to be affected by changes in their implementation. This is not the case when the representation is accessed directly.

In SyB, `gsize` would produce different results because the number and type of node components changes dramatically, *e.g.*:

```
gsize Queue.empty   -- BatchedQueue: BQ Nil Nil
> 3
gsize Queue.empty   -- BootStrappedQueue: E
> 1
```

In Generic Haskell, redundant elements of payload type are added by `gsize` into the total count, *e.g.*:

```
q = foldl (λx y → Queue.enq y x) Queue.empty [7,5,9,4,6]
> PQ [7,5,9] 5 [7,5,9,4,6] 0 []   -- implemented as PhysicistQueue

gsize⟨Queue⟩ (const 1) q
> 8  -- instead of 5
```

7.1.2 Breaking the law

Consider the ADT of ordered sets. Among the type's laws there is one indicating that ordered sets have no duplicates, *e.g.*:

```
insert x (insert x s) = insert x s
```

Suppose that lists without duplicates are used as concrete representations:

```
module Set(Set,empty,isEmpty,insert,member) where
data Ord a ⇒ Set a = MkSet [a]
empty  :: Ord a ⇒ Set a
```

```
empty  = Set []
insert :: Ord a ⇒ a → Set a → Set a
insert x (MkSet xs) = MkSet ((sort ∘ nub) (x:xs))
...
```

Function **nub** removes duplicates from a list.

The application of `gmap` may introduce duplicate elements making the result a list that is no longer a valid representation of a set. For example, mapping **const** `5` over the set $\{1, 2, 3\}$ should yield $\{5\}$. This is not the case:

```
s = foldr (λx y → Set.insert x y) Set.empty [1,2,3]
gmap⟨Set⟩ (const 5) s
> MkSet [5,5,5]
```

Ordered sets can also be implemented in terms of boolean vectors, where payload elements are indices, or hash tables, where hashed payload elements are indices (Section 5.6). However, the map function for a vector maps the elements *in* the vector, not the indices, let alone the values to which a (perhaps non-invertible) indexing function has been applied in order to obtain the indices.

Now consider the type of ordered trees of Section 5.7 which can be used to implement binary search trees, ordered sets, and priority queues. Again, the application of map over the implementation can break the structural invariants.

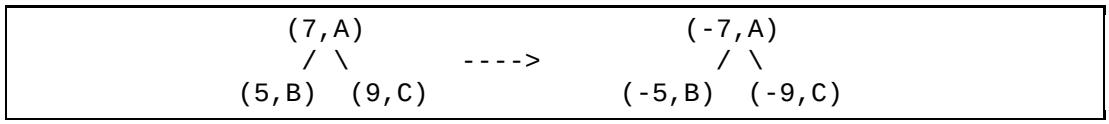


Figure 7.3: Mapping negation over a binary search tree representing a priority queue yields an illegal queue value.

Suppose an ordered tree implements a priority queue where the priority is given by an integer value (Figure 7.3, left tree). If we invert priorities, mapping `λ(x,y)→(-x,y)`, the resulting tree (Figure 7.3, right tree) is not a valid representation of a priority queue. The correct representation would be the mirror tree where the element with highest priority in the left subtree has *at most* the root's priority.

Consider a parameterised `MemoList` ADT supporting the following operators:

```
nil  :: ([a] → a) → MemoList a
```

```
module MemoList (MemoList,nil,cons,head,tail,memo) where
import Prelude hiding (head,tail,null)

data MemoList a = ML ([a] → a) a [a]

nil :: ([a] → a) → MemoList a
nil f  = ML f (f $! []) []

null :: MemoList a → Bool
null (ML _ _ [])     = True
null (ML _ _ (x:xs)) = False

cons :: a → MemoList a → MemoList a
cons x (ML f y ys) = ML f (f $! xs) xs
  where xs = x:ys

head :: MemoList a → a
head (ML _ _ [])     = error "Empty List"
head (ML _ _ (x:xs)) = x

tail :: MemoList a → MemoList a
tail (ML _ _ [])     = error "Empty List"
tail (ML f _ (x:xs)) = ML f (f $! xs) xs

memo :: MemoList a → a
memo (ML _ x _) = x
```

Figure 7.4: A `MemoList` implementation.

```
null :: MemoList a → Bool
cons :: a → MemoList a → MemoList a
head :: MemoList a → a
tail :: MemoList a → MemoList a
memo :: MemoList a → a
```

A function on *ordinary* lists is passed when creating an empty `MemoList` using `nil`. This function remains fixed during operation. The value of this function is recalculated in strict fashion every time an element is inserted (`cons`) or removed (`tail`) from the memo list. The value of the calculation can be obtained in constant time using `memo`. Operators `cons` and `tail` are inefficient because the value of `f` on list elements, which could take linear time to compute, is executed every time these operators are called. Memo lists are meant to be used in situations where there is a high degree of persistence and high demand for the memoised value.

Figure 7.4 shows an obvious implementation. The infix strict application `$!` eagerly evaluates its second argument and then the application of its first argument to the result. The representation value `ML f m xs` represents an `f`-memoised list with the implementation invariant `m` = `f xs`. Ordinary list operators can be readily programmed, *e.g.*:

```
mlength :: MemoList a → Int
mlength ml = if null ml then 0 else 1 + mlength (tail ml)
```

And we expect typical list equations to hold:

```
mlength (mconcat xs ys) == mlength xs + mlength ys
```

(Unfortunately, the type-checker cannot stop `mconcat` from concatenating memoised lists built using different memoised functions *of the same type*, for function equality is undecidable.)

Generic Haskell's `gsize` counts the memoised value.[1] But more worryingly, the application of `gmap` can easily break the implementation invariants and produce concrete values that do not represent values of the ADT:

```
mlA = cons 1 (cons 1 (cons 1 (nil sum)))
mlB = cons 1 (cons -2 (cons 3 (nil max)))

gsize⟨MemoList⟩ (const 1) mlA
> 4                    -- should have been 3
gmap⟨MemoList⟩ (+1) mlA
> ML ? 4 [2,2,2]       -- sum yields 6 not 4
gmap⟨MemoList⟩ negate mlB
> ML ? -3 [-1,2,-3]    -- max yields 2 not -3
```

Broadly speaking, the instance of `gmap` for `MemoList` behaves thus:

```
gmap⟨MemoList⟩ g (ML f m xs) = ML f (g m) (gsize⟨List⟩ g xs)
```

The implementation invariant breaks when `g m` ≠ `f` (`gmap⟨List⟩ g xs`).

[1] In point of fact, `gsize` (Figure 6.5) must provide a case for the arrow type operator or the application would produce a run-time error: memoised lists contain functions as data.

7.1.3 On mapping over abstract types

Some readers may wonder whether 'map' is justified in the case of non-free types. Categorially, a type is a functor if we can define its map function that satisfies the functorial laws:

$$\text{map id} = \text{id} \tag{7.1}$$

$$\text{map } (f \circ g) = \text{map } f \circ \text{map } g \tag{7.2}$$

The bits of category theory described in Chapter 3 assumed an unboundedly polymorphic world. It is because of this parametricity assumption that properties of polymorphic programs (natural transformations) can be obtained directly from their types (functors) [Rey74, Wad89].

The map function for *unbounded* ADTs (Section 5.10) must respect the number and position of elements. However, for *bounded* ADTs this does not have to be the case: there are context-dependent properties such as ordering, lack of repetition, etc, that must be preserved by map.

Let us concretise the point. Think of unbounded ADTs such as lists, stacks, FIFO queues, etc. For these ADTs the following equation is upheld by map:

```
map f (con x y) == con (f x) (map f y)
```

Here `con` stands for the binary constructor. Replace `con` by `cons` in the case of lists, by `push` in the case of stacks, and by `enq` in the case of FIFO queues. The fact that stacks and queues are subject to more equations is an orthogonal issue that relates to how **`map`** is actually defined in terms of the available observers. In particular, FIFO queues do not satisfy the product law:

```
con (exl q) (exr q) == q
```

where `con` is `enq`, `exl` is `front`, and `exr` is `deq` (Section 9.4).

The map equation may not hold for bounded ADTs, as demonstrated by all the examples in Section 7.1.2. Replace `con` by `insert` in ordered sets or `enq` in priority queues and the equation only holds when **`map`** preserves the order. Consequently, in the previous examples `gmap` is not the right function to apply to these ADTs, but a law-abiding `gmap` that preserves the semantic properties of the ADTs.

Is such function really a map? It is if it satisfies the functorial laws. Take ordered sets, for instance. First, let us make `Set` an instance of **`Functor`**. We use a *multi-parameter* type class **`Functor`** because `Set`'s implementation type is constrained by **`Ord`** and therefore we cannot make `Set` an instance of the **`Functor`** class provided by the Haskell prelude [MJP97]:

```
class Functor f a b where
  map :: (a → b) → f a → f b

instance (Ord a, Ord b) ⇒ Functor Set a b where
  map f (MkSet xs) = MkSet ((sort ∘ nub ∘ map f) xs)
```

Notice the overloading: **`map`** on the right hand side is map on lists.

`Set` is a functor not because it has been made an instance of **`Functor`** but because the definition of **`map`** satisfies the functorial laws. Let us use the following abbreviation:

ϕ `=` **`sort`** $\circ$ **`nub`**

Equation (7.1) is trivial to prove: if the set is empty then **`map`** returns another empty set, and the identity of an empty set is an empty set. If the set is not empty then the identity is mapped over the list which is not changed.

Equation (7.2) is also trivial to prove for the case of empty sets. For the non-empty case, let us first expand the left hand side of Equation (7.2):

```
    map (f ∘ g) (MkSet xs)
=       { def. of map for Set }
    MkSet ((φ ∘ map (f ∘ g)) xs)
=       { List is a functor }
    MkSet ((φ ∘ map f ∘ map g) xs)
```

Let us now expand the right hand side:

```
    map f ∘ map g (MkSet xs)
=       { def. of map for Set }
    map f (MkSet ((φ ∘ map g) xs))
=       { def. of map for Set }
    MkSet ( (φ ∘ map f) ((φ ∘ map g) xs) )
=       { composition }
```

```
MkSet ((φ ∘ map f ∘ φ ∘ map g) xs)
```

Both expansions are equal if the following holds:

```
φ ∘ map f ∘ map g = φ ∘ map f ∘ φ ∘ map g
```

We can prove this is the case using the definition of ϕ at this point.

However, let us indulge in a more general discussion: What is ϕ? It is a function that is applied to a value of the implementation type in order to make it satisfy the implementation invariants. In the case of ordered sets, it sorts and removes repetitions from the list.

We can provide a better definition of such *repairing* functions in terms of the whole implementation type. For ordered sets:

```
φ (MkSet xs) = MkSet (φ xs)
```

A map on sets is obtained by mapping over the payload type (list) and then applying φ to re-establish the implementation invariants:

```
map f (MkSet xs) = φ (MkSet (map f xs))
```

In balanced trees, φ performs the balancing. In binary search trees, φ turns a `BTree` into a binary search `BTree`, etc.

Recall the implementation type of memo lists from Section 7.1.2:

```
data MemoList a = ML ([a] → a) a [a]
```

The map for `MemoList` is defined as follows:

```
instance Functor MemoList where
  fmap f (ML g x xs) = ML g (g ys) ys
     where ys = map f xs
```

In other words:

```
fmap f (ML g x xs) = φ (ML g x (map f xs))
  where φ (ML g x xs) = ML g (φ g xs) xs
        φ g xs        = g xs
```

Again, the payload part of the concrete type is mapped and the implementation in-

variant is maintained by φ. Notice that `g` and `x` are implementation clutter and `g` is a function component that cannot be mapped by Generic Haskell's `gmap` (Section 6.1.2). However, the ADT is a functor, for we can define a map that satisfies the functorial laws.

Summarising, a map for a bounded ADT may be defined in terms of a function that maps over the payload parts of the implementation type and another function φ that re-establishes implementation invariants. The ADT is a functor if the functorial laws are satisfied.

We conclude the section discussing the impact of these issues on Generic Programming. `Set` has been made an instance of **`Functor`** by defining **`map`** in terms of the implementation type. Therefore, it is assumed that the instance declaration has been written *by the ADT implementor.* However, our aim is to program a *generic* map *outside* the ADT. If we attempt to define it in terms of the implementation type then we have to somehow figure out φ *polytypically* for any given ADT. This is not only a Herculean task; if the representation changes, it is also useless, for φ is no longer valid. However, the reader may have noticed that φ's job is already performed by `insert`! Interface operators seem to be part of the solution.

7.2 Don't abstract, export.

Section 6.1.12 discussed the drawbacks of providing fixed values for units and base types in polytypic function definitions. Programming with ADTs worsens the situation. It is impossible to give a *meaningful* case for all possible non-parametric (kind-$*$) ADTs in polytypic function definitions. It may not be possible physically or logically to turn them into parameterised ADTs by abstracting over the payload type.

An example: an event-driven GUI system keeps a queue of events. The type has been defined in a module:

```
module EventQueue (EventQueue,empty,isEmpty,enq,deq,front) where
import Event
import Queue
data EventQueue  = mkEQ (Queue.Queue Event.EventType)
empty            = mkEQ Queue.empty
isEmpty (mkEQ q) = Queue.isEmpty q
```

...

Type `EventQueue` is an ADT whose implementation in terms of a `Queue` ADT is hidden (the module only exports the type name and the operators). `EventQueue` is defined as a new type, not a type synonym, to enforce abstraction further. Programmers writing other parts of the GUI only know about the interface, and write their code accordingly. `EventQueue` is a manifest type. It plays the same role as a kind-$*$ basic type. Generic Haskell's `gsize` on an `EventQueue` value always returns 0 and `gmap` is the identity.

Let us leave aside the fact that `EventQueue` could have been bundled by a third-party provider as part of a pre-compiled library and its representation type would therefore be unknown (the library is closed source) and physically inaccessible. It makes no sense to abstract `EventQueue` into `EventQueue a`, for `a` is always `Event.EventType`; this is tantamount to using `Queue` directly. Abstraction is necessary only to use polytypic functions:

```
type EventQueue = Queue.Queue Event.EventType
q :: EventQueue
gsize⟨Queue.Queue⟩ (const 1) q
```

The type synonym is used everywhere in the program but polytypic function applications. The situation is rather strange: the programmer uses the type synonym and thinks in terms of `EventQueue` but has to use `Queue.Queue` when calling polytypic functions.

Now consider this scenario: after some beta testing, `EventQueue` implementors decide to use a direct implementation in terms of their own fancy queue type:

```
data EventQueue = EmptyQueue | Fancy [Event] Int Blah Blahdiblah ...
```

Polytypic *applications* with `Queue.Queue` become affected by this change.

It would be preferable for polytypic functions to be able to cope with the types in a software design than to adapt the software design to what polytypic functions can or cannot do.

Finally, it will be common for manifest ADTs to be used in the implementation of other manifest ADTs. We are faced with a cascading chain of abstractions which would force programmers to, pretty much, give up encapsulation:

```
module ProcessQueue(ProcessQueue,empty,isEmpty,enq,deq,front) where
import Process
import Queue
data ProcessQueue = PQ (Queue.Queue Process.ProcessType)
...
module ProcessManager(ProcessManager,create,run,kill) where
import Process
import qualified ProcessQueue as ProcQ
data ProcessManager = PM {
  current  :: Process.ProcessType,
  sleeping :: ProcQ.ProcessQueue,
  running  :: ProcQ.ProcessQueue
}
...
```

Another worrying aspect of abstracting over payload is that it might dredge parametricity constraints, affecting interfaces and client code. Using `EventQueue` is simple: programmers create `EventType` values and store them in values of `EventQueue`. Using `Queue` requires knowledge of the type's constraints. If implemented as a binary search tree, the constraint **`Ord`** becomes visible or, worse, other constraints imposed by the representation type implementing `Event.EventType`, some of which may be type classes only known by implementors.

What is needed is a different linguistic mechanism that allows manifest ADTs to indicate or ***export*** payload types. It is straightforward to have `EventQueue`'s interface specify that its payload is `Event.EventType`. It is a different thing to have to work with `Queue` whose payload *we know* it always to be `Event.EventType`. The payload type remains the same even if the implementation of the container changes, or even if the interface changes. Exporting is explored in Chapter 9.

7.3 Buck the representations!

The reader only needs to glance through the functional data structures in [Oka98a] to realise the gap between concrete representation types and abstract types. The former are *bigger*. They contain implementation clutter: values that capture properties of the structure such as size, rank, depth, etc; or parts of the structure itself, or distinguished payload elements, or even part of the data structure's state represented as data (*e.g.*,

Hood-Melville queues).

The examples in this chapter are simple. One could provide more complicated examples of rather obscure representation types but we would have to explain their purpose. Some ADT implementations are quite a feat of engineering and cleverness, and the systematic study of efficient functional data structures is still an ongoing field of research.

At any rate, our examples illustrate the hoary point that accessing concrete representations may produce unintended results and may break implementation invariants and semantics. This is why concrete representations are hidden behind an interface of operators that maintain them and enable construction and observation (a *view*) of the *relevant* data.

Polytypic functions are no different from ordinary functions in this regard. Their 'genericity' is due to structural parametrisation alone.

It is perfectly conceivable for future languages to enable compilers to choose ADT implementations at compile-time based on operator-usage analyses. It is also possible for a program to manipulate simultaneously different implementations of the same ADT as long as they are not mixed up in operations. Finally, implementations may change, but results produced by client code should not.

There are situations in which data seldom changes. A well-designed abstract syntax tree is rarely changed and compilers usually manipulate its concrete representation directly. But more often than not, implementors have to prepare for change. Generic Programming is about making this preparation unnecessary with respect to code: generic functions work for all types or, at least, for a big set of types. Polytypic function definitions *should not change* when the data changes and should provide accurate and meaningful results. In short, polytypic functions *must not* access the concrete definitional structure of ADTs. The reader may wonder how structural polymorphism may be possible at all. It depends on what we mean by 'structure' (Chapter 9).

Chapter 8
Pattern Matching and Data Abstraction

> Pattern matching and data abstraction are important concepts... but they do not fit well together. Pattern matching depends on making public a free data type representation, while data abstraction depends on hiding the representation. [Wad87]

In Chapter 7, we have argued that polytypic programming conflicts with the principle of data abstraction. Pattern matching is another language feature that conflicts with data abstraction, for pattern matching is performed upon unencapsulated, *concrete* data types, and therefore its applicability is limited to *within* the modules implementing ADTs. There are several proposals for reconciling pattern matching with data abstraction and the first thing that comes to mind is to investigate whether they can be of any use in reconciling polytypic programming with data abstraction—polytypic functions pattern-match over concrete definitional structures.

There are two major approaches for reconciling pattern matching and ADTs. The first approach is based on providing *views* of the ADT in terms of *exported* concrete types together with translation functions from the ADT's internal concrete representation to the exported view and vice-versa [Wad87]. The second approach is based on providing only one translation, keeping constructor *operators* and turning some discrimination and selection operators into pattern expressions which, logically, are syntactic sugar for the former [PPN96, WC93].

In this chapter we review why the first approach is not satisfactory (in fact, it has been dropped entirely) and why the second approach is of limited help. We also describe other less well-known, and even less suitable, approaches. We provide the chapter's conclusions upfront in the next section and elaborate the details in the remaining sections.

8.1 Conclusions first

In this chapter we overview the most popular approaches for reconciling pattern matching with data abstraction. These are some of the lessons to be learned from them:

1. Pattern matching is, logically, syntactic sugar for observation.
2. Trying to evolve the value-constructor concept for observation leads to problems. More precisely, construction and observation may not be inverses and therefore must be separated. The latter may be provided by some pattern-matching construct. The former should be performed by ordinary ADT operators.
3. For pattern matching to be effective, it must be possible for computation to take place *at matching time; i.e.*, effective component selection requires computation.
4. Relying on canonical values is deprecated.

Generic Haskell and SyB are oblivious to the second point in the list. We could embark on a project to adapt them accordingly, but this is downplayed by the remaining points. In order to program polytypic functions on ADTs it is necessary to define a uniform notion of structure. Interfaces may provide such structure. The introduction of elaborated pattern-matching mechanisms and their luggage (*e.g.*, changes to the type system, possible undecidabilities, etc) is an extra complication, and there is the problem that construction must take place via operators.

But polytypic functions can be defined in terms of ADT operators. At the end of the day, ADT operators provide a 'view' of an implementation type. In ordinary programming the need for pattern matching is more pressing: there are issues of conciseness, readability, structural definitions and proofs, etc. In Generic Programming these pressures are localised in the definition of generic functions, which can be provided in two parts (construction and observation) based on the structure of interfaces (Chapter 9).

8.2 An overview of pattern matching

Conceptually, data types are either concrete or abstract (Section 4.2). In functional languages, concrete types are either primitive types or algebraic types (Section 6.1.1). At the value level, algebraic types introduce a free algebra (Chapter 5) generated by a

```
Pattern ::= Variable
          | ValueConstructor Pattern*
          | ( Pattern, Pattern (, Pattern)* )
```

Figure 8.1: A simple language of patterns consisting of variables, value constructors applied to patterns, and n-ary tuples of patterns.

set of value constructors which play several roles. More concretely, given the Haskell data type definition template introduced in Section 6.1.1:

$$\textbf{data}\ Q \Rightarrow T\ a_1 \ldots a_n\ =\ C_1\ \tau_{11} \ldots \tau_{1k_1}\ |\ \ldots\ |\ C_m\ \tau_{m1} \ldots \tau_{mk_m}$$

value constructors C_i play the following roles:

1. Introduction (construction), *e.g.*, a term $C_1\ t_{11} \ldots t_{1k_1}$ introduces (constructs) a value of the algebraic type, where t_{1j} is an arbitrary term of type τ_{1j}.

2. Representation (freeness), *e.g.*, the term $C_1\ t_{11} \ldots t_{1k_1}$ represents (denotes) a value. Unlike regular functions, the value computed by an application of a value constructor to its arguments is denoted by the application itself. Thus, values carry their structure explicitly. This is possible because *there are no equations between value constructors that suggest the need of further computation in order to satisfy them.*

3. Elimination, *e.g.*, the ***pattern*** $C_1\ p_{11} \ldots p_{1k_1}$, where p_{1j} are sub-patterns, can be used for *discriminating* among sum values of an algebraic type and for *selecting* the product components. Patterns are allowed in **`case`** expressions of core languages,[1] and in top-level function definitions, lambda abstractions, and let-expressions of most fully-fledged functional languages. A small language of patterns is shown in Figure 8.1.

   Notice that patterns can be ***nested***; an example would be `Cons x (Cons y ys)`. Of particular interest are ***simple patterns*** of the form $C\ x_1 \ldots x_n$ where C is a value constructor and x_i are variables. If the variables are all different the pattern is ***linear***. Writing the same variable in different positions imposes an equality test: the values matched against the various occurrences must be the same.

[1] Case expressions for pattern matching are more general than those of Section 2.7.2 which only pattern-match against values of a sum type.

4. Value constructors also play the less conspicuous role of aids to the type checker and of enabling the definition of *iso-recursive* type operators [Pie02, p275–277,445].

The first three roles are illustrated by the following function definition:

```
insert :: Eq a ⇒ a → List a → List a
insert x Nil         = Cons x Nil
insert x (Cons y ys) = if x == y then Cons y ys
                       else Cons y (insert x ys)
```

which is sugar for a definition by case expression:

```
insert x l = case l of
               Nil         → Cons x Nil
               (Cons y ys) → if x == y then Cons y ys
                             else Cons y (insert x ys)
```

The term `Cons y ys` is used as a pattern in the left hand side of the second arrow and as a term on the right hand side. As a pattern it discriminates whether the second argument to `insert` is constructed using `Cons` and binds the new variable `y` to the value of the first product component, and the new variable `ys` to the value of the second. Finally, there is computation with selected values at each discriminated branch.

Technically, the matching process consists of the application of a boolean predicate (discriminator) followed by an optional process of selection, binding into locally-defined variables, and computation with selected components. As rightly pointed out in [Pal95, p4], unification in logic programming languages is a similar but more expressive construct than pattern matching.

Without pattern matching, `insert` would be written using list operators in a style familiar to LISP programmers:

```
insert x l = if null l            -- discrimination
             then cons x nil      -- computation
             else let y  = head l -- selection and binding
                      ys = tail l
                  in cons y (insert x ys) -- computation
```

This is also the manner in which `insert` would be written *if the list type were abstract.*

With pattern matching, discrimination and selection are expressively combined and can be compiled effectively. The number of discriminated alternatives (number and name of value constructors) is fixed and known at compile time so, depending on the language of patterns, it is possible to check statically whether alternatives overlap or are exhaustive, *i.e.*, whether patterns in case expressions capture all possible forms of values. However, partial operators and polymorphic non-terminating terms are allowed in functional languages, making it possible to write case expressions that produce run-time errors or do not terminate:[2]

```
case l of Nil → head l ...
```

But what goes on to the left of the arrow is safe, *i.e.*, discrimination between the alternatives of a disjoint sum can be exhaustive and selection of product components into local variables is type-safe.

Another benefit of pattern matching is conciseness and expressiveness. The first definition of `insert` is easier to read and understand. It closely follows the structure of the data it works on. Properties on well-ordered, recursive algebraic types can be proven by structural induction [Mit96]. Pattern matching can also be used when defining functions over co-recursive algebraic types and in their proofs [GH05].

However, pattern matching is incompatible with data abstraction, for patterns are meant to capture the *concrete* structure of a value. It has also other disadvantages [Tul00, p3]:

1. Pattern matching imposes an evaluation order: patterns in case expressions are evaluated from left to right and from top to bottom. This has consequences on the semantics of functions and therefore on how they must be defined. A typical example is the **`zip`** function. The reason for an evaluation order comes from how nested patterns are decomposed into nested case expressions with simple patterns.

2. Pattern matching "begs for extension upon extension": there are irrefutable patterns, `as` patterns, guarded patterns, etc.

3. Patterns can be nested and make it difficult for the eye to determine whether they

[2] Undefined terms enable the definition of partial operators. The reader should bear in mind the difference between *undefined* and *non-terminating* terms. The former are stuck terms, the latter have no normal form.

overlap or are exhaustive, *e.g.*:

```
foo Nil          = ...
foo (Cons 1 Nil) = ...
foo (Cons _ xs ) = ...
```

In fact, testing for exhaustiveness may be toggled off or demoted into a warning: "partial but total" auxiliary functions are not atypical. For example, although local function `bar` is partial, it is always supplied a value on its domain:

```
foo Nil = 0
foo xs  = bar xs
 where bar (Cons y ys) = ...
```

4. It is important to differentiate between patterns and terms: patterns introduce new variables and have different semantics, *e.g.*:

```
λl → let x = Nil in case l of x → 0 ...
```

Pattern matching over sums of products demonstrates how *low-level* these data definition mechanisms are and the pitfalls of positional selection. Named records provide a much better abstraction mechanism as shown below—the example is adapted from [Pal95, p9]:

```
data Person = P String String Int ...
birthday :: Person → Person
birthday (P f s a ...) = (P f s (a+1) ...)

data Person = P{name::String, surname::String, age::Int, ...}
birthday :: Person → Person
birthday (person@P{age = a,_}) = person{age ← a + 1}
```

In the last line, pattern matching only requires the value of the `age` field, which gets 'updated' by the function. Notice also the possibility of record subtyping and the closeness to ADT programming.

8.3 Proposals for reconciliation

The following sections outline the most important proposals for reconciling pattern matching with data abstraction.

8.3.1 SML's abstract value constructors

Abstract value constructors are an implemented feature of the 'Standard ML of New Jersey' compiler [AR92]. Strictly speaking, they are not really a proposal for reconciling pattern matching with data abstraction, but can be used for that purpose in restricted situations.

An abstract value constructor `C` is defined as follows:

`C` x_1 ... x_n **`match`** P

where to the left of **`match`** there is a *simple linear pattern* and to the right an arbitrary SML pattern P. The compiler replaces the left hand side by the right hand side, *i.e.*, abstract value constructors are macro-substituted by 'real' patterns.

Abstract value constructors can be used in pattern matching and in construction. This double role imposes restrictions on the simple pattern and on P:

- When used *in pattern matching*, P must be linear and must contain all the variables of the simple pattern. For example, the following abstract value constructors are illegal:

  ```
  C x y match C' x
  C x   match C' x x
  ```

  In the first line, variable `y` is defined and, therefore, possibly used on the right hand side of the arrow in a case expression, but nothing is matched against it when the abstract value constructor is macro-expanded to `C' x`. In the second line, a linear pattern is macro-expanded into a non-linear one.

- When used *in construction*, all the variables in P must occur in the simple pattern. Look at this illegal example:

  ```
  C x match C' x y
  ```

  If the left hand side is macro-expanded to the right-hand side, variable `y` is not bound to any value and the term cannot construct anything.

The following is an example of an abstract value constructor that can be used in pattern matching and construction:

```
data Complex = Complex Real Real
PureReal r match Complex r 0
```

Functions on complex numbers can be defined by pattern matching over `PureReal`.

```
foo (PureReal r)  = ...
foo (Complex r i) = ...
```

We can write `foo`'s body in the second case under the premise that `i` $\neq 0$.

Abstract value constructors are not only limited in expressibility because of their use in patterns and construction, but also because selection does not involve computation. Recall the FIFO-queue ADT of Chapter 7, in particular the `BatchedQueue` implementation of Section 7.1.1. It would be very interesting to be able to define two abstract value constructors `Empty` and `Enq` for pattern matching over FIFO-queue values, *e.g.*:

```
Empty    match  BQ [] []
Enq x q  match  BQ (x::xs) r
```

Unfortunately, `Enq`'s definition is illegal and we need some way of specifying computation:

```
q = xs @ (reverse r)
```

8.3.2 Miranda's lawful concrete types

The functional programming language Miranda supports so-called ***lawful algebraic types*** which are algebraic types with equations between value constructors [Tho86]. These equations are actually rewrite rules for transforming values into *canonical* ones. Consequently, rewrite rules must be confluent and terminating.

More precisely, a value of the concrete representation type may not represent a value of the ADT (junk), and multiple values of the concrete representation type may represent the same ADT value (confusion). One way of supporting pattern matching and equality on concrete types is to normalise every constructed value into a *canonical value* so that patterns are matched against, and equality is performed on, canonical values that *uniquely represent* ADT values.[3]

There follows an example of a lawful concrete type for sets implemented as lists or,

[3]Notice that canonical values may still contain implementation clutter.

more precisely, as an algebraic type structurally isomorphic to the `List` type:

```
data Eq a ⇒ Set a = Empty | Insert a (Set a)
Insert x (Insert y s) == if x == y then Insert y s
                         else Insert y (Insert x s)

Insert 1 (Insert 1 (Insert 2 Empty))
> Insert 2 (Insert 1 Empty)
```

Notice the equation involving abstract value constructor `Insert`. The equation shows that value constructor `Insert` plays not only the role of a constructor but also the role of a normalisation function. More precisely, the lawful type has the following normalisation function:

```
insert :: Eq a ⇒ a → Set a → Set a
insert x Empty        = Insert x Empty
insert x (Insert y s) = if x == y then Insert y s
                        else Insert y (insert x s)
```

The problems with this approach are not difficult to see:

1. Checking that rewrite rules are confluent and normalising requires a lot of effort and is, in the general case, undecidable.

2. Normalisation into canonical values is inefficient, forces particular representations, and might not be possible or recommended. Recall the FIFO-queue implementations of Section 7.1.1 which relied on unevaluated data components to achieve their amortised efficiency.

3. There is not that much abstraction from the representation: functions working on values of type `Set a` are defined by pattern matching on `Empty` and `Insert` but we might want to change the implementation. Wadler's views (Section 8.3.3) argues for these two value constructors to be part of a *view* of the `Set` type.

4. Selection involving computation is not possible during pattern matching: normalisation takes place during construction, not matching.

5. There are serious problems with equational reasoning: the fact that patterns are matched against values in canonical form is not reflected in the values themselves. For example, given the following definition:

```
select (Insert x s) = x
```

the following equations hold:

```
   1
==    { select's definition }
   select (Insert 1 (Insert 2 Empty))
==    { Insert's definition }
   select (Insert 2 (Insert 1 Empty))
==    { select's definition }
   2
```

In order to reason with values of the concrete type we have to reduce them to normal form and therefore know the details of the implementation. In particular, the clients of `Set` should not be obliged to know details of normalisation.

8.3.3 Wadler's views

A lawful type in Miranda is a subset of a concrete type: its canonical forms. In contrast, ***Wadler's views*** [Wad87] specify an *isomorphism* between subsets of concrete types. This becomes particularly useful when dealing with ADTs, for they can be implemented by many concrete types. ADT designers may choose one of these types as the *implementation type* and allow clients to work with (possibly many) *view types* isomorphic to (a subset of) the implementation type. Value constructors of the view type can be used in pattern matching and construction. Because of this double role, there must be a correspondence between each view type and implementation type, and vice versa.

An illustrative example is perhaps the type of natural numbers. The Peano representation in terms of `Zero` and `Succ` is handy but inefficient. Programmers usually work with the base type of positive integers.[4] An ADT of natural numbers can be implemented in terms of positive integers but *viewed* in terms of its Peano representation. A possible syntax for declaring this follows:

```
view Int = Zero | Succ Int
```

[4]Surprisingly, natural numbers are rarely offered as a base type by most functional programming languages.

```
  where
   in n
    | n == 0    = Zero
    | n > 0     = Succ (n-1)
    | otherwise = error "Negative Integer"
   out Zero = 0
   out (Succ n) = n + 1
```

The keyword **view** declares that **Int** can be viewed as a recursive algebraic type with Peano value constructors. Behind the scenes, the compiler translates this view into the following non-recursive type:

```
data View = Zero | Succ Int
```

Functions **in** and **out** are translation functions from the implementation type to the view type and vice versa:

```
in  :: Int → View
out :: View → Int
```

We need two functions because Peano value constructors may occur in pattern matching (which requires calls to **in**) and construction (which requires calls to **out**).

The factorial function can be defined on the Peano view:

```
factorial :: Int → Int
factorial Zero     = Succ Zero
factorial (Succ n) = (Succ n) * factorial n
```

The definition can be translated by the compiler to a factorial on integers by inserting calls to **in** and **out** at appropriate places:

```
factorial :: Int → Int
factorial n = case (in n) of
   Zero   → out (Succ (out Zero))
   Succ n → (out (Succ n)) * factorial n
```

Functions **in** and **out** are similar in spirit to the embedding-projection pairs of Section 6.1.4. These functions must be inverses of each other and this cannot be checked by a compiler.

In particular, **out** must be total and injective or otherwise it would introduce junk and

confusion respectively: there would be values of the view type that are not represented in the implementation type, and there would be values of the view type that would have the same representation in the implementation type, thus introducing an implicit equation between the former values.

Also, **`in`** must be injective and its domain must be **`out`**'s image. If the domain is a proper superset, there are implicit equations in the implementation type. If the domain is a proper subset, there is junk in the implementation type.

Moreover, it might not be possible for a view type and an implementation type to satisfy them: for instance, it might be the case that a value of the view type is *always* representable by multiple values of the implementation type. Palao [Pal95, p32] illustrates this situation using complex numbers where the implementation type is the cartesian representation and the view type the polar representation. Multiple representations introduce implicit equations which hinder equational reasoning. This is the main reason why Wadler's views were not included in the Haskell definition. Other problems with views are [Pal95, Chp4]:

1. The need to take into account the side conditions of functions **`in`** and **`out`** during equational reasoning.

2. The fact that seemingly total functions are indeed partial. For example, in the `factorial` example, with the Peano representation the function triggers a run-time error if fed a negative integer. Haskell's type system cannot check statically whether `Succ` is always applied to positive integers during construction.

3. It might be the case that values of the view type should be given in some canonical form (*e.g.*, complex numbers in polar representation), again introducing implicit equations.

4. There is a logical separation between pattern matching and construction. Using value constructors for both makes no sense in many situations. Take for example a `BatchedQueue` implementation of queues (Chapter 7):

   ```
   data BatchedQueue a = BQ [a] [a]
   data View a = EmptyQ | Front a (Queue a)
   ```

   Queue elements must be inserted at the rear of the queue, yet value constructor

`Front` can be used for construction. Notice that introducing an extra value constructor:

```
data View a = EmptyQ | Front a (Queue a) | Enq (Queue a) a
```

would also introduce an implicit equation: the same queue value can be pattern-matched against a `Front` pattern and a `Enq` pattern. These two value constructors are not free among themselves. In general, all possible observations cannot be captured by a single view.

5. However, it is illegal, for implementation reasons, to do pattern matching on different view types simultaneously. In the `BatchedQueue` example, we could have provided the `Enq` pattern in a different view but we would not be able to pattern match against `Front` and `Enq` simultaneously. This is unproblematic for simple queues but not so for double-ended queues, where insertion can take place either at the front or at the rear.

6. Finally, there are many possible ways in which different ADT operations could be expressed in terms of view types; implementing all of them could be too expensive.

Most problems with Wadler's views are due to the double role of value constructors in view types. This is pointed out in [WC93], where view types are restricted to pattern matching alone, with construction carried out by operators. The **`out`** function disappears and there is no restriction on **`in`**. However, pattern matching over different view types is still illegal because of implementation reasons: the representation type is transformed into the view type *before* the matching is performed.

8.3.4 Palao's Active Patterns

According to Palao *et al* [PPN96, p112], the limitations of the previous approaches stem from trying to "*evolve* the [value] constructor concept instead of starting the problem from scratch". For instance, Wadler's views are a way for programmers to move across possible implementation types, but observation is performed by means of a value constructor. In non-free types, construction and observation may not be inverses. For example, FIFO queues are constructed by inserting elements at the rear whereas selection from non-empty FIFO queues takes place at the front. Construction and observation must be separated.

As underlined in Section 8.2, observation consists of discrimination followed optionally by selection and binding of selected values into new local variables. Pattern matching is just syntactic sugar for this, with added checks for overlapping and exhaustive cases. Nested patterns only add expressibility to nested observation.

This is the idea behind ***Palao's Active Patterns*** [Pal95, PPN96], which can be regarded as an extension of SML's abstract value constructors (Section 8.3.1) where there can be computation *after* the matching is performed (hence the 'active') and their use in construction is banned.

The language of Active Patterns is built from ordinary patterns, ***active destructors***, and compositions of Active Patterns via the `@` operator, which is explained shortly:

AP `::=` *Variable*
`|` *ValueConstructor* *AP**
`|` *ActiveDestructor* *AP**
`|` *AP* `@` *AP*

The main advantages of Active Patterns are their expressibility, their support for equational reasoning, and their smooth integration with algebraic specifications, the latter an important aspect that is ignored by the previous approaches. ADT operators can be replaced, expressed, or accompanied by active destructors whose axiomatic semantics are defined in terms of the operators themselves.

An active destructor consists of a label together with optional positional arguments: the label denotes an alternative and the positional arguments are *expressions* that select components. The translation goes from the implementation type to the active destructor (the 'view'), which *is not* a concrete type, and translation takes place after matching.

For example, given an ADT of complex numbers:

```
module Complex(Complex,realPart,imgPart,modulus,argument) where
data Complex = Complex Real Real
...
```

any operator could be provided as an active destructor. For example:

```
RealPart r  match  Complex r _
Modulus m   match  Complex r i  where m = sqrt (r^2 + i^2)
```

The first active destructor, `RealPart`, is a projection identical to an SML abstract value constructor. The second, `Modulus`, is also a projection but the projected value `m` involves a computation which is performed *after* matching. The reader should not be conned by the notation: if `Modulus` were an ordinary value constructor then `m` would match `Modulus`'s argument. In contrast, `m` is an *output* value and, therefore, the type of `Modulus` *is not* **`Real`** $\rightarrow$ `Complex`. It is not a function and cannot be used for construction. (Types for active destructors are mentioned at the end of the section.)

Informally, the operational semantics of the matching process is as follows. Suppose function `foo` has one case involving `Modulus`:

```
foo (Modulus m') = E
```

where E is an expression where `m'` may occur free. In the application `foo e`, the value of `e` is pattern-matched against the ordinary pattern `Complex r i`. If the matching succeeds, then we have in `m'` the modulus of the complex number—the value of `m` in the active destructor's definition.

Pattern matching occurs behind the scenes, respecting data abstraction: the matching of `e`'s value against the concrete type and the computation involved in getting the modulus is hidden from `foo`'s writer, who only cares about having the value of the modulus in `m'` when the matching succeeds.

In code, `foo`'s definition involving `Modulus` is equivalent to:

```
foo = λe → case e of
             Complex r i → E where m' = sqrt (r^2 + i^2)
```

Variables `r` and `i` only occur free in the definition of `m'`, and `m'` may occur free in E. It becomes clear now that `Modulus` is just an abstract label. It does not play any role in the compiled code.

We have used a local variable `m` in the definition of `Modulus` but the selecting expression can be written directly in the active destructor:

```
Modulus (sqrt (r^2 + i^2))  match  Complex r i
```

FIFO queues illustrate the expressibility of active destructors, which can be provided by the interface. For example, in a `BatchedQueue` implementation:

```
EmptyQ   match BQ [] []
Front x  match BQ (x:_) _
Deq q    match BQ [_] r    where q = BQ (reverse r) []
Deq q    match BQ (_:f) r  where q = BQ f r
```

Because matching takes place over the ordinary patterns after the **match** keyword, we can provide multiple definitions of the same active destructor, in this case `Deq`. We could have provided a single active destructor `Deq'` that selects both the front and the remaining queue.

The following function uses the previous active destructors:

```
sizeQ :: Queue a → Int
sizeQ EmptyQ  = 0
sizeQ (Deq q) = 1 + sizeQ q
```

To belabour the point, active destructors `EmptyQ` and `Deq` hide the representation type `BQ`. A `Queue` value could be implemented as a physicist's queue or what have you. Function `sizeQ` would not be affected as long as active destructors are defined for the new implementation.

The following function illustrates the use of `@`:

```
showQ :: Show a ⇒ Queue a → String
showQ EmptyQ            = ""
showQ (Front x)@(Deq q) = show x ++ showQ q
```

Here, the `@` operator matches `showQ`'s argument against `Front` and `Queue`, obtaining the appropriate values for `x` and `q` if the matching succeeds.

We can define the aforementioned `Deq'` active constructor in terms of `Front` and `Deq` using `@`:

```
Deq' x q  match (Front x)@(Deq q)
```

Indeed, the pattern on the right can be an Active Pattern (but active destructors cannot be directly recursive). Now:

```
showQ :: Show a ⇒ Queue a → String
showQ EmptyQ     = ""
showQ (Deq' x q) = show x ++ showQ q
```

In functional languages, `@` is used for '`as` patterns' where the ordinary pattern `x@`p matches an argument against the ordinary pattern p but binds `x` to the argument. Generalising from the fact that variable `x` is itself a pattern, given two Active Patterns p_1 and p_2, the Active Pattern p_1`@`p_2 is a conjunction of patterns which succeeds only if the argument matches both p_1 and p_2. For ordinary patterns this operator is less useful: a value matches p_1`@`p_2 when both patterns have the same value constructor.

Unlike conventional value constructors, active destructors need not be free among themselves, so `Enq` can be used in conjunction with any other active destructor:

```
Enq x q  match  BQ f (x:xs) where q = BQ f xs

extremes :: Queue a → (a,a)
extremes (Front x q)@(Enq y p) = (x,y)
```

An interesting aspect of Active Patterns is that whether observation is provided in terms of operators or active destructors is not an irrevocable decision. New active destructors can be defined in terms of available ones (*e.g.*, `Deq'`) or in terms of existing operators. In fact, active destructors could be defined by ADT *clients*, not implementors, purely in terms of operators, *e.g.*:

```
EmptyQ   match q, if isEmpty q
Front x  match q, if not (isEmpty q)  where x  = front q
Deq q'   match q, if not (isEmpty q)  where q' = deq q
```

Here we have made use of *guards*. Matching against an active destructor succeeds if matching against the pattern on the right succeeds and the guard is satisfied. Here the pattern on the right is a variable, `q`, and matching against it always succeeds. *This example clearly shows that Active Patterns are sugar for discrimination and selection operators.*

The general form of an active destructor definition is:

```
C e11 ... e1n match p1, if G1 where D1
...
C em1 ... emn match pm, if Gm where Dm
```

where `C` is the active destructor name, e_{ij} are expressions, p_i are Active Patterns (in which `C` cannot appear at the top level), G_i are guards, and D_i are declarations

providing bindings for variables in e_{ij}. A guard G_i may use variables in D_i and p_i. Notice that in all cases the active destructor must have the same 'arity'. Unlike value constructors, active destructors cannot be partially applied.

Active Patterns in function definitions must be linear. Nested Active Patterns are possible because e_{ij} are arbitrary expressions. For example, here the queue's payload is a binary-tree node:

```
foo (Front (Node x l r))@(Deq q) = ...
```

The value 'returned to' `Front` is pattern-matched against ordinary value constructor `Node`.

Active Patterns can be integrated into algebraic specifications: active destructors can be defined in terms of conditional equations involving ADT operators. Recall the FIFO-queue example above. The active constructors all follow the pattern:

$$\texttt{C } \texttt{v}_1 \ldots \texttt{v}_n \textbf{ match v, if } G \textbf{ where } \begin{array}{l} \texttt{v}_1 = \texttt{s}_1\ \texttt{v} \\ \ldots \\ \texttt{v}_n = \texttt{s}_n\ \texttt{v} \end{array}$$

Equational reasoning proceeds by checking guards and substituting selection expressions $\texttt{s}_i$, which involve ADT operators, by active destructor variables (see [PPN96, p118] and [Pal95, Sec5.3] for examples). Using equational reasoning it is possible to prove, for particular functions, whether Active Patterns are exhaustive and do not overlap.

Two different compilation algorithms for transforming Active Patterns to case expressions with simple patterns are given in [Pal95, Sec5.5]. It is not clearly specified whether the algorithms check that Active Patterns are exhaustive and do not overlap. Active Patterns are exhaustive if ordinary patterns and guards are exhaustive. This is in general undecidable: guards are unrestricted boolean expressions. The ability to compose patterns could make checking for exhaustiveness also difficult to the eye, as Active Destructors could be conjugated in different fashion via the `@` operator. If matching failure occurs, the run-time system can only provide information about which active destructor failed and at which point; it cannot provide information involving the concrete type without compromising abstraction.

Active destructors can be first-class citizens if the type system provides a type for

them. A proposal is given in [Pal95, Sec5.7.1] together with type-checking rules. Type inference in a Hindley-Damas-Milner type system is only conjectured.

8.3.5 Other proposals

Erwig's Active Patterns [Erw96] must not be confused with Palao's. The former also allow computation at matching time, rearranging the concrete type to some desired pattern, but can be used for construction and functions compute directly with representations.

First-class patterns [Tul00] are an attempt at providing a combinator language for patterns, which are now functions of type `a` → **`Maybe`** `b`. Case expressions are restricted to be exhaustive and cannot contain nested patterns. Pattern combinators are built using some base combinators, operators for composing patterns (*e.g.*, or-match, then-match, parallel-match, etc.), and an operator for lifting value constructors to pattern combinators. Syntactic sugar is offered in order to make first-class pattern expressions more readable.

SML views [Oka98b] carry the ideas in [Wad87] and [WC93] to Standard ML, its module system, and its call-by-value semantics with effects.

Chapter 9
F-views and Polytypic Extensional Programming

> From the practical point of view, not only for economy of implementation but also for convenience in use, the logically simplest representation is not always the best [Str00, p38]

In Chapter 6 we have overviewed the two main polytypic language extensions of Haskell. In Chapter 7 we have argued that the idea in which they are based, structural polymorphism, conflicts with data abstraction and is therefore limited in its applicability and genericity.

Structural polymorphism is founded on a regularity: the structure of a function follows the structure of the data. Data abstraction destroys the regularity. Abstract values are represented by a subset of concrete values, those that satisfy implementation invariants.

Data abstraction is upheld by client code, whether polytypic or not, when ADTs are accessed through a *public interface*. Interfaces supply operators that satisfy implementation invariants and deal away with clutter. The question we must ask is whether ADT interfaces offer a sufficiently regular description of *structure* that may enable the definition of polytypic functions. Before trying to answer the question, let us discuss other alternative solutions.

9.1 An examination of possible approaches

We consider some ways of dealing with the problems raised in Chapter 7 and contrast their advantages and disadvantages.

1. However tempting, canonical representations are a blind alley. We have already touched upon their drawbacks in Section 8.3.2.

2. For many, ADTs should provide the relevant functionality, *i.e.*, come equipped with their own map, foldr, and foldl operators which are expected to be efficient due to their privileged access to the representation. This approach has several drawbacks. Firstly, there is no Generic Programming here. Secondly, we have already discussed

fold's problems with respect to control abstraction (Sections 4.3 and 6.2.1). Thirdly, ADT implementors cannot anticipate all the possible functionality—recall the arguments against polytypic extension (Section 7.1). Lastly, regarding efficiency, the implementation of map as an ADT operator may require reshuffling the whole structure. Think of ordered sets implemented as ordered lists, of heaps implemented as binary search trees, of dictionaries as hash tables, etc. The folds follow suit, for map can be programmed in terms of them. The efficient implementation of map may require a representation tailored for that purpose.

In contrast to providing full functionality, ADTs can provide a *minimal* or *narrow interface*. Control abstraction can be provided in terms of external and generally applicable (*i.e.* generic) functions. We consider maps and catamorphisms examples of such functions. In this sense, we adhere to the philosophy of the C++ STL [MS96, MS94], but for us genericity means *polytypism*, not just polymorphism.

3. An intermediate solution is the ***iterator*** concept proposed by the C++ STL. An iterator is an abstraction of a pointer which is manipulated via operators offered by and implemented within the ADT. Catamorphisms can be defined externally using iterators.

   The iterator approach has its drawbacks. First, iterators are tailored to specific ADTs and are not polytypic. Second, the pointer abstraction only enables linear traversals, *e.g.*, top-down breadth-first, bottom-up breadth-first, top-down depth-first, etc. Third, for type-safety reasons one iterator is needed per payload type.

   In a purely functional setting an iterator corresponds to a function that flattens the ADT into some linear concrete type, for example, a list. Its inverse, a ***co-iterator*** builds an ADT from a list. What is desired is a *polytypic* iterator that extracts payload from *any* ADT to particular concrete types, not just lists. Dually, what is desired is a coiterator that can build a value of the ADT from values of those concrete types.

4. It might be possible for a polytypic function to discern automatically whether a piece of data is clutter and to abstract over clutter without losing information. It might even be possible for compilers to check, with the help of assertions, whether polytypic functions break implementation invariants. But even if such a feat were feasible and decidable in theory, it would be useless in practice. Making changes to

ADT implementations forces the recompilation of client code for clutter and security checks. And if the checks fail, what should we do?

5. Lastly, we could move into richer type languages where concrete types faithfully encode abstract types and capture structural properties *as data*. In other words, we could move to languages where we can express within a concrete type what otherwise has to be expressed via operators and their semantics. This path leads to dependent types [Pie05, Hof97]. Unfortunately, dependent types are designed and tailored to the structure of the problem at hand, and this hinders their reusability. We have to rely again on some form of Generic Programming and introduce some notion of data abstraction to cope with data change [AMM05]. This is a research topic of its own. In this thesis we content ourselves with making polytypic programming cope with ADTs in the present state of affairs.

9.2 Extensional Programming: design goals

ADT interfaces provide a *view* of payload data. A function computing with an ADT is computing with its payload values alone. How these values are stored internally is irrelevant and opaque to client code. For lack of better terminology, let us call this form of programming via interfaces ***Extensional Programming***.

Is *Generic* Extensional Programming possible? The answer is positive. The following are our assumptions and goals:

1. We assume ADTs are first order and specified algebraically (Chapter 5). The programming language at hand (for us, Haskell) need not support algebraic specifications, but we assume they have been developed in the *design* of every ADT. Algebraic specifications enable programmers to reason about their ADTs and are a contract for implementors. We will show that algebraic specifications are also needed in the development of polytypic functions.

2. We assume ADT interfaces are narrow and provide the minimum necessary constructors and observers (discriminators and selectors). Polytypic functions will be written *outside* the ADT using the first-order operators.

3. 'Structure' must be defined in terms of interfaces so that polytypic functions are

structurally polymorphic but also uphold encapsulation and respect the ADT's semantics.

4. Polytypic extension must be supported. Programmers must be able to use specialised operators when available. There is no need to generate, say, an instance of `gmap` for an ADT if it comes equipped with its map operator.

5. It must be possible to define extensional polytypic functions on *manifest* ADTs. The map for a queue of integers, say, must not be the identity. The solution must not rely on abstraction over the payload type (recall Section 7.2).

The *functor* defined by an ADT's *interface* can be put to use as the required notion of structure that enables the definition of polytypic functions on the ADT. The elaboration of the details make up the bulk of the chapter. In Section 9.3 we explain some of the ideas using so-called 'linear' ADTs as running examples. From Section 9.9 onwards we generalise and show how to define typed polytypic functions that work on arbitrary ADTs. Section 9.12 discusses polytypic extension and Section 9.13 discusses support for manifest ADTs.

9.3 Preliminaries: F-algebras and linear ADTs

Recall the notion of F-Algebra from Section A.3.1. The polynomial functor F provides a specification of 'structure'. However, we have complained in Section A.3.1 that the mediating S-function is not informative about operator names, and that the same functor can capture the signature of theories with different equations.

For example, many ADTs are characterised by the signature shown in the first box of Figure 9.1. The second box shows its rendition as a Haskell type class.

We call an ADT ***linear*** if there is a signature morphism (Definition A.1.8) from it to the `LINEAR` signature. Linear ADTs are described by the equation:

$$L(X) \;=\; \mathbf{1} + X \times L(X)$$

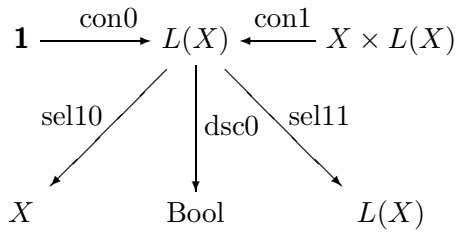
and may satisfy various laws. The equation can be expressed as the fixed point of the functor $F(Y) = \mathbf{1} + X \times Y$:

$$L(X) = F(L(X))$$

```
signature LINEAR
sorts Linear
use BOOL
param Elem
ops
  con0  : → Linear
  con1  : Elem Linear → Linear
  dsc0  : Linear → Bool
  sel10 : Linear → Elem
  sel11 : Linear → Linear
```

```
module Linear(Linear(..)) where
class Linear l where
  con0  :: l a
  con1  :: a → l a → l a
  dsc0  :: l a → Bool
  sel10 :: l a → a
  sel11 :: l a → l a
```

Figure 9.1: Signature and Haskell class definition of linear ADTs.



List			Stack			FIFO Queue		
dsc0	=	**null**	dsc0	=	isEmptyS	dsc0	=	isEmptyQ
con0	=	Nil	con0	=	emptyStack	con0	=	emptyQ
con1	=	Cons	con1	=	push	con1	=	enq
sel10	=	**head**	sel10	=	tos	sel10	=	front
sel11	=	**tail**	sel11	=	pop	sel11	=	deq

Figure 9.2: Lists, stacks, and FIFO queues are examples of linear ADTs.

Figure 9.2 (first box) depicts the signature diagrammatically. There are two constructors con0 and con1, two selectors sel10 and sel11, and one discriminator dsc0. Each constructor takes a product argument: con0 takes a nullary product (Section A.3.1 and 3.6) and con1 takes a binary product.

$L(X)$ is the coproduct of the product types of constructor arguments. The numbers in constructor names denote the position of their argument's product type in $L(X)$ from left to right, starting from 0. In Haskell, operator con1 has a curried type (Section 5.4). Dually, there are observer operators: a discriminator dsc0 associated with the only coproduct and two selectors sel10 and sel11 associated with the binary product. There is nothing to select from a value constructed with con0.

The signatures of several ADTs can be mapped by a signature morphism to the signature of the diagram. Figure 9.2 (second box) shows the mapping for lists, stacks, and FIFO queues.[1]

Perusing [Oka98a], we also find that the signatures of the following ADTs can be mapped by a signature morphism to the diagram:

- *Catenable lists*, which have an efficient concatenation operator but its interface supplies all ordinary list operators.
- *Priority queues*, for they offer the same operators as FIFO queues, only that internally elements are stored according to priority (a function on elements).
- *Ordered sets* and *ordered bags*. In general, these ADTs only provide an interrogator (membership test) and possibly a removal operator. For the mapping to work we need operators that may be assigned to sel10 and sel11. The most natural two would be:

  ```
  choice :: Set a → a
  remove :: a → Set a → Set a
  ```

  Function `choice` is a ***deterministic choice*** operator that given two equal sets returns the same element from the set. For ordered sets, it can just return the minimum element. Function `remove` removes a given element from the set. We can assume similar operators for ordered bags. With these operators at hand, ordered

[1] Signature morphisms are examples of *adapters* [GHJV95].

sets and ordered bags become linear ADTs:

```
sel10 = choice
sel11 = λs → remove (choice s) s
```

The `choice` operator is absent from ordinary sets and bags. These ADTs only require equality on their payload type and `choice` is non-deterministic, *i.e.*, it may be the case that $s_1 = s_2$ but (`choice` s_1) $\neq$ (`choice` s_2).

- *Sortable collections* [Oka98a, p202], if a `choice` is implemented.

- *Heaps*, whose Haskell signature follows (adapted from [Oka98a, p197]):

```
emptyH    :: Heap a
isEmptyH  :: Heap a → Bool
insert    :: a → Heap a → Heap a
merge     :: Heap a → Heap a → Heap a
findMin   :: Heap a → a
deleteMin :: Heap a → Heap a
```

  with the following mapping of operators:

```
dsc0  = isEmptyH
con0  = emptyH
con1  = insert
sel10 = findMin
sel11 = deleteMin
```

  Heaps differ from ordered sets in that (1) only the minimum element can be removed from a heap whereas any element can be removed from an ordered set, and (2) heaps may contain repeated elements. Heaps can be implemented functionally in many ways [Oka98a]: leftist heaps, splay heaps, skew binomial heaps, bootstrapped heaps, pairing heaps, etc.

- Finally, *finite maps*, *tables*, and *dictionaries* can be mapped too but only if they offer discriminators and selectors. In [Oka98a, p204] we only find the following operators:

```
emptyT :: Table k a
insert :: k → a → Table k a → Table k a
lookup :: k → Table k a → Maybe a
```

  Function `lookup` is an interrogator. We need the following operators:

```
isEmptyT :: Table k a → Bool
choice   :: Table k a → (k,a)
remove   :: (k,a) → Table k a → Table k a
```

where `choice` is deterministic. We can then provide the following mapping of operators:

```
dsc0  = isEmptyT
con0  = emptyT
con1  = curry insert
sel10 = choice
sel11 = λt → remove (choice t) t
```

Double-ended queues (or 'deques') are examples of ADTs with ***multiple constructors*** that have the same argument types. In particular, there are alternative ways of mapping deque operators to con1, sel10, and sel11. The list of deque operators follows:

```
isEmptyD :: Deque a → Bool
emptyD   :: Deque a
enqfront :: a → Deque a → Deque a
enqrear  :: a → Deque a → Deque a
deqfront :: Deque a → Deque a
deqrear  :: Deque a → Deque a
front    :: Deque a → a
rear     :: Deque a → a
```

The following mappings turn deques into stacks:

```
dsc0  = isEmptyD
con0  = emptyD
con1  = enqfront
sel10 = front
sel11 = deqfront
```

```
dsc0  = isEmptyD
con0  = emptyD
con1  = enqrear
sel10 = rear
sel11 = deqrear
```

The following mappings turn deques into FIFO queues:

```
dsc0  = isEmptyD
con0  = emptyD
con1  = enqrear
sel10 = front
sel11 = deqfront
```

```
dsc0  = isEmptyD
con0  = emptyD
con1  = enqfront
sel10 = rear
sel11 = deqrear
```

9.4 Construction vs Observation

Construction and observation in ADTs may not be inverses. Consequently, it is not possible in general to program polytypic functions on them in a single definition following the pattern given in Figure 6.21.

Let us recall the case for products which involves selection and construction (coproducts involve discrimination):

$$\begin{array}{rcl} g\langle A\times B\rangle & = & \theta \circ (p_A\ g\langle A\rangle) \times (p_B\ g\langle B\rangle) \\ \text{where } l\times r & = & (l \circ \text{exl}) \vartriangle (r \circ \text{exr}) \\ p_A\ g & = & \mathit{pafter}_A \circ g \circ \mathit{pbefore}_A \\ p_B\ g & = & \mathit{pafter}_B \circ g \circ \mathit{pbefore}_B \end{array}$$

In linear ADTs, arrows exl and exr correspond to sel10 and sel11 respectively, and con1 corresponds to `prod` in the definition of $\triangledown$ (recall Figure 3.3). When construction and selection are not inverses in products, the following is the case:

$$\begin{array}{rcl} \text{sel10 (con1 } x\ y) & \neq & x \\ \text{sel11 (con1 } x\ y) & \neq & y \end{array}$$

That is:

```
exl (prod x y) ≠ x
exr (prod x y) ≠ y
```

However, according to the definitions of Figure 3.3, the equations:

```
exl (prod x y) = x
exr (prod x y) = y
prod (exl p) (exr p) = p
```

are product laws. More precisely, the first two equations are equivalent to:

$$\text{exl} \circ (f \vartriangle g) = f \tag{9.1}$$

$$\text{exr} \circ (f \vartriangle g) = g \tag{9.2}$$

For instance:

$$\begin{array}{rcl} & & \text{exl} \circ (f \vartriangle g) = f \\ = & & \{\ \text{extensionality}\ \} \end{array}$$

$$
\begin{array}{ll}
 & \text{exl} \circ (f \vartriangle g)\ x\ =\ f\ x \\
= & \quad \{\ \text{def. of } \vartriangle\ \} \\
 & \text{exl} \circ \text{prod}\ (f\ x)\ (g\ x)\ =\ f\ x \\
= & \quad \{\ \text{generalisation and def. of composition}\ \} \\
 & \text{exl}\ (\text{prod}\ x\ y)\ =\ x
\end{array}
$$

Consequently, when construction and selection in products are not inverses the product laws are not satisfied.

FIFO queues are examples of ADTs where this occurs. According to Figure 9.2, we have the following mapping:

```
prod = enq
exl  = front
exr  = deq
```

However, the product law:

```
enq (front q) (deq q) = q
```

is only satisfied by empty queues. For non-empty queues what is satisfied is the following:

```
enq (front q) (deq q) = deq (enq (front q) q)
```

which is derivable from the queue law (Figure 5.6):

```
¬ (emptyQ? q) ⇒ deq (enq x q) = enq x (deq q)
```

by working under the assumption that the queue is not empty, by substituting `front q` for `x` on both sides, and by reversing the equation. The equation can be expressed in point-free style thus:

```
enq ∘ (front △ deq) = deq ∘ enq ∘ (front △ id)
```

Clearly, `enq` $\neq$ `(front △ deq)`$^{-1}$.

In Figure 6.21, the case for products $g\langle A \times B\rangle$ assumes the product laws hold. Figure 6.21 cannot express the map function for FIFO queues. Such function must satisfy:

```
map_Fifo f (enq x q) = enq (f x) (map_Fifo f q)
```

There are two natural ways of writing `map_Fifo`. The first uses queue reversal to

account for the fact that product laws do not hold:

```
map_Fifo :: (a → b) → Fifo a → Fifo b
map_Fifo f = reverseQ ∘ map_Fifo' f
  where map_Fifo' f q = if (isEmptyQ q) then q
                        else let fq = front q
                                 dq = deq q
                             in enq (f fq) (map_Fifo' f dq)
```

The second uses an accumulating parameter, performing 'reversal' during construction:

```
map_Fifo :: (a → b) → Fifo a → Fifo b
map_Fifo f q = map_Fifo' f q emptyQ
where
  map_Fifo' f q = λac → if (isEmptyQ ac) then ac
                        else let fq = front q
                                 dq = deq q
                             in map_Fifo' f dq (enq (f fq) ac)
```

In the first definition, `map_Fifo'` is a catamorphism. However, `map_Fifo'` is an auxiliary function. Function `map_Fifo` is not a catamorphism but the composition of a partially applied catamorphism (`reverseQ` = `map_Fifo' id`) to `map_Fifo'`. (We recall that, in general, catamorphisms are not closed under composition [GNP95].)

In the second definition, `map_Fifo'` is a catamorphism (we prove this at the end of the section), but it is also an auxiliary function. The original `map_Fifo` is not a catamorphism.

Consequently, polytypic `gmap` written according to Figure 6.21 cannot express `map_Fifo`.

Let us illustrate the problem from another angle. Recall the notion of representation type in Generic Haskell (Section 6.1.2). We can define a representation type for linear ADTs and concomitant embedding-projection pairs *using the operators of the linear interface:*

```
type Linear' a = Unit + a × (Linear a)

from_Linear :: ∀ a. Linear a → Linear' a
from_Linear l = if dsc0 l then Inl Unit
                else Inr (sel10 l, sel11 l)
```

```
to_Linear :: ∀ a. Linear' a → Linear a
to_Linear (Inl u)     = con0
to_Linear (Inr (l,r)) = con1 l r
```

Unfortunately, for some linear ADTs, *e.g.* FIFO queues, the following is not the case:

```
to_Linear ∘ from_Linear == id
```

(The equation is certainly not the case for most *bounded* ADTs. Think of ordered sets, for example. We postpone their discussion until Section 9.6).

We conclude the section with a proof that `map_Fifo'` in the second definition of `map_Fifo` is a catamorphism. The proof uses the universality property of catamorphisms [Hut99]. In what follows, `g` abbreviates `map_Fifo' f`:

```
g = (|c▽d|) ⇔
  isEmptyQ q ⇒ g q = c
¬ isEmptyQ q ⇒ g q = d (front q) (g (deq q))
```

First, the proof for empty queues:

```
   c = g q
=     { q empty }
   c = g emptyQ
=     { def. of g }
   c = λac → ac
=     { polymorphism }
   c = id
```

Now the proof for non-empty queues:

```
   d (front q) (g (deq q)) = g q
=     { q non-empty and def. of g }
   d (front q) (g (deq q)) = λac → g (deq q) (enq (f (front q)) ac)
=     { x = front q and y = deq q }
   d x (g y) = λac → g y (enq (f x) ac)
=     { generalising z = g y }
   d x z = λac → z (enq (f x) ac)
```

Thus:

```
map_Fifo' f = (| id ▽ (λx z ac → z (enq (f x) ac)) |)
```

9.4.1 Finding dual operators in lists

In mathematics, "often duality is associated with some sort of general operation, where finding the 'dual' of an object twice retrieves the original object".[2] Inspired by the notion of duality in boolean algebras (*e.g.*, De Morgan principles) it is possible to define a function `dual` that when applied to a list operator returns its dual operator [Tur90]:

```
dual head  == last
dual tail  == init
dual Cons  == snoc
dual Nil   == Nil
dual foldl == foldr ∘ flip
dual foldr == foldl ∘ flip
```

[Jon95a] shows how to define function `dual` using type classes. Class `Dual` is the class of types with a function that maps values to their duals:

```
class Dual a where
  dual :: a → a
```

with the proof obligation that:

```
dual ∘ dual == id
```

Extending duality to function types allows us to find duals of functions:

```
instance (Dual a, Dual b) ⇒ Dual (a → b) where
  dual f = dual ∘ f ∘ dual
```

It is easy to prove the following equations:

```
dual (f x)   = dual f ∘ dual x
dual (f ∘ g) = dual f ∘ dual g
```

Duality in lists involves list reversal:

```
instance Dual a ⇒ Dual (List a) where
  dual = reverse ∘ map dual
```

[2] Source: Wikipedia.com.

9.4.2 Finding dual operators in linear ADTs

We can try to define instances of `dual` for other linear ADTs. For instance, for FIFO queues there are several possibilities:

```
prod = enq                prod = dual enq
exl  = dual front         exl  = front
exr  = dual deq           exr  = deq
```

With these mappings the product laws are satisfied. Glossing over the representation-type machinery (Section 6) and the fact that in Figure 6.21 function g is defined by pattern matching, the following definitions of `map_Fifo` could be instances of $g\langle\texttt{Fifo}\rangle$, where $g = \texttt{gmap}$:

```
map_Fifo f q = if (isEmptyQ q) then q
               else let fq = (dual front) q
                        dq = (dual deq) q
                    in enq (f fq) (map_Fifo f dq)

map_Fifo f q = if (isEmptyQ q) then q
               else let fq = front q
                        dq = deq q
                    in (dual enq) (f fq) (map_Fifo f dq)
```

However, there are several problems. First, `dual` and `map_Fifo` are mutually recursive:

```
instance Dual a ⇒ Dual (Fifo a) where
  dual = reverseQ ∘ map_Fifo dual
```

In lists, `dual` and list map are not mutually recursive. The above instance declaration does not type check. The inferred type of `map_Fifo` is:

```
(a → a) → Fifo a → Fifo a
```

which is not general enough. Second, `dual front`, `dual deq`, and `dual enq` cannot be identified with any FIFO operator. Third, if it run, `map_Fifo` would be extremely inefficient: every call to `dual` reverses the queue.

9.5 Insertion and extraction for *unbounded* linear ADTs

In order to define polytypic functions on ADTs, construction and observation must be separated. In other words, the pattern in Figure 6.21 must be 'de-fused'. Observation can be defined as the process of extracting payload *into something* and construction as the process of inserting payload *from something*. There remains to find a *something* that affords a uniform and general definition of insertion and extraction, and to study whether these operations can be defined polytypically. We begin the study focusing on *linear* ADTs first.

9.5.1 Choosing the concrete type and the operators

Given an *unbounded* ADT whose interface can be mapped to `LINEAR` by a signature morphism, it is possible to write extraction and insertion functions from/to the ADT to/from a concrete type *with the same interface functor*, such that:

1. The extraction function produces a concrete-type replica of the ADT. The value of the concrete type constructed must mirror the *logical* positioning of payload in the ADT. (Observer operators determine the way in which payload is extracted.)
2. The insertion function is the *left inverse* of the extraction function. However, the extraction function is not, in general, the left inverse of the insertion function: the linear ADT satisfies more laws.

For linear ADTs, the obvious choice of concrete type is the list type. Let us call extraction and insertion functions `extracT` and `inserT` respectively:[3]

```
extracT :: Linear a → Linear a
extracT t = if dsc0 t then c_con0
            else c_con1 (sel10 t) (extracT (sel11 t))

inserT :: Linear a → Linear a
inserT t = if c_dsc0 t then con0
           else con1 (c_sel10 t) (inserT (c_sel11 t))
```

[3]We have capitalised their last letter to avoid name clashes with ordinary ADT operators. In Section 9.6 we define `inserT` for ordered sets which already have an `insert` operator.

We have distinguished ADT operators from list operators by prefixing the latter with the symbol `c_`, which stands for 'concrete'. For instance, `c_sel10` is the `sel10` operator in the concrete type (lists, for now).

For many linear ADTs (*e.g.*, FIFO queues and stacks) there is only one possible signature morphism giving values to `con0`, `con1`, etc. For ADTs with multiple constructors (*e.g.*, double-ended queues), there may be more. For concrete types, there are also several possible signature morphisms giving values to `c_con0`, etc. These must be chosen so that `inserT` and `extracT` satisfy the requirements stated at the beginning of the section. In other words, the following 'product laws' must be satisfied:

```
(c_sel10 ∘ extracT) (con1   x y) = x
(c_sel11 ∘ extracT) (con1   x y) = y
(sel10   ∘ inserT)  (c_con1 x y) = x
(sel11   ∘ inserT)  (c_con1 x y) = y
```

The laws of the ADT must be used for this task, *i.e.*, programmers must use algebraic specifications in the definition of `inserT` and `extracT`.

Recall the algebraic specifications of FIFO queues and stacks given in Figure 5.6 and Figure 5.5 respectively. In FIFO queues we dispose of `enq`, `front` and `deq`. We must find the appropriate `c_con1`, `c_sel10`, and `c_sel11` in the list type. Functions `front`, `deq`, and `enq` satisfy, *in lists*, the same laws as **`head`**, **`tail`** and `snoc`, respectively. Observation and construction are not inverses; queue reversal is needed and this is captured by `dual`. Thus, for extraction, the following mapping satisfies the product laws:

```
c_con1  = dual snoc
sel10   = front
sel11   = deq
```

A list replica of the FIFO queue is constructed. For insertion, the following mapping satisfies the product laws:

```
con1    = enq
c_sel10 = dual head
c_sel11 = dual tail
```

Insertion and extraction functions for FIFO queues are shown below:

```
extracT :: Fifo a → List a
extracT q = if isEmptyQ q then Nil
            else (dual snoc) (front q) (extracT (deq q))

inserT  :: List a → Fifo a
inserT l = if null l then emptyQ
           else enq (dual head l) (inserT (dual tail l))
```

Similarly, we dispose of `tos`, `pop`, and `push` in stacks, which satisfy, *in lists*, the same laws as **`head`**, **`tail`**, and `Cons` respectively. Fortunately, observation and construction are inverses. For extraction, the following mapping satisfies the product laws:

```
c_con1 = Cons
sel10  = tos
sel11  = pop
```

For insertion, the following mapping satisfies the product laws:

```
con1    = push
c_sel10 = head
c_sel11 = tail
```

Insertion and extraction functions for stacks are shown below:

```
extracT :: Stack a → List a
extracT s = if isEmptyS s then Nil
            else Cons (tos s) (extracT (pop s))

inserT  :: List a → Stack a
inserT l = if null l then emptyS
           else push (head l) (inserT (tail l))
```

9.5.2 Parameterising on signature morphisms

Calls to `dual` are terribly inefficient, they involve calls to **`reverse`** and **`map`** (Section 9.4.1). Fortunately, a concrete type is equipped with inverse observers for every constructor, or they can be programmed for this purpose.

Functions `inserT` and `extracT` must be parametric on *two* signature morphisms, one mapping ADT operators to linear operators and another mapping concrete-type oper-

ators to linear operators.

In this section we show a Haskell implementation and discuss its limitations.

First, we define `LinearADT`, a type class that describes the operators of the linear interface:

```
class LinearADT f where
  dsc0  :: f a → Bool
  con0  :: f a
  con1  :: a → f a → f a
  sel10 :: f a → a
  sel11 :: f a → f a
```

We then define `LinearCDT`, a type class that describes the operators of a concrete type with a linear interface:

```
class LinearCDT c_f where
  c_dsc0  :: c_f a → Bool
  c_con0  :: c_f a
  c_con1  :: a → c_f a → c_f a
  c_sel10 :: c_f a → a
  c_sel11 :: c_f a → c_f a
```

We re-define `inserT` and `extracT` in terms of these type classes:

```
extracT :: ∀ a. (LinearADT f, LinearCDT c_f) ⇒ f a → c_f a
extracT t = if dsc0 t then c_con0
            else c_con1 (sel10 t) (extracT (sel11 t))

inserT :: ∀ a. (LinearADT f, LinearCDT c_f) ⇒ c_f a → f a
inserT t = if c_dsc0 t then con0
           else con1 (c_sel10 t) (inserT (c_sel11 t))
```

Programmers have to provide the appropriate signature morphisms, *i.e.*, to declare their ADTs instances of `LinearADT`, and to declare their concrete types instances of `LinearCDT`. Programmers must use ADT laws when choosing operators so as to make `inserT` the left inverse of `extracT`:

```
instance LinearADT Stack where
  dsc0  = isEmptyS
```

```
  con0  = emptyS
  con1  = push
  sel10 = tos
  sel11 = pop

instance LinearCDT List where
  c_dsc0  = null
  c_con0  = Nil
  c_con1  = Cons
  c_sel10 = head
  c_sel11 = tail

instance LinearADT Fifo where
  dsc0  = isEmptyQ
  con0  = emptyQ
  con1  = enq
  sel10 = front
  sel11 = deq

instance LinearCDT List where
  c_dsc0  = null
  c_con0  = Nil
  c_con1  = Cons
  c_sel10 = last
  c_sel11 = init
```

The first `List` instance is to be used with stacks whereas the second instance is to be used with FIFO queues. Unfortunately, there are two problems to tackle:

1. There are two *overlapping instances* of `LinearCDT List`. Given an application:

   ```
   (extracT q) :: List Int
   ```

   where `q` has type `Fifo Int`, the compiler cannot determine which instance of `LinearCDT List` to use.

2. We have only dealt with unbounded ADTs with unconstrained payload.

The Haskell language does not allow us to name **`instance`** declarations and to refer to them by name. That would enable us to define different instances for the same concrete

type and also for the same ADT when it has multiple product constructors and there are several ways of making it conform to the linear interface. Some examples:

```
instance LinearADT MemoList where
  con0  = MemoList.nil max
  ...

instance LinearADT MemoList where
  con0  = MemoList.nil sum
  ...

instance LinearADT Deque where
  dsc0  = isEmptyD
  con0  = emptyD
  con1  = enqfront
  sel10 = rear
  sel11 = deqrear

instance LinearADT Deque where
  dsc0  = isEmptyD
  con0  = emptyD
  con1  = enqrear
  sel10 = front
  sel11 = deqfront
```

We tackle this problem in Section 9.10.2. First we deal with bounded linear ADTs.

9.6 Insertion and extraction for *bounded* linear ADTs

Extraction in bounded ADTs behaves in the same way as in unbounded ADTs: it extracts data in a deterministic order imposed by the choice of discriminators and selectors. In contrast, constructors are 'clever' and arrange the payload internally: think of `insert` in ordered sets. Although product laws may not be satisfied, there is no need to find inverse observers in the concrete type of 'clever' ADT constructors.

Indeed, an unbounded ADT with functorial interface F can be viewed as a concrete type with interface F where the laws restrict the way in which payload is inserted or selected from the type. In contrast, the laws of a bounded ADT impose context-

dependent restrictions that rely on properties of the payload type. Such restrictions are taken into account by 'clever' constructors.

For example, extraction and insertion for ordered sets can be defined as follows:

```
extracT :: ∀ a. Ord a ⇒ Set a → List a
extracT s = if isEmptyS s then Nil
            else Cons (choice s) (extracT (remove (choice s) s))

inserT  :: ∀ a. Ord a ⇒ List a → Set a
inserT l = if null l then emptySet
           else insert (head l) (inserT (tail l))
```

It does not matter whether `inserT` is defined otherwise as:

```
inserT  :: ∀ a. Ord a ⇒ List a → Set a
inserT l = if null l then emptySet
           else insert (last l) (inserT (init l))
```

Payload elements are arranged internally by `insert` according to order, and only elements not already in the set make it. In both cases, `inserT` is the left inverse of `extracT`.

There is an obstacle: the presence of the **`Ord`** constraint. It not only appears in the types of `extracT` and `inserT`, but also forces us to define `LinearADT` and `LinearCDT` as *multi-parameter* type classes (recall the discussion in Section 7.1.3). A possible solution is to introduce γ-abstraction (Section 6.1.11):

```
extracT :: γq. ∀a. (q a, LinearADT f, LinearCDT c_f) ⇒ f a → c_f a
inserT  :: γq. ∀a. (q a, LinearADT f, LinearCDT c_f) ⇒ c_f a → f a
```

Fortunately, we can encode these functions in Haskell using a technique proposed in [Hug99]. The only hurdle remaining is the lack of overlapping instances. In Section 9.8 we present the details of the encoding and show how to define generic functions on linear ADTs in terms of `inserT` and `extracT`. The section uses ***extensional equality*** as an example of generic function. We first explain what extensional equality means in the following section.

9.7 Extensional equality

Extensional equality compares ADTs by comparing their payload contents, not their internal representation. Extensional equality on two values `x` and `y` of the same linear ADT can be defined as follows:

```
geqLinear eqa x y = qeq⟨List⟩ eqa (extracT x) (extracT y)
```

Function `eqLinear` is defined under the assumption that the following laws hold:

```
(dsc0 x) ∧  (dsc0 y) ⇒ x = y
(dsc0 x) ∧ ¬(dsc0 y) ∨ ¬(dsc0 x) ∧ (dsc0 y) ⇒ x ≠ y
¬(dsc0 x) ∧ ¬(dsc0 y) ⇒
   x = y ⇔ (sel10 x = sel10 y ∧ sel11 x = sel11 y)
```

For instance, ordered sets with deterministic `choice` and `remove` satisfy the conditions. We can define (extensional) equality for ordered sets provided the payload type supports ordinary equality:

```
instance Eq a ⇒ Eq (Set a) where
  (==) sx sy = let x = choice sx
                   y = choice sy
               in x == y && (remove x sx) == (remove y sy)
```

It is possible to test whether two different ADTs that conform to the linear interface are extensionally equal. For example, it is possible to test whether a stack and an ordered set are extensionally equal by extracting their payload into two values of the same concrete type and comparing them for ordinary equality. Unfortunately, this general notion of extensional equality requires signature morphisms for every ADT involved, two in this case. We come back to this in Section 9.15.

9.8 Encoding generic functions on linear ADTs in Haskell

This section shows a Haskell implementation[4] of `inserT`, `extracT`, and of generic size, map, and equality functions for linear ADTs, whether bounded or unbounded.

To handle constraints we encode both bounded and unbounded ADTs as ***restricted types*** [Hug99]. The key idea is to encode constraints using *explicit* dictionaries in-

[4]We have compiled the code using the Glasgow Haskell Compiler v6.2.1, which supports multi-parameter type classes and explicit kind annotations.

stead of the implicit dictionaries created by the compiler, and to define `LinearADT` and `LinearCDT` as multi-parameter type classes where one parameter is an explicit dictionary. ADTs and concrete types that are instances of these classes are parametric on constraints, and so are functions defined on them.

```
data TopD a = TopD{ my_id :: a → a }
data EqD  a = EqD{ eq :: a → a → Bool }
data OrdD a = OrdD{ lt :: a → a → Bool, eqOrd :: EqD a }

class Sat t where dict :: t

instance Sat (TopD a) where      -- universal constraint
    dict = TopD{ my_id = id }

instance Eq a ⇒ Sat (EqD a) where
    dict = EqD { eq = (==) }

instance Ord a ⇒ Sat (OrdD a) where
    dict = OrdD{ lt = (<), eqOrd = dict }
```

Figure 9.3: Explicit dictionaries and `Sat` proxy.

First, we explain the encoding of explicit dictionaries. The first three lines in Figure 9.3 declare the explicit dictionary types `TopD`, `EqD`, and `OrdD`. The first is a 'universal' dictionary which is associated with unbounded ADTs. It has to be provided because `LinearADT` and `LinearCDT` will expect an explicit dictionary argument. However, unbounded ADTs do not make use of the dictionary's 'dummy' operator. An `EqD` dictionary has an equality operator `eq` and an `OrdD` dictionary 'extends' an `EqD` dictionary with a comparison operator `lt` (less than). An **`Eq`** constraint is encoded by the explicit dictionary `EqD`, and an **`Ord`** constraint by the explicit dictionary `OrdD`.

Class `Sat` is a *proxy*, *i.e.*, a type class that is used by programmers to tell the compiler that an explicit dictionary exists. More precisely, a type `a` has a dictionary `D` (or is `D`-constrained) if `D a` is an instance of `Sat`. Figure 9.3 shows how `TopD`, `EqD`, and `OrdD` are made instances of `Sat`. Notice that the last two dictionaries use operators in *implicit* dictionaries.

Figure 9.4 shows type classes `LinearADT` and `LinearCDT` which now take the explicit dictionary parameter `cxt`. The dictionary appears as a 'constraint' in the type-signatures of operators.

```
class LinearADT l cxt where
    dsc0  :: Sat (cxt a) ⇒ l cxt a → Bool
    con0  :: Sat (cxt a) ⇒ l cxt a
    con1  :: Sat (cxt a) ⇒ a → l cxt a → l cxt a
    sel10 :: Sat (cxt a) ⇒ l cxt a → a
    sel11 :: Sat (cxt a) ⇒ l cxt a → l cxt a

class LinearCDT l cxt where
    c_dsc0  :: Sat (cxt a) ⇒ l cxt a → Bool
    c_con0  :: Sat (cxt a) ⇒ l cxt a
    c_con1  :: Sat (cxt a) ⇒ a → l cxt a → l cxt a
    c_sel10 :: Sat (cxt a) ⇒ l cxt a → a
    c_sel11 :: Sat (cxt a) ⇒ l cxt a → l cxt a
```

Figure 9.4: Type classes `LinearADT` and `LinearCDT`.

Functions `extracT` and `inserT` are shown in Figure 9.5. Function `extracT` takes an ADT argument that is an instance of `LinearADT` and whose payload is constrained on `cxt`. It returns a concrete type that is an instance of `LinearCDT` whose payload is also constrained on `cxt`. Function `inserT` is the left-inverse operation.

```
extracT :: (LinearADT l cxt, LinearCDT l' cxt, Sat (cxt a))
        ⇒ l cxt a → l' cxt a

extracT l = if dsc0 l then c_con0
            else c_con1 (sel10 l) (extracT (sel11 l))

inserT :: (LinearADT l cxt, LinearCDT l' cxt, Sat (cxt a))
        ⇒ l' cxt a → l cxt a

inserT l' = if c_dsc0 l' then con0
            else con1 (c_sel10 l') (inserT (c_sel11 l'))
```

Figure 9.5: Generic functions `extracT` and `inserT`.

The first box in Figure 9.6 shows the definition of type `List`. We cannot use ordinary Haskell lists because an instance of `LinearCDT` must have the same constraints as the ADT that is made an instance of `LinearADT`, and Haskell's built-in list type is not a restricted type. Type `List` is also parametric on an explicit dictionary `cxt`, whose kind is written explicitly because it cannot be properly inferred by the compiler (it infers kind $*$ by default). `List` operators also have to be programmed from scratch. For each ordinary list operator we define a `List` one whose name is prefixed by the letter `r` (from 'restricted'). The second box in Figure 9.6 shows two examples. The third box shows

the definitions of map and size for `List`. The definition of equality has been omitted for reasons of space.

Figure 9.7 shows the FIFO-queue interface `QueueClass` and a batched implementation (Chapter 7). The `QueueClass` interface is defined within a module named `QueueClassM`. (ADT operators will be qualified by module name in later figures.) As with the `List` type, the implementation type `BatchedQueue` is parametric on `cxt`.

Figure 9.8 shows the stack interface `StackClass` and an implementation in terms of Haskell's built-in list type. The `StackClass` interface is defined within a module named `StackClassM`.

Notice that a `TopD` dictionary is associated with FIFO queues whereas a `TopD'` dictionary is associated with stacks. Two dictionaries are needed because overlapping instances of `LinearCDT List TopD` are illegal.

Figure 9.9 shows the ordered-set interface `SetClass` and one possible implementation in terms of ordered (Haskell) lists. The `SetClass` interface is defined within a module named `SetClassM`. `SetList` is made an instance of `SetClass` with constraint `OrdD`. Notice how explicit-dictionary operators are used in the implementation of `insert` and `remove`.

Figure 9.10 shows the encoding of signature morphisms. Ordered sets, FIFO queues, and stacks are made instances of the `LinearADT` class with the relevant constraints. There are several `LinearCDT List` instances associated with these linear ADTs. The association is established by the shared explicit dictionary.

Finally, Figure 9.11 defines map, size, and equality for linear ADTs. The concrete type has been fixed to `List`, which is an instance of class `LinearCDT`.

It would be preferable to use polytypic functions on the concrete type, that is, to define, say, `sizeLinear` as follows:

```
sizeLinear sa = gsize⟨List⟩ sa ∘ extracT
```

Unfortunately, Generic Haskell does not support constrained types (Section 6.1.10) and the instance of `gsize` generated would be the instance for the *restricted* `List` type. In our encoding, we have to use `sizeLinear`.

Admittedly, there is no polytypism in the code. From Section 9.9 onwards we show how

```
data List (cxt :: ∗ → ∗) a = Nil | Cons a (List cxt a)
```

```
rNull :: Sat (cxt a) ⇒ List cxt a → Bool
rNull Nil        = True
rNull (Cons _ _) = False

rHead :: Sat (cxt a) ⇒ List cxt a → a
rHead Nil        = error "rHead: empty list"
rHead (Cons x _) = x
```

```
mapList :: (Sat (cxt a), Sat (cxt b))
        ⇒ (a → b) → List cxt a → List cxt b
mapList f Nil = Nil
mapList f (Cons x xs) = Cons (f x) (mapList f xs)
--
sizeList :: Sat (cxt a) ⇒ (a → Int) → List cxt a → Int
sizeList sa Nil         = 0
sizeList sa (Cons x xs) = sa x + sizeList sa xs
```

Figure 9.6: `List` type and functions `mapList` and `sizeList`.

```
class QueueClass q cxt where
    empty   :: Sat (cxt a) ⇒ q cxt a
    isEmpty :: Sat (cxt a) ⇒ q cxt a → Bool
    enq     :: Sat (cxt a) ⇒ a → q cxt a → q cxt a
    front   :: Sat (cxt a) ⇒ q cxt a → a
    deq     :: Sat (cxt a) ⇒ q cxt a → q cxt a
```

```
data BatchedQueue (cxt :: ∗ → ∗) a = BQ [a] [a]
```

```
instance QueueClass BatchedQueue TopD where
    empty = BQ [] []

    isEmpty (BQ f r) = null f

    enq x (BQ f r) = check f (x:r)

    front (BQ [] _)    = error "Empty Queue"
    front (BQ (x:f) r) = x

    deq (BQ [] _ )   = error "Empty Queue"
    deq (BQ (x:f) r) = check f r
```

Figure 9.7: FIFO-queue interface and a possible implementation.

```
class StackClass s cxt where
    empty   :: Sat (cxt a) ⇒ s cxt a
    isEmpty :: Sat (cxt a) ⇒ s cxt a → Bool
    push    :: Sat (cxt a) ⇒ a → s cxt a → s cxt a
    tos     :: Sat (cxt a) ⇒ s cxt a → a
    pop     :: Sat (cxt a) ⇒ s cxt a → s cxt a
```

```
data Stack (cxt :: ∗ → ∗) a = ST [a]
```

```
instance StackClass Stack TopD' where
    empty           = ST []
    isEmpty (ST xs) = null xs
    push x (ST xs)  = ST (x:xs)
    tos (ST [])     = error "tos: empty stack"
    tos (ST (x:xs)) = x
    pop (ST [])     = error "pop: empty stack"
    pop (ST (x:xs)) = ST xs
```

Figure 9.8: Stack interface and a possible implementation

`inserT`, `extracT`, and functions on ADTs defined in terms of them can be programmed polytypically by generalising the solution presented in this Section.

We conclude the section with examples of usage in Figures 9.12 and 9.13. The reader may want to compare the results with those of Chapter 7. (N.B.: pretty-printing **`show`** functions were defined for every type.)

```
class SetClass s cxt where
    isEmpty :: Sat (cxt a) ⇒ s cxt a → Bool
    empty   :: Sat (cxt a) ⇒ s cxt a
    insert  :: Sat (cxt a) ⇒ a → s cxt a → s cxt a
    choice  :: Sat (cxt a) ⇒ s cxt a → a
    remove  :: Sat (cxt a) ⇒ a → s cxt a → s cxt a
    member  :: Sat (cxt a) ⇒ a → s cxt a → Bool
```

```
data SetList (cxt :: ∗ → ∗) a = SL [a]
```

```
instance SetClass SetList OrdD where
  empty               = SL []

  isEmpty (SL xs)     = null xs

  insert x (SL xs)    = SL (insert' x xs)
      where
      insert' x []         = [x]
      insert' x (l@(y:ys))
            | eq (eqOrd dict) x y = l
            | (lt dict) x y       = (x:y:ys)
            | otherwise           = y : (insert' x ys)

  remove x (SL xs)    = SL (remove' x xs)
      where remove' x []     = []
            remove' x (y:ys) = if eq (eqOrd dict) x y then ys
                               else y : (remove' x ys)

  member x (SL xs)    = any (eq (eqOrd dict) x) xs

  choice (SL [])      = error "choice: empty set"
  choice (SL (x:xs))  = x
```

Figure 9.9: Ordered-set interface and a possible implementation.

```
instance LinearADT SetList OrdD where
    dsc0  = SetClassM.isEmpty
    con0  = SetClassM.empty
    con1  = SetClassM.insert
    sel10 = SetClassM.choice
    sel11 = λs → SetClassM.remove (SetClassM.choice s) s

instance LinearCDT List OrdD where
    c_dsc0  = rNull
    c_con0  = Nil
    c_con1  = Cons
    c_sel10 = rHead
    c_sel11 = rTail
```

```
instance LinearADT BatchedQueue TopD where
    dsc0  = QueueClassM.isEmpty
    con0  = QueueClassM.empty
    con1  = QueueClassM.enq
    sel10 = QueueClassM.front
    sel11 = QueueClassM.deq

instance LinearCDT List TopD where
    c_dsc0  = rNull
    c_con0  = Nil
    c_con1  = Cons
    c_sel10 = rLast
    c_sel11 = rInit
```

```
instance LinearADT Stack TopD' where
    dsc0  = StackClassM.isEmpty
    con0  = StackClassM.empty
    con1  = StackClassM.push
    sel10 = StackClassM.tos
    sel11 = StackClassM.pop

instance LinearCDT List TopD' where
    c_dsc0  = rNull
    c_con0  = Nil
    c_con1  = Cons
    c_sel10 = rHead
    c_sel11 = rTail
```

Figure 9.10: Ordered sets, FIFO queues, and stacks are linear ADTs.

```
mapLinear :: (LinearADT l cxt, LinearCDT List cxt,
              Sat (cxt a), Sat (cxt b))
          ⇒ (a → b) → l cxt a → l cxt b

mapLinear f = inserT ∘ mapList f ∘ extracT
--
sizeLinear :: (LinearADT l cxt, LinearCDT List cxt, Sat (cxt a))
           ⇒ (a → Int) → l cxt a → Int

sizeLinear sa = sizeList sa ∘ extracT
--
eqLinear :: (LinearADT l cxt, LinearCDT List cxt, Sat (cxt a))
         ⇒ (a → a → Bool) → l cxt a → l cxt a → Bool

eqLinear eqa lx ly = eqList eqa (extracT lx) (extracT ly)
```

Figure 9.11: Map, size, and equality as generic functions on `LinearADT`s.

```
  s :: SetList OrdD Int s = foldr (λx y → SetClassM.insert x y)
  SetClassM.empty [5,1,2,4,3,2,1] > { 1,2,3,4,5 }

s0 = mapLinear (const 0) s
> {0}

mapLinear negate s
> {-5,-4,-3,-2,-1}

sizeLinear (const 1) s
> 5

sizeLinear (const 1) s0
> 1

eqLinear (==) s s
> True

eqLinear (==) s (remove (choice s) s)
> False

eqLinear (==) s (insert (choice s) s)
> True
```

Figure 9.12: Computing with ordered sets.

```
q :: BatchedQueue TopD Int
q = foldl (λx y → QueueClassM.enq y x) QueueClassM.empty [2,5,1,6]
> <2,5,1,6>

mapLinear negate q
> <-2,-5,-1,-6>

k :: Stack TopD' Int
k = foldr (λx y → StackClassM.push x y) StackClassM.empty [1,2,3]
> 1,2,3|

mapLinear negate k
> -1,-2,-3|

eqLinear (==) k (pop k)
> False
```

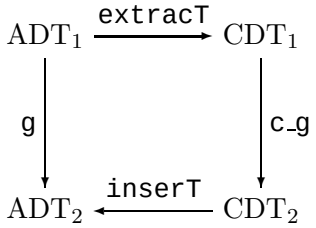
Figure 9.13: Computing with FIFO queues and stacks.

9.9 Extensional Programming = EC[I]

In this and the following sections we generalise our solution for linear ADTs to arbitrary ADTs that can be made to conform to some functorial interface. We also show that `inserT`, `extracT`, and other functions on ADTs can be defined polytypically.

The previous sections have introduced the notion of Extensional Programming and made the case for the separation of insertion and extraction when attempting the definition of generic functions on ADT values.

Let us abbreviate and call **EC[I]** the model of computation with ADTs where Extensional Programming is carried out in terms of Extraction to a concrete type, Computation on this type, and optional Insertion. The following diagram depicts its general form:



The acronym CDT stands for 'concrete data type'. Function `g` takes an ADT_1 value and returns an ADT_2 value. It is implemented in terms of `c_g`, `inserT`, and `extracT`. Function `c_g` takes a CDT_1 value and returns a CDT_2 value. Function `extracT` returns a CDT_1 value with ADT_1's payload, and `inserT` takes a CDT_2 value and produces an ADT_2 value using CDT_2's payload.

Notice the similarity with the principles of the C++ STL where ADTs are ***containers*** with payload and ***iterators*** afford to decouple functions from containers. In the EC[I] model, iterators are replaced by concrete types.

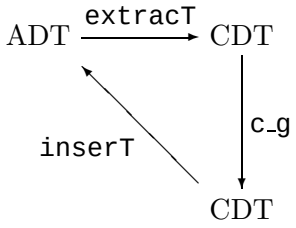
In type-unifying computations, there is no insertion and therefore CDT_2 is a manifest type like **`Int`** or **`Bool`**. In type-preserving computations, CDT_1 and CDT_2 need not be the same type, but both must be parametric on the same payload types in order to pass payload around.

What is desired is a *polytypic* EC[I] model where all the arrows are polytypic functions:

$$\texttt{g}\langle \mathrm{ADT}_1, \mathrm{CDT}_1, \mathrm{ADT}_2, \mathrm{CDT}_2 \rangle = \texttt{inserT}\langle \mathrm{ADT}_2, \mathrm{CDT}_2 \rangle \circ \texttt{c\_g}\langle \mathrm{CDT}_1 \rangle \circ \texttt{extracT}\langle \mathrm{ADT}_1, \mathrm{CDT}_1 \rangle$$

Every function except `c_g` is polytypic on more than one argument. This function is an ordinary Generic Haskell function. Extraction and insertion functions need to know the functorial structure of their source and target types. Consequently, CDT_1 and CDT_2 must be provided as arguments to `g` or otherwise their values would be fixed in `g`'s body. ADT_i and CDT_i must conform to the same functorial interface. However, ADT_1 and ADT_2 need not conform to the same functorial interface.

We will consider a simplified EC[I] model where ADT_1 and ADT_2 are the same abstract type and CDT_1 and CDT_2 are the same concrete type:



That is:

$$\texttt{g}\langle \mathrm{ADT}, \mathrm{CDT}\rangle \texttt{ = inserT}\langle \mathrm{ADT}, \mathrm{CDT}\rangle \circ \texttt{c\_g}\langle \mathrm{CDT}\rangle \circ \texttt{extracT}\langle \mathrm{ADT}, \mathrm{CDT}\rangle$$

The reasons are simple:

1. All polytypic functions on ADTs take the same number and type of arguments.

2. Functions `inserT` and `extracT` are polytypic on the structure of interface functors, and choosing a free CDT_1 with the same functorial interface as CDT_2 is choosing the same concrete type, names of value constructors and operators notwithstanding. Function `inserT` will be a catamorphism, not an anamorphism [MFP91].

3. When CDT and ADT have the same functorial interface then structural information cannot be lost. Structural information is lost if `extracT` is left without a corresponding `inserT`, making type-preserving functions undefinable. Furthermore, `inserT` must be the left inverse of `extracT`:

   ```
   inserT ∘ extracT == id
   ```

In the case of unbounded ADTs, the CDT must provide all the required operators so that the above equation can be satisfied. In the case of bounded ADTs, there is no

need to worry, for there are 'clever' constructors that properly reconstruct the ADT (Section 9.6).

Nonetheless, for some type-unifying computations, such as calculating the size or testing for equality, loosing the information required for building back the original ADT is not a problem (Section 9.14).

9.10 Polytypic extraction and insertion

Insertion and extraction functions are structurally polymorphic on the functorial structure of an interface and, consequently, their definition can be generated automatically by a compiler. Such functorial structure is declared by the generic programmer in what we call an *F-view*.

9.10.1 *F*-views

An *F*-view is a language extension for declaring the functorial structure of ADT interfaces. We introduce the syntax using some examples. The following *F*-view is similar to the type class `LinearADT` of Section 9.8:

```
fview Linear a = 1 + a × (Linear a)
```

The structure of the *F*-view automatically determines the following operators:

```
dsc0  :: γq. ∀a. q a ⇒ Linear a → Bool
con0  :: γq. ∀a. q a ⇒ Linear a
con1  :: γq. ∀a. q a ⇒ a → Linear a → Linear a
sel10 :: γq. ∀a. q a ⇒ Linear a → a
sel11 :: γq. ∀a. q a ⇒ Linear a → Linear a
```

The discriminator `dsc0` comes from the presence of the coproduct. There are two products and hence two constructors `con0` and `con1`. There is no selector for a nullary product and there are two selectors `sel10` and `sel11` for the binary product.

Every *F*-view `F` automatically determines another *F*-view `c_F` where every operator `op` in `F` is named `c_op` in `c_F`:

```
fview c_Linear a = 1 + a × (c_Linear a)
c_dsc0  :: γq. ∀a. q a ⇒ c_Linear a → Bool
c_con0  :: γq. ∀a. q a ⇒ c_Linear a
```

```
  c_con1  :: γq. ∀a. q a ⇒ a → c_Linear a → c_Linear a
  c_sel10 :: γq. ∀a. q a ⇒ c_Linear a → a
  c_sel11 :: γq. ∀a. q a ⇒ c_Linear a → c_Linear a
```

Other examples of F-views are shown in Figure 9.14. Reading an F-view declaration from left to right, every ith coproduct has an associated discriminator `dsc`i. Every ith product has one constructor `con`i with selectors `sel`$i0$ to `sel`im, where m is the arity of the product.

```
fview Bin1 a = 1 + a × (Bin1 a) × (Bin1 a)
   dsc0  :: γq. ∀a. q a ⇒ Bin1 a → Bool
   con0  :: γq. ∀a. q a ⇒ Bin1 a
   con1  :: γq. ∀a. q a ⇒ a → Bin1 a → Bin1 a → Bin1 a
   sel10 :: γq. ∀a. q a ⇒ Bin1 a → a
   sel11 :: γq. ∀a. q a ⇒ Bin1 a → Bin1 a
   sel12 :: γq. ∀a. q a ⇒ Bin1 a → Bin1 a

fview Bin2 a b = 1 + a + b × (Bin2 a b) × (Bin2 a b)
   dsc0  :: γq. ∀a. q a ⇒ Bin2 a b → Bool
   dsc1  :: γq. ∀a. q a ⇒ Bin2 a b → Bool
   con0  :: γq. ∀a. q a ⇒ Bin2 a b
   con1  :: γq. ∀a. q a ⇒ a → Bin2 a b
   sel10 :: γq. ∀a. q a ⇒ Bin2 a b → a
   con2  :: γq. ∀a. q a ⇒ b → Bin2 a b → Bin2 a b → Bin2 a b
   sel20 :: γq. ∀a. q a ⇒ Bin2 a b → b
   sel21 :: γq. ∀a. q a ⇒ Bin2 a b → Bin2 a b
   sel22 :: γq. ∀a. q a ⇒ Bin2 a b → Bin2 a b

fview Composite3 a b c = a × b × c
   con0  :: γq. ∀a. q a ⇒ a → b → c → Composite3 a b c
   sel00 :: γq. ∀a. q a ⇒ Composite3 a b c → a
   sel01 :: γq. ∀a. q a ⇒ Composite3 a b c → b
   sel02 :: γq. ∀a. q a ⇒ Composite3 a b c → c
```

Figure 9.14: Examples of F-view declarations and their implicitly-defined operators.

9.10.2 Named signature morphisms

ADT interfaces can be made to conform to F-views by providing ***named signature morphisms***. We have already showed in Section 9.5.2 examples of signature morphisms. In this section we explain the new syntax by example.

```
Set instance Linear by SetL where
  dsc0 = isEmptySet
```

```
  con0 = emptySet
  ...
```

The syntax declares that there is a signature morphism `SetL` from the ADT `Set` to the `Linear` F-view. A signature morphism between a concrete type and `c_Linear` must be declared in similar fashion:

```
List instance c_Linear by L1 where
  c_dsc0  = null
  c_con0  = Nil
  c_con1  = Cons
  c_sel10 = head
  c_sel11 = tail

List instance c_Linear by L2 where
  c_dsc0  = null
  c_con0  = Nil
  c_con1  = Cons
  c_sel10 = last
  c_sel11 = init
```

The two morphisms have different names and therefore can cohabit in a program. In Section 9.8, F-views were encoded as type classes and signature morphisms were encoded as instance declarations. In that setting, overlapping instances were not allowed. We show a possible implementation of F-views and named signature morphisms in Section 9.10.3.

To the compiler, every named signature morphism has two associated meta-functions called *type* and *view*. The first returns the type that is made an instance of the F-view, *e.g.*:

```
type SetL = Set
type L2 = List
```

The second returns a representation of the functorial structure declared by the F-view, *e.g.*:

```
view Linear = 1 + a × (Linear a)
view Composite3 = a × b × c
```

9.10.3 Implementing F-views and named signature morphisms

F-views and named signature morphisms are language extensions. In this section we show that they are reasonable and feasible extensions by indicating a possible implementation.

An F-view declaration can be translated by the compiler to a multi-parameter type class where one parameter is an explicit dictionary parameter (Section 9.8) and another is a *name* parameter, *i.e.*, a type encoding a signature morphism *name*. Every function in the class takes an extra argument of that type.

Named signature morphisms can be translated by the compiler to instances of the type class that provide actual values for dictionaries and 'name' types.

Figure 9.15 shows the translation for F-view `c_Linear` and named signature morphisms `L1` and `L2`.

```
class C_Linear l cxt n where
  c_dsc0  :: ∀ a. Sat (cxt a) ⇒ n → l cxt a → Bool
  c_con0  :: ∀ a. Sat (cxt a) ⇒ n → l cxt a
  c_con1  :: ∀ a. Sat (cxt a) ⇒ n → a → l cxt a → l cxt a
  c_sel10 :: ∀ a. Sat (cxt a) ⇒ n → l cxt a → a
  c_sel11 :: ∀ a. Sat (cxt a) ⇒ n → l cxt a → l cxt a

data L1 = L1
data L2 = L2

instance C_Linear List TopD L1 where
  c_dsc0  = const null
  c_con0  = cons Nil
  c_con1  = const Cons
  c_sel10 = const head
  c_sel11 = const tail

instance C_Linear List TopD L2 where
  c_dsc0  = const null
  c_con0  = cons Nil
  c_con1  = const Cons
  c_sel10 = const last
  c_sel11 = const init
```

Figure 9.15: Possible implementation of F-views and named signature morphisms.

The F-view is compiled to a type class `C_Linear` which takes an explicit dictionary

parameter `cxt` and a 'name' parameter `n`. Every operator has an extra parameter of type `n`.

The signature-morphism *names* `L1` and `L2` are compiled to new types with nullary value constructors of the same name. The signature morphisms are implemented as instances of `C_Linear` where operators discard their first argument.

The types `L1` and `L2` help the compiler resolve the overloading. More precisely, operators `c_`*op* only occur in the bodies of `inserT` and `extracT`, which take signature-morphism arguments (we discuss the details of this in the next section). When `inserT` or `extracT` are passed signature morphism `L1` as an actual argument, the compiler generates their bodies writing `c_`*op* `L1` where a call to `c_`*op* would have been expected. The compiler can deduce from the application that `c_`*op* must be of type `L1` $\rightarrow \tau$ for some type τ. The other possibility, `L2` $\rightarrow \tau$ does not type-check. Similarly, when `inserT` or `extracT` are passed signature morphism `L2` as an actual argument, what are generated are calls to `c_`*op* `L2`.

9.10.4 Polyaric types and instance generation

We now define `inserT` and `extracT` polytypically. Their types are parametric on the arity of the *type* component of a signature morphism and their bodies are generated following the functorial structure provided by the *view* component. Operator names are also obtained from signature-morphism arguments.

More precisely, `inserT` and `extracT` are parametric on two signature morphisms that provide all the information. One morphism maps an ADT to an F-view F and another maps a CDT to the related F-view c_F. The bodies are structurally polymorphic on the functor defined by the F-view. Like Generic Haskell, we follow a generative approach and generate instances of `inserT` and `extracT` for actual signature-morphism arguments in polytypic applications. Generic programmers do not have to specify anything. The type-signatures and bodies of instances are generated automatically by the compiler.

We introduce the notion of ***polyaric*** type, *i.e.*, a type that is parametric on the *arity* of an ADT, not the kind. The reason for this is that we deal with first-order ADTs

whose kind is described by the grammar:

$$\kappa \ ::= \ * \mid * \rightarrow \kappa$$

Functions `extracT` and `inserT` possess polyaric types, and so will polytypic functions defined in terms of them.

Types `ExtracT` and `InserT` are respectively the polyaric types of `extracT` and `inserT`:

$$\begin{array}{lcl}
\texttt{ExtracT}\langle n\rangle\ t_1\ t_2 & :: & \gamma q.\ \forall\ \overline{a}.\ q\ \overline{a} \Rightarrow t_1\ \overline{a}\ \rightarrow\ t_2\ \overline{a} \\
\texttt{InserT}\langle n\rangle\ t_1\ t_2 & :: & \gamma q.\ \forall\ \overline{a}.\ q\ \overline{a} \Rightarrow t_2\ \overline{a}\ \rightarrow\ t_1\ \overline{a}
\end{array}$$

$$\begin{array}{lcl}
\texttt{extracT}\langle f, c_f\rangle & :: & \texttt{ExtracT}\langle (\mathit{arity} \circ \mathit{type})\ f\rangle\ (\mathit{type}\ f)\ (\mathit{type}\ c_f)\ \Delta(\mathit{type}\ f) \\
\texttt{insertT}\langle f, c_f\rangle & :: & \texttt{InserT}\langle (\mathit{arity} \circ \mathit{type})\ c_f\rangle\ (\mathit{type}\ f)\ (\mathit{type}\ c_f)\ \Delta(\mathit{type}\ f)
\end{array}$$

where $n > 0$ and $\overline{a} \stackrel{\text{def}}{=} a_1 \ldots a_n$. The case $n = 0$ applies to manifest ADTs and cannot be explained until we introduce our notion of exporting in Section 9.13.

Let `F` and `C_F` be two signature morphisms. The polytypic application `extracT⟨F,C_F⟩` in the program triggers the generation of the instance of `extracT` for those signature morphisms, namely, `extracT_F_C_F`, whose definition is generated by:

$$\mathit{genCopro}(\texttt{t}, \mathit{view}(\texttt{F}))$$

where `t` is a chosen value-parameter name, $\mathit{view}(\texttt{F}) = P_0 + \ldots + P_n$ for some $n \geq 0$ coproducts of products $P_i = X_{i0} \times \ldots \times X_{im}$ for some $m \geq 0$, and *genCopro* is a compiler meta-function whose definition for extraction is shown in Figure 9.16.

```
genCopro(t, P_0 + ... + P_n) =
  if (dsc0 t) then genPro(t, P_0)
  ...
  else if (dsc(n−1) t) then genPro(t, P_{n−1})
  else genPro(t, P_n)

genPro(t, 1_i)                = c_coni
genPro(t, X_i0 × ... × X_im)  = c_coni genTerm(t, X_i0) ... genTerm(t, X_im)

genTerm(t, a_ij)     = selij t
genTerm(t, (F ā)_ij) = extracT (selij t)
```

Figure 9.16: Meta-function *genCopro* generates the body of `extracT` at compile-time following the structure specified by its second argument.

An almost identical meta-function is defined for `inserT` where `inserT` occurs instead

of `extracT` and *c_op* operators occur instead of *op* operators and vice versa.

9.10.5 Generation examples

We illustrate the generation process for ordered sets and FIFO queues. First, the signature morphisms:

```
Set instance Linear by SetF where
  dsc0  = isEmptyS
  con0  = emptyS
  con1  = insert
  sel10 = choice
  sel11 = λs → remove (choice s) s

Queue instance Linear by QueueF where
  dsc0  = isEmptyQ
  con0  = emptyQ
  con1  = enq
  sel10 = front
  sel11 = deq
```

For the concrete type, we use signature morphism `L2` from Section 9.10.2. The polytypic application `extracT⟨SetF,L2⟩` triggers the generation of a `extracT_SetF_L2` instance whose type is given by expanding the polyaric type:

```
ExtracT⟨1⟩ Set List [Ord]
```

The resulting type-signature is:

```
extracT_SetF_L2 :: ∀ a. Ord a ⇒ Set a → List a
```

Similarly, the polytypic application `inserT⟨SetF,L2⟩` triggers the generation of an `inserT_SetF_L2` instance whose type is given by expanding the polyaric type:

```
InserT⟨1⟩ Set List [Ord]
```

which yields the type-signature:

```
inserT_SetF_L2 :: ∀ a. Ord a ⇒ List a → Set a
```

The bodies of `extracT_SetF_L2` and `inserT_SetF_L2` are generated by the compile-time evaluation of:

$$genCopro(\texttt{t}, 1 + a \times (\texttt{Linear}\ a))$$

where F-view operator names are replaced by the actual ones provided by the signature morphisms:

```
extracT_SetF_L2 t =
  if isEmptyS t then Nil
  else Cons (choice t) (extracT ((λs → remove (choice s) s) t)

inserT_SetF_L2 t =
  if null t then emptyS
  else insert (last t) (inserT (init t))
```

(If F-views and named signature morphisms are implemented as suggested in Section 9.10.3, list and set operators in the previous definitions must be applied to an `L2` argument first.)

FIFO queues have no payload constraints. The following polytypic applications:

```
extracT⟨QueueF,L2⟩
inserT⟨QueueF,L2⟩
```

trigger the generation of the following type-signatures:

```
extracT_QueueF_L2 :: ExtracT⟨1⟩ Queue List []
inserT_QueueF_L2  :: InserT⟨1⟩ Queue List []
```

and the generation of the bodies using the operators provided by the signature morphisms:

```
extracT_QueueF_L2 :: ∀ a. Queue a → List a
extracT_QueueF_L2 t = if isEmptyQ t then Nil
                      else Cons (front t) (extracT (deq t))

inserT_QueueF_L2 :: ∀ a. List a → Queue a
inserT_QueueF_L2 t = if null t then emptyQ
                     else enq (last t) (inserT (init t))
```

9.11 Defining polytypic functions

Polytypic functions such as `gsize`, `gmap`, or `geq` can now be programmed in terms of `inserT` and `extracT`. Polytypic functions on first-order ADTs possess polyaric types.

The generic programmer first defines the polyaric type of the function:

$$\begin{array}{lcl} G\langle 0\rangle\ \overline{t} & = & \tau \\ G\langle n\rangle\ \overline{t} & = & \forall\ \overline{a}.\ G\langle 0\rangle\ \overline{a}\ \rightarrow\ G\langle n-1\rangle\ \overline{(t\ a)} \end{array}$$

which is translated automatically by the compiler into a context-parametric version:

$$\begin{array}{lcl} G\langle 0\rangle\ \overline{t} & = & \gamma\ q.\ \tau \\ G\langle n\rangle\ \overline{t} & = & \gamma\ q.\ \forall\ \overline{a}.\ q\ \overline{a} \Rightarrow G\langle 0\rangle\ \overline{a}\ \epsilon\ \rightarrow\ G\langle n-1\rangle\ \overline{(t\ a)}\ q \end{array}$$

A polyaric type can be converted into a polykinded type by mapping arity arguments to kind arguments:

$$\begin{array}{lcl} \mathit{kindOf}(0) & = & * \\ \mathit{kindOf}(n) & = & *\ \rightarrow\ \mathit{KindOf}(n-1) \end{array}$$

A polyaric type can be expanded by first transforming it into a polykinded type and then using the type rules described in Section 6.1.11.

The generic programmer then defines the body of the function:

$$\begin{array}{lcl} g\langle f, c\_f\rangle & :: & G\langle f\rangle\ f \\ g\langle f, c\_f\rangle\ \pi g\ \ \overline{x} & = & B(\ \overline{x},\ \texttt{inserT}\langle f, c\_f\rangle,\ g\langle c\_f\rangle\ \pi g,\ \texttt{extracT}\langle f, c\_f\rangle\ ) \end{array}$$

Several remarks are in order:

- The polytypic function is parametric on two named signature morphisms, one mapping the ADT to an F-view and another mapping the CDT to the implicit F-view generated by the former. For example, in the polytypic application $g\langle M_1, M_2\rangle$, M_1 is a signature morphism mapping the operators in $\mathit{type}(M_1)$ to $\mathit{view}(M_1)$ whereas signature morphism M_2 maps the operators in $\mathit{type}(M_2)$ to $\mathit{view}(M_2)$, such that $\mathit{view}(M_1)$ is F, for some F-view F, and $\mathit{view}(M_2)$ is c_F.

- We deprecate the transformation of n-ary (co)products in F-views to associations of binary ones. In a generative approach, it is possible and reasonable to define polytypic functions and their types as templates that are parametric on arity. Thus, function g takes a vector of function arguments $\pi g = g_1 \ldots g_n$, where $n = \mathit{arity}(\mathit{type}(f))$.

The special symbol π is a language extension. The type of each g_i argument is given by the polyaric type, that is:

$$g_i \;::\; \gamma q.\, \forall\, \overline{a}.\; q\; \overline{a} \;\Rightarrow\; G\langle 0\rangle\; \overline{a}\; q$$

- The body of $g\langle f, c\_f\rangle$ is a function B of the argument variables $\overline{x}$, of the `inserT` and `extracT` for the same signature morphisms, and also of the ordinary Generic Haskell function $g\langle c\_f\rangle$ applied to the vector πg.
- The polytypic application $g\langle F, C\_F\rangle$ triggers the generation of the instance $g\_F\_C\_F$ whose type-signature is given by expanding:

  $$G\langle (\mathit{arity} \circ \mathit{type})F\rangle\; \overline{(\mathit{type}\; F)}\; \Delta(\mathit{type}\; F)$$

  The generated body is:

  $$B(\; \overline{x},\; \texttt{inserT\_}F\texttt{\_}C\texttt{\_}F,\; g\langle T\rangle\; g_1 \ldots g_n,\; \texttt{extracT\_}F\texttt{\_}C\texttt{\_}F\;)$$

  where $T = \mathit{type}(C\_F)$ and $n = \mathit{arity}(T)$. For simplicity, generated instances contain calls to polytypic functions on the concrete type. It would be possible to generate instances for those functions simultaneously, *i.e.*, to generate g_T whose type-signature is given by:

  $$G\langle (\mathit{kindOf} \circ \mathit{arity} \circ \mathit{type})C\_F\rangle\; \overline{(\mathit{type}\; C\_F)}\; \Delta(\mathit{type}\; C\_F)$$

  where G is the context-parametric *polykinded* type of the ordinary Generic Haskell function.
- Polytypic function definitions can be type-checked. The Generic Haskell compiler relies on the Haskell compiler to do the job, *i.e.*, it generates the types and bodies of instances and lets the Haskell compiler check that they match. We also follow this approach.

Figure 9.17 shows the definition of `gsize`, `gmap`, and `geq`.

We conclude the section with generation examples for ordered sets.

The polytypic application `gsize⟨SetF,L1⟩` triggers the generation of the instance:

```
GSIZE⟨0⟩ t = γq. t → Int
GSIZE⟨n⟩ t = γq. ∀a. q a ⇒ GSIZE⟨0⟩ a ϵ → GSIZE⟨n-1⟩ (t a) q

gsize⟨f,c_f⟩ :: GSIZE⟨f⟩ f
gsize⟨f,c_f⟩ πg t = gsize⟨c_f⟩ πg ∘ extracT⟨f,c_f⟩ t
```

```
GMAP⟨0⟩ t1 t2 = γq. t1 → t2
GMAP⟨n⟩ t1 t2 = γq. ∀a1 a2. q a1 a2 ⇒
     GMAP⟨0⟩ a1 a2 ϵ → GMAP⟨n-1⟩ (t1 a1) (t2 a2) q

gmap⟨f,c_f⟩ :: GMAP⟨f⟩ f
gmap⟨f,c_f⟩ πg t = inserT⟨f,c_f⟩ ∘ gmap⟨c_f⟩ πg ∘ extracT⟨f,c_f⟩ t
```

```
GEQ⟨0⟩ t = γq. t → t → Int
GEQ⟨n⟩ t = γq. ∀a. q a ⇒ GEQ⟨0⟩ a ϵ → GEQ⟨n-1⟩ (t a) q

geq⟨f,c_f⟩ :: GEQ⟨f⟩ f
geq⟨f,c_f⟩ πg t1 t2 = geq⟨c_f⟩ πg (extracT⟨f,c_f⟩ t1)
                                  (extracT⟨f,c_f⟩ t2)
```

Figure 9.17: Polytypic `gsize`, `gmap`, `geq` defined in terms of `inserT` and `extracT`.

```
gsize_SetF_L1 :: GSIZE⟨1⟩ Set [Ord]
```

that is:

```
gsize_SetF_L1 :: ∀ a. Ord a ⇒ (a → Int) → Set a → Int
gsize_SetF_L1 g t = gsize⟨List⟩ g ∘ extracT_SetF_L1 t
```

Generated instances contain calls to polytypic functions on the concrete type. If instances for them are generated simultaneously then the result is:

```
gsize_SetF_L1 :: ∀ a. Ord a ⇒ (a → Int) → Set a → Int
gsize_SetF_L1 g t = gsize_List g ∘ extracT_SetF_L1 t
```

The polytypic application `gmap⟨SetF,L1⟩` triggers the generation of the `gmap` instance:

```
gmap_SetF_L1 :: GMAP⟨1⟩ Set [Ord]
```

that is:

```
gmap_SetF_L1 :: ∀ a. (Ord a, Ord b) ⇒ (a → b) → Set a → Set b
gmap_SetF_L1 g t = inserT_SetF_L1 ∘ gmap⟨List⟩ g ∘ extracT_SetF_L1 t
```

Finally, the polytypic application `geq⟨SetF,L1⟩` triggers the generation of the `geq` instance:

```
geq_SetF_L1 :: GEQ⟨1⟩ Set [Ord]
```

that is:

```
geq_SetF_L1 :: ∀ a. Ord a ⇒ (a → a → Bool) → Set a → Set a → Bool
geq_SetF_L1 g t1 t2 =
   geq⟨List⟩ g (extracT_SetF_L1 t1) (extracT_SetF_F1 t2)
```

9.12 Polytypic extension

Polytypic extension can be accommodated in our system in a similar way in which template specialisation is done in C++. Polytypic extension amounts to providing a definition of specific instances of polytypic functions for specific ADTs. Specialised functions can be provided for `inserT`, `extracT`, and already defined polytypic functions with polyaric types.

Suppose, for example, that ordered sets come equipped with the following operators:

```
enumerate   :: ∀ a. Ord a ⇒ Set a → List a
fromList    :: ∀ a. Ord a ⇒ List a → Set a
cardinality :: ∀ a. Ord a ⇒ Set a → Int
```

It makes sense to use these operators in the definitions of `inserT`, `extracT`, and `gsize` for ordered sets. This can be specified by the generic programmer thus:

```
instance extracT⟨type=Set,type=List⟩ = enumerate
instance inserT⟨type=Set,type=List⟩  = fromList
instance gsize⟨type=Set,type=List⟩   = cardinality
```

We reuse the keyword **`instance`** to avoid multiplication of keywords. The first declaration instructs the compiler to use `enumerate` instead of generating an `extracT` instance when the *type* attribute of its first signature-morphism argument is `Set` and the *type* attribute of its second signature-morphism argument is `List`. The declaration only mentions types, not F-views, because `extracT` is not generated. The compiler must make sure that `enumerate`'s type-signature matches that of an instance of `extracT` for sets and lists, that is, it must check that:

```
ExtracT⟨1⟩ Set [Ord]
```

expands to:

$\forall$ `a. Ord a` $\Rightarrow$ `Set a` $\rightarrow$ `List a`

which is the case.

Similarly, the second declaration instructs the compiler to use `fromList` instead of generating an instance of `inserT`. Finally, the third declaration instructs the compiler to use `cardinality` instead of generating an instance of `gsize` when the types in the signature morphisms are `Set` and `List`. Again, the compiler must check that type-signatures match, which is the case in these examples.

9.13 Exporting

The difference between exporting and abstracting over payload is somewhat analogous to the difference between lambda abstractions and let-expressions. Exporting in F-views is based on carrying this difference to the type level.

Recall the `EventQueue` example of Section 7.2. To make it conform to the `Linear` F-view we have to specify that the payload type is fixed. The keyword **`export`** is used for this purpose:

```
EventQueue instance Linear by EventQL where
  export a = Event.EventType
  dsc0 = EventQueue.isEmpty
  con0 = EventQueue.empty
  ...
```

The declaration informs the compiler that type variable `a` in the functorial structure of `Linear` will always be `Event.EventType`.

We can explain now how polyaric types work when the *type* attribute of a signature morphism has 0 arity, a discussion that was postponed in Section 9.10.4. These are the base cases of polyaric types `InserT` and `ExtracT`:

$$\texttt{ExtracT}\langle 0\rangle\ t_1\ t_2 \quad :: \quad t_1 \ \rightarrow\ t_2\ (\mathit{payload}\ t_1)$$
$$\texttt{InserT}\langle 0\rangle\ t_1\ t_2 \quad :: \quad t_2\ (\mathit{payload}\ t_1) \ \rightarrow\ t_1$$

Compile-time function *payload* returns the payload type specified in an **`export`** declaration for its type argument.

Let us show some examples of particular instantiations:

```
extracT⟨EventQL,L1⟩ :: ExtracT⟨0⟩ EventQueue List []
```

In this example:

$$\begin{aligned} \mathit{type}(\texttt{EventQL}) &= \texttt{EventQueue} \\ \mathit{arity}(\texttt{EventQueue}) &= 0 \\ \mathit{payload}(\mathit{type}(\texttt{EventQL})) &= \texttt{Event.EventType} \end{aligned}$$

The result of the expansion is:

```
extracT_EventQL_L1 :: EventQueue → List Event.EventType
```

Also:

```
inserT⟨EventQL,L1⟩ :: InserT⟨0⟩ EventQueue List []
```

after expansion:

```
inserT_EventQL_L1 :: List Event.EventType → EventQueue
```

Let us show what should happen with polytypic functions:

```
gsize⟨EventQL,L1⟩ :: export (GSIZE⟨1⟩ EventQueue [])
                            EventQueue [Event.EventType]
```

The compile-time meta-function *export* expands its first argument, the polyaric type `GSIZE`, and on the resulting type-signature replaces `EventQueue a` by `EventQueue` and it replaces all remaining occurrences of `a` by `Event.EventType`. The resulting type-signature of the `gsize` instance is shown below:

```
gsize_EventQL_L1 ::
  (Event.EventType → Int) → EventQueue → Int
```

In general, let $P\langle n\rangle\ T\ [\,]$ expand to type-signature S. The evaluation of:

$$\mathit{export}\ S\ T\ [E_1, \ldots, E_n]$$

where $\overline{E} = \mathit{payload}(T)$, yields the type-signature:

$$[(T\ \overline{a})/T][\overline{a}/\overline{E}]S$$

Exporting is essential in dealing with composite manifest ADTs. Take for example a `Date` ADT whose implementation type is hidden. It comes with the following operators:

```
mkDate   :: Day → Month → Year → Date
getDay   :: Date → Dat
getMonth :: Date → Month
getYear  :: Date → Year
```

We can compute with dates by defining a signature morphism using the export facility. Recall the `Composite3` F-view from Section 9.10.1:

```
Date instance Composite3 by DateC3 where
  export a = Day
         b = Month
         c = Year
  con0  = mkDate
  sel00 = getDay
  sel01 = getMonth
  sel02 = getYear

type Tuple3 a b c = (a,b,c)
tuple3 x y z = (x,y,z)
tuple30 (x,y,z) = x
tuple31 (x,y,z) = y
tuple32 (x,y,z) = z

Tuple3 instance c_Composite3 by Tuple3F where
  c_con0  = tuple3
  c_sel00 = tuple30
  c_sel01 = tuple31
  c_sel02 = tuple32

mapDate :: (Day → Day) → (Month → Month) → (Year → Year)
           → Date → Date
mapDate = gmap⟨DateC3,Tuple3F⟩
```

In Haskell, n-ary tuples are constructed using bracket notation and there are only predefined selectors for binary tuples, namely, **`fst`** and **`snd`**. The reader will agree that concrete types of the form `Tuple`n for $n > 0$ could be assumed by the generic programmer and their type, constructors, and selectors be generated automatically by the compiler.

Tuple types are also convenient for dealing with *random access* structures such as arrays of fixed length. For example:

```
type Array3 a = Array Int a

Array3 instance Composite3 by ArrayComposite3 where
  con0  = λx y z → array (0,2) [(0,x),(1,y),(2,z)]
  sel00 = (!0)
  sel01 = (!1)
  sel02 = (!2)

mapArray3 :: (a → b) → Array3 a → Array3 b
mapArray3 = gmap⟨Array3,Tuple3F⟩
```

9.14 Forgetful extraction

In type-unifying computations in which there is no need for insertion, it is possible to cheat and extract payload from an ADT into a concrete type with different functorial structure; more precisely, to a concrete type with a polynomial functor of *lower coefficient* but equal number of payload. For instance, it is possible to extract data from an ADT conforming to `Bin1` (defined in Section 9.10.1) to a list. The trick is to make `List` conform to `c_Bin1`, which is possible because list selectors are not used when `inserT` is omitted:

```
List instance C_Bin1 by ListBin1 where
  c_dsc0  = null
  c_con0  = Nil
  c_con1  = λx y z → Cons x (y ++ z)
  c_sel10 = λx → error "attempting to select"
  c_sel11 = λx → error "attempting to select"
  c_sel12 = λx → error "attempting to select"
```

Combining ADTs and CDTs with different functorial interfaces entails an exponential increase in the number of signature-morphism arguments. We come back to this topic in Chapter 10.

9.15 Passing and comparing payload between ADTs

Because `inserT` and `extracT` are parametric on signature morphisms for abstract and concrete types, it is possible to write non-generic functions that 'copy' payload between different ADTs as long as the functorial structure is the same and the intermediate concrete type is the same, even if the two signature morphisms for the concrete type differ. A simple example:

```
set2queue :: Ord a ⇒ Set a → Queue a
set2queue = inserT⟨QueueF,L2⟩ ∘ extracT⟨SetF,L1⟩
```

Notice that *type*(`L1`)=*type*(`L2`) and that queue selectors in `L2` are never used and list selectors in `L1` are never used.

It is not possible however to write payload-copy functions polytypically because insertion, extraction, and polytypic functions with polyaric types are parametric only on *one* signature morphism associated with an ADT.

It is also possible to compare payload between ADTs, *i.e.*, to program extensional equality (Section 9.7):

```
let x = extracT⟨SetF,L1⟩
    y = extracT⟨QueueF,L1⟩
in x == y
```

Notice that the same signature morphism is used for both extraction functions.

Chapter 10
Future Work

> I have no illusions about the prospects of the theory I am proposing: it will suffer the inevitable fate of being proven wrong in many, or most, details ... what I am hoping for is that it will be found to contain a shadowy pattern of truth... [Koe89, p18]

Programming with ADTs is programming with their data contents, not their hidden structure. But for structural polymorphism to be possible we need some notion of structure.

In this thesis we have shown how polytypic programming can be reconciled with data abstraction. More precisely, inspired by the concept of F-Algebra and signature morphism, we have proposed a way for generic programmers to define 'structure' in terms of ADT interfaces and to define polytypic functions that are parametric on such structure.

We list some possibilities for future work and research:

1. The language extensions proposed from Section 9.9 onwards have to be implemented. The reader should bear in mind, however, that the Generic Haskell compiler is not an open-source project and the inclusion of any extension into an official release requires the authorisation (and, for practical purposes, the collaboration) with the design team.

2. Our proposal has focused on first-order ADTs. It would be interesting to investigate ADTs that are higher-order (take parameterised ADTs as parameters) or have higher-order operators. Polyaric types would have to be 'upgraded' to special polykinded types that capture not only the arity but also the order of ADTs. Also, the reduction rules for context-parametric polykinded types must take into account the possibility that higher-order ADTs may be passed constrained ADTs as arguments.

   Another issue to address is that, with higher-order operators, discriminators and partial selectors disappear in favour of higher-order eliminators. For instance, partial selectors **`null`**, **`head`**, and **`tail`** in lists may be replaced by eliminator:

```
caseList :: ∀ a b. (Unit → c) → ((a,List a) → c) → List a → c
caseList f g l = if null l then f unit
                 else g (head l, tail l)
```

Functions on lists can be programmed in terms of eliminators:

```
length :: List a → Int
length = caseList (const 0) ((+1) ∘ length ∘ snd)
```

And so would polytypic functions be programmed using eliminators. F-views and named signature morphisms would have to be adapted accordingly.

3. Objects can be seen as first-class ADTs with higher-order operators. We believe the development of polytypic programming in object-oriented languages like C++ or Java will have to follow an EC[I] model. In this thesis we have not taken into account the possibility of subtyping. Haskell does not support subtyping but this may change in the future. Polytypic programming in object-oriented languages is certainly a path worth investigating and carrying the ideas of polytypic extensional programming to these languages is an interesting starting point.

4. We would like to carry out the ideas presented here to the SyB approach. First, instances of `gfoldl` could be generated automatically following the definitional structure of user-defined F-views. More precisely, in SyB there are instances of `gfoldl` for linear ADTs for which there exists an extraction function `toList` and an insertion function `fromList`, *e.g.*:

   ```
   instance (Data a, Ord a) ⇒ Data (OrdSet a) where
     gfoldl k z s = k (z fromList) (toList s)
     ...
   ```

   If extraction and insertion functions are not offered by the interface they could be obtained automatically using `inserT` and `extracT`:

   ```
   instance (Data a, Ord a) ⇒ Data (OrdSet a) where
     gfoldl k z s = k (z inserT⟨SetF,L1⟩) (extracT⟨SetF,L1⟩ s)
     ...
   ```

   Notice it would be possible to use insertion and extraction functions from/to other concrete types than lists.

   Second, generic functions on ADTs could be generated automatically by the compiler

as polytypically-extended versions of ordinary SyB generic functions. More precisely, the instance:

```
instance Size a ⇒ Size (OrdSet a) where
  gsize = gsize ∘ extracT⟨SetF,L1⟩
```

could be generated automatically from the polytypic application:

```
gsize⟨SetF,L1⟩
```

provided the programmer has given the definition:

```
gsize⟨f,c_f⟩ = gsize⟨type(c_f)⟩ ∘ extract⟨f,c_f⟩
```

which tells the compiler that `gsize⟨type(c_f)⟩` is to be the instance of `gsize` for the concrete type in `c_f`.

5. The separation of insertion and extraction calls for code optimisation techniques in the form of fusion [AS05]. In particular, it is important to investigate whether insertion and extraction in bounded ADTs can be fused.

6. The possibility of using different F-views in polytypic definitions could be investigated. There are some problems to tackle. For instance, it is possible to extract payload from a tree into a list in the order imposed by calls to discriminators and selectors, but there are many ways of constructing a tree from a list, and perhaps they cannot be captured uniformly.

   Furthermore, sometimes it is not possible to define a suitable F-view for an ADT. A simple example would be a `Tree` ADT with the following operators:

   ```
   isEmptyT :: Tree a → Bool
   emptyT   :: Tree a
   insert   :: a → Tree a → Tree a
   node     :: Tree a → a
   left     :: Tree a → Tree a
   right    :: Tree a → Tree a
   ```

   The functor described by constructors is $\mathtt{T}\,a = \mathbf{1} + a \times (\mathtt{T}\,a)$ whereas the one described by discriminators and selectors is $\mathtt{T}\,a = \mathbf{1} + a \times (\mathtt{T}\,a) \times (\mathtt{T}\,a)$. We may argue that it is possible to wrap the `Tree` interface behind a `Heap` one (Section 9.3). Or we may argue that it is possible to define a `joinT` operator that merges two trees into a single tree so that `Tree`'s interface can be mapped to the `Linear` F-view thus:

```
Tree instance Linear by TreeL where
  dsc0  = isEmptyT
  con0  = emptyT
  con1  = insert
  sel10 = node
  sel11 = λt → joinT (left t) (right t)
```

We may also argue that it is possible to rely on polytypic extension:

```
instance gmap⟨type=Tree,type=List⟩ g =
  inserT⟨TreeL,L1⟩ ∘ flattenT ∘ gmap⟨BTree⟩ g
∘ extracT⟨TreeF,BinTreeF⟩
```

where:

```
flattenT :: BTree a → List a
```

is a function that flattens the tree into a list.

However, it seems reasonable to investigate the possibility of defining polytypic F-view *transformers* that would enable programmers to write polytypic functions on ADTs using different F-views by means of these transformers.

7. Polytypic functions are not parametric on base types in Generic Haskell, nor in our scheme (Section 6.1.12).

Part III

Appendix

Appendix A
Details from Chapter 5

A.1 Our specification formalism, set-theoretically

Formal or symbolic systems are typically presented (not necessarily contrived or grasped) in four steps:

1. Define the syntax of terms and the context-dependent information, *e.g.*, sort rules that inductively define the set of well-formed, well-sorted terms.

2. Define the semantics syntactically by means of relations between terms. An ***axiomatic semantics*** defines equations between terms which are axioms in a proof system of syntactic equality. An axiomatic semantics also provides a specification of what is wanted: entities conforming to the specification are ***models***. An ***operational semantics*** defines a reduction relation between terms, usually by directing the equations which become rewrite rules.

3. Investigate the ***denotational semantics***, *i.e.*, provide a translation to another formal system that provides the semantics: syntactic terms are mapped to semantic values and syntactic relations to semantic relations. This mapping is interesting if more than one syntactic term may stand for the same semantic value. The translation is provided by an interpreter (poshly called 'semantic function') [Sto77, Sch94, Ten76], which is expressed in a meta-language with care to avoid circularity [Rey98]. The transformation of interpreters into operational semantics or abstract machines has been studied recently, *e.g.* [ABDM03].

4. Make sure the translation works, *i.e.*, that the syntactic and semantic formalisms are the same. More precisely, prove the ***syntactic consistency*** of the proof system (roughly, it cannot prove all possible equations between terms), the ***semantic consistency*** (roughly, there is some equation that is not satisfied by the models), the ***soundness*** (if two terms are related by an equation their meanings are the same), the ***semantic completeness*** (if two meanings are equal, the syntactic equation

between their symbolising terms can be proven syntactically), and ***syntactic completeness*** (the set of equations given is enough to prove all desirable equations between terms).

Consistency and soundness can be proven from properties of the rewrite system. For instance, if the system of induced rewrite rules is strongly normalising (every term has a normal form) and confluent (the normal form is unique), then it is consistent and we can prove an equation between terms by testing whether they have the same normal form. Also, soundness can be used to prove that an equation is not syntactically provable by finding a counterexample, *i.e.*, model that satisfies the axioms but not the equation.

In Chapter 5 we have described our formal system of algebraic specifications by example. In this appendix we provide its set-theoretic formalisation. In Section A.3 we also describe its categorial rendition, which is more concise because the low-level details provided by the set-theoretic formulation are abstracted away by the concepts wielded. These details are important for us because we are concerned with the structure of the objects involved. Moreover, explicit signatures and laws are essential in our approach to Generic Programming.

The set-theoretic formalisation is presented in painstaking detail for reasons of precision which inexorably entails a bit of verbosity that might make 'the obvious seem impressive' and might look 'overly abstract' [BG82]. Of particular interest is our definition of Σ-Algebra and F-Algebra. We have borrowed some concepts and notation from [BG82, Bai89, Ber01, Hei96, Mar98, Mit96]. Our presentation makes heavy use of inductive definitions in natural-deduction style interspersed with discursive expositions of the intuitions involved.

Our basic formalism can be described informally in a few sentences: all the elements present in *all* specifications are collected into sets: a set of all the defined sorts, a set of all operator symbols, a set of all equations, a set of well-sorted *closed* terms that can be formed from proper and constant operators, a set of well-formed *open* terms that can be formed by allowing free variables to be terms, etc. Operators and terms are classified according to their sort-signature and sort respectively. Equations are axioms in a proof system of syntactic equality. Issues of consistency, soundness, and completeness are discussed thereafter. Algebras provide meanings to specifications and

are also described set-theoretically.

A.1.1 Signatures and sorts

It is convenient to use the set of sorts as an indexing device for other sets. Box A.1 (page 263) provides the relevant machinery. The following definition allows us to index the set of operators conveniently.

DEFINITION A.1.1 Given a set S, we define S^* as the smallest language (set of strings) defined by the following inductive rules:

$$\frac{}{\epsilon \in S^*} \qquad \frac{s \in S}{\text{quote}(s) \in S^*} \qquad \frac{x \in S^* \qquad y \in S^*}{xy \in S^*}$$

where ϵ is the empty string, function quote returns the symbol of its argument as a string, and string concatenation is denoted by string juxtaposition.

We define S^+ as the set $S^* \times S$:

$$\frac{w \in S^* \qquad s \in S}{(w, s) \in S^+}$$

□

We write ws as syntactic sugar for (w, s). In what follows w will always range over strings of S^* and s over symbols in S. Sort-signatures have the form $w \to s$ with the convention that w is treated as a string and the whitespace separating argument sorts is ignored. For example, the `plus` operator in Figure 5.3 has sort-signature $NatNat \to Nat$.

DEFINITION A.1.2 A ***signature*** Σ is a pair (S, Ω) where S is a finite set of ***sorts*** and Ω is an S^+-set of ***operator-symbol*** sets, *i.e.*, $\Omega = \{\Omega_{ws} \mid w \in S^* \wedge s \in S\}$. Operator symbols are sorted according to their sort-signature, *i.e.*, an operator $\sigma : w \to s$ is in the set Ω_{ws}. □

It is assumed that operator symbols have a unique sort-signature (this holds in the previous definition because Ω_{ws} is a set). For simplicity, we also assume that all operator

BOX A.1: S-sets and their operations

A non-empty and enumerable *set of sets* C can be indexed by a non-empty and enumerable set S if there is a total surjection $i : S \to C$. To abbreviate, we say C is an S-***set*** and for every $s{\in}S$ we write C_s for $i(s)$. Thus the following equality: $C \;=\; \{C_s \mid s \in S\}$

We define S-set operations as pointwise liftings of set operations. Let C, D be S-sets:

S-***null set*** :	0	$\stackrel{\text{def}}{=}$	$\{\, \emptyset_s \mid s \in S \,\}$
S-***inclusion*** :	$C \mathbin{\dot\subseteq} D$	iff	$\forall s{\in}S.\ C_s \subseteq D_s$
S-***product***	$C \mathbin{\dot\times} D$	$\stackrel{\text{def}}{=}$	$\{\, C_s \times D_s \mid s \in S \,\}$
S-***union***	$C \mathbin{\dot\cup} D$	$\stackrel{\text{def}}{=}$	$\{\, C_s \cup D_s \mid s \in S \,\}$
S-***arrow***	$C \mathbin{\dot\to} D$	$\stackrel{\text{def}}{=}$	$\{\, C_s \to D_s \mid s \in S \,\}$

Notice that $0 \neq \emptyset$, hence the change of name and symbol.

An S-***function*** $f : C \dot\to D$ is an S-set of functions $\{f_s : C_s \to D_s \mid s{\in}S\}$ which is an *injective*, *surjective*, or *bijective* S-function iff *every* f_s is an injective, surjective, or bijective function respectively. Other notions such as S-composition can be defined in a similar fashion.

An S-***relation*** R is an S-subset of an S-product: $R \mathbin{\dot\subseteq} C \dot\times D$, *i.e.*, for every $s{\in}S$, $R_s \subseteq C_s \times D_s$ is a relation. R is an *equivalence* or an *order* S-relation if *every* R_s is an equivalence or an order relation respectively. R is defined as an S-relation on C when $R \mathbin{\dot\subseteq} C \dot\times C$.

Let R be an S-equivalence relation, the S-***quotient*** is defined as follows: $(C \dot/ R)_s \stackrel{\text{def}}{=} C_s/R_s$ where C_s/R_s denotes the partition of the set C_s into the equivalence classes generated by R_s.

In the following pages we 'drop dots' and overload set operators and S-set operators. Whether an operator is an S-set operator or a set operator will be determined from whether its arguments are S-sets or not.

symbols are different (not overloaded) but this is not a strict requirement, for sort-signatures are not inferred (Section 4.7.2). Because the number of operators in a specification is finite, many sets Ω_{ws} will be empty. In subsequent definitions predicates of the form $\sigma{\in}\Omega_{ws}$ are placed as rule premises.

A.1.2 Terms, sort-assignments and substitution

Given a signature we can define the set of closed terms (with no free variables), but because equations involve free variables it is better to define the set of open terms from the start and define the set of closed terms as a special case. Variables are also sorted. For convenience, instead of assuming the existence of an S-set of variables we make use of the notion of sort-assignment, *i.e.*, a finite set of pairs that is the extension of a finite function from variables to sorts.

DEFINITION A.1.3 A ***sort-assignment*** $\Gamma : X \to S$ is a finite function from a finite set of variables to the finite set of sorts. □

We use sort-assignments both intensionally (*i.e.*, $\Gamma(x)$ is a sort) and extensionally as sets of pairs. Being a set, X has no repetitions.

DEFINITION A.1.4 Given a signature $\Sigma = (S, \Omega)$ and a sort-assignment Γ we define the S-set of ***open terms***:

$$Term(\Sigma,\Gamma) \;=\; \{\; Term(\Sigma,\Gamma)_s \mid s \in S \;\}$$

for every sort s inductively as follows:

$$\frac{\Gamma(x) = s}{x \in Term(\Sigma,\Gamma)_s} \qquad\qquad \frac{\sigma \in \Omega_s}{\sigma \in Term(\Sigma,\Gamma)_s}$$

$$\frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in Term(\Sigma,\Gamma)_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{\sigma\ t_1\ \ldots\ t_n \;\; \in Term(\Sigma,\Gamma)_s}$$

The S-set of closed terms $Term(\Sigma,\emptyset)$ is thus defined as a special case. It follows from the above definition that $Term(\Sigma,\emptyset) \subseteq Term(\Sigma,\Gamma)$. □

Because of the uniform treatment of constant and proper operators, the second inductive rule in Definition A.1.4 could be embedded in the last one by making $n \geq 0$, but it would lose the flavour of a definition by structural induction. Notice that Definition A.1.4 defines the set of terms by induction on the following recursive grammar of terms:

$$t \ ::= \ x \mid \sigma \mid \sigma \ t_1 \ldots t_n$$

where x is a meta-variable standing for any object variable, σ is a meta-variable standing for any constant operator symbol in the second alternative, and for a proper operator of arity $n > 0$ in the last alternative. In subsequent definitions or proofs, the phrase 'by structural induction on terms' will mean by induction on this grammar.

Figure A.1 (page 266) formally describes the signature of a specification example of strings and characters in a given sort-assignment.

The S-set of closed terms can also be defined in terms of a substitution of all free variables in open terms for closed terms. Since terms have no bound variables, the ***set of free variables of a term*** can be defined easily.

DEFINITION A.1.5 Given a term $t \in \mathit{Term}(\Sigma, \Gamma)_s$, the set $\mathrm{FV}(t)$ of free variables of t is defined inductively on the structure of t as follows:

$$\frac{x \in \mathit{Term}(\Sigma, \Gamma)_s}{\mathrm{FV}(x) \stackrel{\text{def}}{=} \{x\}} \qquad \frac{\sigma \in \mathit{Term}(\Sigma, \Gamma)_s}{\mathrm{FV}(\sigma) \stackrel{\text{def}}{=} \emptyset}$$

$$\frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in \mathit{Term}(\Sigma, \Gamma)_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{\mathrm{FV}(\sigma \ t_1 \ldots t_n) \stackrel{\text{def}}{=} \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)}$$

□

We first define the substitution of free variables for *open* terms and then define the substitution of free variables for *closed* terms as a special case.

DEFINITION A.1.6 Given term t of sort s in which variable x may occur free, and another term t' with the same sort as x, the substitution of t' for x in t produces another term of the same sort as t. A ***substitution*** is a function from open terms to

```
signature STRING                          signature CHAR
sorts String                              sorts Char
use CHAR                                  ops
ops                                        ch0   : → Char
 empty : → String                          ...
 pre   : Char String → String              ch255 : → Char
end                                       end
```

$$
\begin{array}{rlll}
\Sigma \stackrel{\text{def}}{=} (S, \Omega) \text{ where} & S & \stackrel{\text{def}}{=} & \{\mathit{Char}, \mathit{String}\} \\
& \Omega & = & \{\Omega_{\mathit{Char}}, \Omega_{\mathit{String}}, \Omega_{\mathit{CharStringString}}\} \\
& \Omega_{\mathit{Char}} & \stackrel{\text{def}}{=} & \{\mathrm{ch0}, \ldots, \mathrm{ch255}\} \\
& \Omega_{\mathit{String}} & \stackrel{\text{def}}{=} & \{\mathrm{empty}\} \\
& \Omega_{\mathit{CharStringString}} & \stackrel{\text{def}}{=} & \{\mathrm{pre}\}
\end{array}
$$

$$
\begin{array}{rcl}
\Gamma & \stackrel{\text{def}}{=} & \{\ (c, \mathit{Char}),\ (s, \mathit{String})\ \} \\
\mathit{Term}(\Sigma, \emptyset)_{\mathit{Char}} & \stackrel{\text{def}}{=} & \{\ \mathrm{ch0}, \ldots, \mathrm{ch255}\ \} \\
\mathit{Term}(\Sigma, \Gamma)_{\mathit{Char}} & = & \{c\} \cup \mathit{Term}(\Sigma, \emptyset)_{\mathit{Char}} \\
\mathit{Term}(\Sigma, \Gamma)_{\mathit{String}} & \stackrel{\text{def}}{=} & \{\ s,\ \mathrm{empty},\ \mathrm{pre}\ c\ s,\ \mathrm{pre}\ c\ \mathrm{empty},\ \mathrm{pre\ ch0\ empty}, \\
& & \quad \mathrm{pre\ ch1\ empty},\ \ldots,\ \mathrm{pre}\ c\ (\mathrm{pre\ ch0\ empty}),\ \ldots\}
\end{array}
$$

Figure A.1: Strings and characters in a given sort-assignment.

open terms which we denote as $[t'/x]t$. The following definition provides its 'type':

$$\frac{t \in \mathit{Term}(\Sigma, \Gamma \cup \{(x, s')\})_s \qquad t' \in \mathit{Term}(\Sigma, \Gamma)_{s'}}{[t'/x]t \in \mathit{Term}(\Sigma, \Gamma)_s}$$

Γ is enlarged to guarantee the existence of variable x with sort s'. Notice that Γ is a function so for the union to yield a valid sort-assignment there cannot be a binding for x in Γ. This device will be used in subsequent definitions.

The 'body' of substitution is defined inductively as follows:

$$\frac{x \in \mathrm{Dom}(\Gamma)}{[t'/x]x \stackrel{\mathrm{def}}{=} t'} \qquad \frac{y \in \mathrm{Dom}(\Gamma) \qquad y \neq x}{[t'/x]y \stackrel{\mathrm{def}}{=} y} \qquad \frac{\sigma \in \mathit{Term}(\Sigma, \Gamma)_s}{[t'/x]\sigma \stackrel{\mathrm{def}}{=} \sigma}$$

$$\frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in \mathit{Term}(\Sigma, \Gamma)_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{[t'/x](\sigma\ t_1 \ldots t_n) \stackrel{\mathrm{def}}{=} \sigma\ ([t'/x]t_1) \ldots ([t'/x]t_n)}$$

The substitution of a variable for a *closed* term is a special case where $t' \in \mathit{Term}(\Sigma, \emptyset)_{s'}$. □

Substituting all remaining free variables in t again for closed terms produces a term in $\mathit{Term}(\Sigma, \emptyset)_s$. Recall that for each s, $\mathit{Term}(\Sigma, \emptyset)_s \subseteq \mathit{Term}(\Sigma, \Gamma)_s$ and therefore all possible substitutions of open-term variables for closed terms will produce closed terms in $\mathit{Term}(\Sigma, \emptyset)_s$. This is consistent with our use of universally quantified variables as standing for the possibly countably infinite closed terms of the same sort, which enables us to write a finite set of equations involving variables (perhaps we should call them equation *schemas*) instead of a countably infinite number of them involving closed terms.

A.1.3 Algebras

The semantics of a signature Σ (no equations) is provided by an algebra $\mathcal{A}$. Free variables for equations are added by a theory. In order to save definitions, let us define the meaning of a signature in the presence of a sort-assignment (which becomes necessary when equations are added later on). The semantics is formalised by defining a 'symbol-mapping' S-function that maps sorts to carriers and operator symbols to

algebraic operators, plus a semantic S-function that provides the meaning (value in a carrier) of a term when given the value-assignment that provides the meanings (values) of its free variables.

DEFINITION A.1.7 A Σ-***Algebra*** $\mathcal{A}$ for a signature $\Sigma \stackrel{\text{def}}{=} (S, \Omega)$ and sort-assignment Γ is a pair $(|\mathcal{A}|, \Omega^{\mathcal{A}})$ where:

1. $|\mathcal{A}|$ is an S-set of ***carrier*** sets, each providing values for each sort in S.

2. $\Omega^{\mathcal{A}}$ is an S^+-set of ***algebraic operators***. It is common mathematical practice to use cartesian products in the ***carrier-signatures*** of algebraic operators, *i.e.*,

$$\frac{\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \ldots s_n s}}{\sigma^{\mathcal{A}} \ : \ |\mathcal{A}|_{s_1} \times \ldots \times |\mathcal{A}|_{s_n} \to |\mathcal{A}|_s}$$

3. There is an overloaded ***symbol-mapping*** S-***function***[1], I, that consistently maps sorts to carriers and operator symbols to algebraic operators. That is, $I : S \to |\mathcal{A}|$ and $I : \Omega \to \Omega^{\mathcal{A}}$ such that:

$$\frac{s \in S}{I(s) \ \stackrel{\text{def}}{=} \ |\mathcal{A}|_s} \qquad \frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad n \geq 0}{I(\sigma) \in \Omega^{\mathcal{A}}_{s_1 \ldots s_n s}}$$

   In other words:

$$\frac{\sigma \ : \ s_1 \ldots s_n \to s \qquad n \geq 0}{I(\sigma) \ : \ I(s_1) \ldots I(s_n) \to I(s)}$$

4. There is a ***value-assignment*** or finite function $\rho : X \to \bigcup_s |\mathcal{A}|_s$ from variables to values of the carriers that *conforms* to Γ, *i.e.*: $\rho(x) \in |\mathcal{A}|_{\Gamma(x)}$.

5. There is a ***semantic*** S-***function*** h where:

$$h_s : \mathit{Term}(\Sigma, \Gamma)_s \to (X \to \bigcup_s |\mathcal{A}|_s) \to |\mathcal{A}|_s$$

   It has become standard practice in denoting semantic functions to use the symbol of the entity providing the semantics and a double-bracket notation for semantic

[1] Or if the reader prefers, two S-functions with overloaded name.

function application in order to underline the fact that arguments are syntactic entities that are interpreted in a particular semantic universe [Mit96, Sto77, Ten76]. In the rest of the chapter we write $\mathcal{A}_s[\![\cdot]\!]$ instead of h_s and $\mathcal{A}_s[\![t]\!]\rho$ instead of $(h_s(t))(\rho)$, breaking our convention of using explicit parentheses for application at the meta-level only for this function (Section 2.6).

$\mathcal{A}_s[\![\cdot]\!]$ is inductively defined on the structure of terms as follows:

$$\frac{\Gamma(x) = s}{\mathcal{A}_s[\![x]\!]\rho \stackrel{\text{def}}{=} \rho(x)}\ \text{SEM1} \qquad \frac{\sigma \in \Omega_s}{\mathcal{A}_s[\![\sigma]\!]\rho \stackrel{\text{def}}{=} I(\sigma)}\ \text{SEM2}$$

$$\frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in \mathit{Term}(\Sigma,\Gamma)_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{\mathcal{A}_s[\![\sigma\ t_1\ \ldots\ t_n]\!]\rho \stackrel{\text{def}}{=} (I(\sigma))(\mathcal{A}_{s_1}[\![t_1]\!]\rho, \ldots, \mathcal{A}_{s_n}[\![t_n]\!]\rho)}\ \text{SEM3}$$

□

The following property follows from the definition. In words, the property states that if a term of sort s is in the set of open terms then its meaning with respect to a value-assignment giving meaning to its free variables is in the carrier giving meaning to s:

$$\frac{t \in \mathit{Term}(\Sigma,\Gamma)_s}{\mathcal{A}_s[\![t]\!]\rho \in |\mathcal{A}|_s}$$

In some presentations the set of variables X is an S-set, and consequently Γ and ρ are S-functions; in particular $\rho_s : X_s \to |\mathcal{A}|_s$. In this setting, the semantic S-function can be seen as an extension of value-assignments for variables to value-assignments for terms (domain extension), and is sometimes denoted as $\rho^{\#}$ (see also Section A.3).

Figure A.2 (page 271) shows two possible algebras giving meaning to the algebraic specification of characters and strings of Figure A.1. The first box defines the S-sets $|\mathcal{A}|$ and $\Omega^{\mathcal{A}}$. It also defines that, as required by Definition A.1.7(3), $I(s) \stackrel{\text{def}}{=} |\mathcal{A}|_s$ for all $s \in \{\mathit{Char}, \mathit{String}\}$. Finally, the value-assignment ρ is empty because there are no variables or equations in Figure A.1. These definitions are the same for the two algebras given in the second and third boxes.

In the second box the carrier for characters is the (extended) ASCII table and the carrier for strings is the set of strings that can be formed with characters from the ASCII table. The reader may want to recall Definition A.1.1 where the star operator, ϵ, and string concatenation are defined.

Notice that the definition of $\Omega^{\mathcal{A}}_{Char}$ is informal. The carrier $|\mathcal{A}|_{Char}$ contains constant values, not algebraic operators. We can consider constants $c \in |\mathcal{A}|_{Char}$ as functions $c : |\mathcal{A}|_{\mathbf{1}} \to |\mathcal{A}|_{Char}$ by positing the existence of an extra sort name $\mathbf{1}$ with carrier $|\mathcal{A}|_{\mathbf{1}}$ which is the carrier of nullary cartesian products (see Sections 3.6 and A.3.1).

There are two algebraic operators with signature $|\mathcal{A}|_{Char} \times |\mathcal{A}|_{String} \to |\mathcal{A}|_{String}$ called 'append' and 'prepend' whose description follows:

$$\frac{c \in \text{ASCII} \qquad s \in \text{ASCII}^*}{\text{append}(c, s) \in \text{ASCII}^* \qquad \text{append}(c, s) \stackrel{\text{def}}{=} s(c\epsilon)}$$

$$\frac{c \in \text{ASCII} \qquad s \in \text{ASCII}^*}{\text{prepend}(c, s) \in \text{ASCII}^* \qquad \text{prepend}(c, s) \stackrel{\text{def}}{=} cs}$$

I maps `pre` to prepend, but it could have mapped it to append.

In the third box, the carrier for characters is a singleton set with element \$. All `ch`i operators are mapped by I to \$. Therefore, if the specification had the equations `ch0 = ch1`, ..., `ch254 = ch255`, this algebra would satisfy them whereas the algebra in the second box would not (Section A.1.6). Again, $\Omega^{\mathcal{A}}_{Char}$ is informally defined to be $|\mathcal{A}|_{Char}$, but bear in mind that formally there is a lifting of constants to functions. The meaning for strings is provided by stacks. The empty string is the empty stack [ ], and `pre` is mapped to the push operator for stacks. Thus, $|\mathcal{A}|_{String}$ contains all the stack values that can be formed by stacking \$s.

A.1.4 Substitution lemma

Each S-indexed semantic function is compositional: the meaning of a term depends on the meaning of its subterms in a value-assignment. As expected, the ***substitution lemma*** holds: the meaning of $[t'/x]t$ is the same as the meaning of the term t in a value-assignment that assigns to x the meaning of t'. To state the substitution lemma we need the notion of value-assignment *enlargement* denoted $\rho\langle x \mapsto v\rangle$, where ρ is

$$
\begin{array}{rcl}
|\mathcal{A}| & \stackrel{\text{def}}{=} & \{\ |\mathcal{A}|_{Char},\ |\mathcal{A}|_{String}\ \} \\
\Omega^{\mathcal{A}} & \stackrel{\text{def}}{=} & \{\ \Omega^{\mathcal{A}}_{Char},\ \Omega^{\mathcal{A}}_{String},\ \Omega^{\mathcal{A}}_{CharStringString}\ \} \\
I(Char) & \stackrel{\text{def}}{=} & |\mathcal{A}|_{Char} \\
I(String) & \stackrel{\text{def}}{=} & |\mathcal{A}|_{String} \\
\rho & \stackrel{\text{def}}{=} & \{\ \}
\end{array}
$$

$$
\begin{array}{rcl}
|\mathcal{A}|_{Char} & \stackrel{\text{def}}{=} & \text{ASCII} \\
|\mathcal{A}|_{String} & \stackrel{\text{def}}{=} & \text{ASCII}^* \\
\Omega^{\mathcal{A}}_{Char} & \stackrel{\text{def}}{=} & |\mathcal{A}|_{Char} \\
\Omega^{\mathcal{A}}_{String} & \stackrel{\text{def}}{=} & \{\ \epsilon\ \} \\
\Omega^{\mathcal{A}}_{CharStringString} & \stackrel{\text{def}}{=} & \{\ \text{append},\ \text{prepend}\ \} \\
I(\texttt{ch0}) & \stackrel{\text{def}}{=} & \text{NUL} \\
I(\texttt{ch1}) & \stackrel{\text{def}}{=} & \text{SOH} \\
\vdots & & \\
I(\texttt{ch255}) & \stackrel{\text{def}}{=} & \\
I(\texttt{empty}) & \stackrel{\text{def}}{=} & \epsilon \\
I(\texttt{pre}) & \stackrel{\text{def}}{=} & \text{prepend}
\end{array}
$$

$$
\begin{array}{rcl}
|\mathcal{A}|_{Char} & \stackrel{\text{def}}{=} & \{\ \$\ \} \\
|\mathcal{A}|_{String} & \stackrel{\text{def}}{=} & \{\ [\ ], [\$], [\$\$], [\$\$\$], \ldots,\ \} \\
\Omega^{\mathcal{A}}_{Char} & \stackrel{\text{def}}{=} & |\mathcal{A}|_{Char} \\
\Omega^{\mathcal{A}}_{String} & \stackrel{\text{def}}{=} & \{\ [\ ]\ \} \\
\Omega^{\mathcal{A}}_{CharStringString} & \stackrel{\text{def}}{=} & \{\ \text{push}\ \} \\
I(\texttt{ch0}) & \stackrel{\text{def}}{=} & \$ \\
\vdots & & \\
I(\texttt{ch255}) & \stackrel{\text{def}}{=} & \$ \\
I(\texttt{empty}) & \stackrel{\text{def}}{=} & [\ ] \\
I(\texttt{pre}) & \stackrel{\text{def}}{=} & \text{push}
\end{array}
$$

Figure A.2: Two possible algebraic semantics for the specification of Figure A.1.

enlarged with the pair (x, v) such that $v \in |\mathcal{A}|_{\Gamma(x)}$. Context enlargement binds stronger than application. An enlargement creates a new value-assignment that overlays any previous binding for x, that is:

$$\rho\langle x \mapsto v\rangle(y) \stackrel{\text{def}}{=} \text{if } y = x \text{ then } v \text{ else } \rho(y)$$

LEMMA A.1.1 (SUBSTITUTION) Let $\Gamma = \Gamma' \cup \{(x, s')\}$

$$\frac{t \in \mathit{Term}(\Sigma, \Gamma)_s \qquad t' \in \mathit{Term}(\Sigma, \Gamma')_{s'} \qquad [t'/x]t \in \mathit{Term}(\Sigma, \Gamma')_s}{\mathcal{A}_s[\![\ [t'/x]t\ ]\!]\rho \ = \ \mathcal{A}_s[\![t]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle}$$

(Because Γ' has lower cardinality than Γ, open terms can be turned into closed terms by repeated substitution.)

PROOF: by straightforward induction on t (assuming the preconditions hold):

- t is x:

$$\begin{array}{ll} & \mathcal{A}_s[\![\ [t'/x]x\ ]\!]\rho \\ = & \quad \{\ \text{Substitution}\ \} \\ & \mathcal{A}_s[\![\ t'\ ]\!]\rho \\ = & \quad \{\ x \notin \mathrm{FV}(t')\ \} \\ & \mathcal{A}_s[\![\ x\ ]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle \end{array}$$

- t is y and $y \neq x$:

$$\begin{array}{ll} & \mathcal{A}_s[\![\ [t'/x]y\ ]\!]\rho \\ = & \quad \{\ \text{Substitution}\ \} \\ & \mathcal{A}_s[\![\ y\ ]\!]\rho \\ = & \quad \{\ x \notin \mathrm{FV}(y)\ \} \\ & \mathcal{A}_s[\![\ y\ ]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle \end{array}$$

- t is σ and $\sigma \in \Omega_s$:

$$\begin{array}{ll} & \mathcal{A}_s[\![\ [t'/x]\sigma\ ]\!]\rho \\ = & \quad \{\ \text{Substitution}\ \} \\ & \mathcal{A}_s[\![\ \sigma\ ]\!]\rho \\ = & \quad \{\ x \notin \mathrm{FV}(\sigma)\ \} \\ & \mathcal{A}_s[\![\ \sigma\ ]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle \end{array}$$

- t is $\sigma\ t_1 \ldots t_n$:

$$
\begin{array}{ll}
 & \mathcal{A}_s[\![\ [t'/x](\sigma\ t_1 \ldots t_n)\ ]\!]\rho \\
= & \quad \{\ \text{Substitution}\ \} \\
 & \mathcal{A}_s[\![\ \sigma\ ([t'/x]t_1)\ \ldots\ ([t'/x]t_n)\ ]\!]\rho \\
= & \quad \{\ \text{Definition of } \mathcal{A}_s[\![\cdot]\!]\rho\ \} \\
 & (I(\sigma))(\mathcal{A}_s[\![[t'/x]t_1]\!]\rho, \ldots, \mathcal{A}_s[\![[t'/x]t_n]\!]\rho \\
= & \quad \{\ \text{Induction hypothesis}\ \} \\
 & (I(\sigma))(\mathcal{A}_s[\![t_1]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle, \ldots, \mathcal{A}_s[\![t_n]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle) \\
= & \quad \{\ \text{Definition of } \mathcal{A}_s[\![\cdot]\!]\rho\ \} \\
 & \mathcal{A}_s[\![\ \sigma\ t_1 \ldots t_n\ ]\!]\rho\langle x \mapsto \mathcal{A}_s[\![t']\!]\rho\rangle
\end{array}
$$

□

A.1.5 Signature morphisms

Before discussing theories and homomorphisms we introduce the notion of signature morphism. A signature morphism is similar to a symbol-mapping S-function (Definition A.1.7), but sorts and operators in one signature are mapped to sorts and operators in another signature instead of to carriers and algebraic operators.

DEFINITION A.1.8 A ***signature morphism*** $M : \Sigma \to \Sigma'$ between two signatures $\Sigma \stackrel{\text{def}}{=} (S, \Omega)$ and $\Sigma' \stackrel{\text{def}}{=} (S', \Omega')$ is an overloaded S-function that consistently maps sorts to sorts and operator symbols to operator symbols:

$$
\frac{s \in S}{M(s) \in S'} \qquad\qquad \frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad n \geq 0}{M(\sigma) \in \Omega'_{M(s_1)\ldots M(s_n)M(s)}}
$$

$$
\text{that is :} \quad \frac{\sigma\ :\ s_1 \ldots s_n \to s \qquad n \geq 0}{M(\sigma)\ :\ M(s_1) \ldots M(s_n) \to M(s)}
$$

□

The set of open terms $Term(\Sigma', \Gamma)$ can be defined from $Term(\Sigma, \Gamma)$ using a signature morphism M by a sort of 'syntactic S-function' similar in spirit to $\mathcal{A}[\![\cdot]\!]$ only that it maps every term in $Term(\Sigma, \Gamma)_s$ to a term in $Term(\Sigma', \Gamma)_{M(s)}$, for every sort $s \in S$. In

order to be precise about the fact that every variable in $\mathrm{Dom}(\Gamma)$ is mapped to itself, and to be able to define also the set of open terms $Term(\Sigma', \Gamma')$ where $\Gamma' \neq \Gamma$, we need another device, a ***variable-assignment*** $\nu : \mathrm{Dom}(\Gamma) \to \mathrm{Dom}(\Gamma')$ such that if $\Gamma(x) = s$ then $\Gamma'(\nu(x)) = M(s)$. We do not use variable-assignments again and only mention them for the sake of precision.

A.1.6 Theories and homomorphisms

Variables have been taken into account in the previous definitions which assumed the existence of a sort-assignment Γ. A basic theory adds *equations* to a signature. Since we consider *conditional equations* in Section 5.5, we generalise in advance and speak about *laws* instead of equations. Equations are between terms of the same sort and therefore the laws form an S-set. Equations $t_1 = t_2$ are formalised by triples of the form $\langle t_1, t_2, \Gamma \rangle$. Mentioning the sort-assignment explicitly is more precise and will be necessary in order to specify the proof system of which the equations are axioms, in particular Leibniz's rule (*i.e.*, substitution of equals for equals) where equality is made compatible with substitution.

Definition A.1.9 A ***basic*** Σ***-Theory*** (or just theory) $\mathcal{T}$ is a pair (Σ, L) where Σ is a signature and L is an S-set of ***laws*** (equations) where:

$$L_s \quad \subseteq \quad \{\ \langle t_1, t_2, \Gamma \rangle \mid t_1 \in Term(\Sigma, \Gamma)_s \wedge t_2 \in Term(\Sigma, \Gamma)_s\ \}$$

In what follows we shall use syntactic sugar and write $\langle t_1 = t_2,\ \Gamma \rangle$ instead of $\langle t_1, t_2, \Gamma \rangle$. □

The following is an example equation:

$$\langle\ \mathrm{pop}\ (\mathrm{push}\ x\ s) = s\ ,\ \{(x, Nat), (s, Stack)\}\ \rangle$$

Equations introduce equivalence relations among terms and constitute axioms in a proof system of syntactic equality which we now define precisely. (We follow standard notation and write $\vdash \phi$ if the formula ϕ can be proven in the proof system from the axioms by application of rules of inference.)

First, equality is an equivalence relation so we can write down the following rules of inference:

$$\frac{}{\vdash\ \langle t = t,\ \Gamma\rangle}\ \text{REF} \qquad \frac{\vdash\ \langle t_1 = t_2,\ \Gamma\rangle}{\vdash\ \langle t_2 = t_1,\ \Gamma\rangle}\ \text{SYM} \qquad \frac{\vdash\ \langle t_1 = t_2,\ \Gamma\rangle \qquad \vdash\ \langle t_2 = t_3,\ \Gamma\rangle}{\vdash\ \langle t_1 = t_3,\ \Gamma\rangle}\ \text{TRS}$$

Second, equality must be compatible with operator symbols:

$$\frac{\vdash\ \langle t_1 = t'_1,\ \Gamma\rangle\ \cdots\ \vdash\ \langle t_n = t'_n,\ \Gamma\rangle \qquad \sigma \in \Omega_{s_1 \ldots s_n s} \qquad n \geq 0}{\vdash\ \langle \sigma\ t_1 \ldots t_n\ =\ \sigma\ t'_1 \ldots t'_n,\ \Gamma\rangle}\ \text{OPS}$$

When $n = 0$, the equation $\langle \sigma = \sigma,\ \Gamma\rangle$ is trivially true.

Third, equality must be compatible with substitution, which provides Leibniz's rule of inference:

$$\frac{\vdash\ \langle t_1 = t_2,\ \Gamma \cup \{(x, s')\}\rangle \qquad \vdash\ \langle t'_1 = t'_2,\ \Gamma\rangle \qquad t'_1, t'_2 \in \mathit{Term}(\Sigma, \Gamma)_{s'}}{\vdash\ \langle [t'_1/x]t_1 = [t'_2/x]t_2,\ \Gamma\rangle}\ \text{SUB}$$

The sort-assignment is enlarged in the premise to guarantee that there is a sort for variable x. Notice the rule is more general than the typical one where the same term is replaced on both sides in the consequent.

Sort-assignments can be extended in premises or conclusions by adding extra variables:

$$\frac{\vdash\ \langle t_1 = t_2,\ \Gamma\rangle}{\vdash\ \langle t_1 = t_2,\ \Gamma \cup \{(x, s)\}\rangle}\ \text{ENL1} \qquad \frac{\vdash\ \langle t_1 = t_2,\ \Gamma \cup \{(x, s)\}\rangle}{\vdash\ \langle t_1 = t_2,\ \Gamma\rangle}\ \text{ENL2}$$

We have not provided a proof system at the semantic level for it would only contain Rule REF: more than one term may stand for the same value but values are all distinct. The following three definitions make precise the meaning of a theory.

DEFINITION A.1.10 An algebra $\mathcal{A}$ *satisfies an equation in the value-assignment* ρ when the semantic S-function assigns the same meaning to both terms. That is, for any t_1

and t_2 in $Term(\Sigma, \Gamma)_s$:

$$\frac{\mathcal{A}_s[\![t_1]\!]\rho = \mathcal{A}_s[\![t_2]\!]\rho}{\mathcal{A} \models_\rho \langle t_1 = t_2,\ \Gamma \rangle}$$

An algebra *satisfies the laws in a value-assignment* when it satisfies *all* the laws in L. The algebra is a ***model*** if it satisfies *all* equations in *all* possible value-assignments. □

Notice that the notion of *validity* (an equation that is satisfied by all algebras) is in our case uninteresting as we are only concerned with equations that are satisfied by specific classes of algebras.

We are now in a position to discuss consistency, soundness, and completeness. At the semantic level there is no notion of proof but of ***semantic implication***: an S-set of equations L semantically implies another equation (not in L) if that equation is satisfied by the models that satisfy L. The S-set of equations L must be closed under semantic implication (semantic completeness) and syntactic provability (syntactic completeness). It must also be semantically consistent (*i.e.*, there is some equation not semantically implied by L) and syntactically consistent (*i.e.*, there is some unprovable equation). A proof system where all equations are provable would prove contradictions. A model where all equations are semantically implied would be a model of contradictions.

Completeness, consistency, and soundness are proven with respect to the given equational axioms of a specification. It is the task of the specification designer to prove these properties formally, certainly a non-trivial task. Soundness is the easiest: the equational proof system is sound when syntactically provable equations are satisfied by all models. A proof of soundness for the equational system can be found in [Mit96, p165]. Briefly, axioms and rules of inference are proven sound (if the antecedent is sound so must be the consequent) so that syntactic proofs can only produce sound equations. Consequently, given a specification, designers only need to prove the soundness of axioms.

Proving completeness and consistency can get intricate in the case of complex specifications. There is however a systematic method for writing a set of equations that will most probably axiomatise the imagined type [Mar98, p189-190]. This method has been used in the production of all specification examples discussed in this work. Briefly, the

method consists in (1) identifying the set of constructor operators that generate all the terms representing values of the type, (2) writing equations between constructors to make equal values congruent, and (3) writing equations that specify the behaviour of the remaining operators on constructor terms.

Completeness is essential to guarantee that we have specified the intended model. However, we want our specification to be satisfied by a *unique* model (up to isomorphism). We are interested in a least-model completeness, namely, that equations satisfied by the least model are syntactically provable. This form of completeness fails for algebras with empty carriers[2] (which we bar by fiat), or laws with disjunctions (which is not our case) [Mit96, p157-179]. The least model is unique (up to isomorphism) and in order to define it we need the notion of *homomorphism* (literally, 'same-form-ism') between algebras.

In the absence of equations, the S-sets $Term(\Sigma, \emptyset)$ and $Term(\Sigma, \Gamma)$ are themselves algebras where the symbol-mapping and semantic S-functions are identities and where the carrier for sort s is $Term(\Sigma, \emptyset)_s$. This construction receives the name of ***free algebra*** from the fact that an algebra is obtained for free only from terms, where syntactically different terms are taken as semantically different values. This algebra constitutes a least model.

In the presence of equations and sort-assignment Γ, terms are classified into equivalence classes. It is common practice to denote by $[t]$ the ***equivalence class*** of term t which contains all those terms equal to t as established by the equations in L or those that can be proven syntactically from them. Let us denote by $\equiv_L$ the equivalence relation on terms generated by the proof system, that is:

$$t \equiv_L t' \quad \Leftrightarrow \quad \vdash \langle t = t',\ \Gamma \rangle$$

Inference rule OPS states that syntactic equality is compatible with the operators of the algebra, making $\equiv_L$ a ***congruence relation*** on terms. In this case, the S-set of closed terms partitioned by equivalence classes constitutes the least model. The following definitions elaborate the technical scaffolding.

DEFINITION A.1.11 A Σ-***homomorphism*** between two algebras $\mathcal{A}$ and $\mathcal{B}$ is an over-

[2]The set of sorts is not empty so if carriers are empty there cannot exist a value-assignment conforming to any sort-assignment.

loaded total map $H : \mathcal{A} \to \mathcal{B}$ consistently mapping values and operators in $\mathcal{A}$ to values and operators in $\mathcal{B}$. More precisely, H stands for an S-function $H : |\mathcal{A}| \to |\mathcal{B}|$ and an S^+-function $H : \Omega^{\mathcal{A}} \to \Omega^{\mathcal{B}}$ such that:

$$\frac{v \in |\mathcal{A}|_s}{H_s(v) \in |\mathcal{B}|_s} \qquad \frac{\sigma^{\mathcal{A}} \ : \ |\mathcal{A}|_{s_1} \times \ldots \times |\mathcal{A}|_{s_n} \to |\mathcal{A}|_s \qquad n \geq 0}{H_{s_1 \ldots s_n s}(\sigma^{\mathcal{A}}) \ : \ H_{s_1}(|\mathcal{A}|_{s_1}) \times \ldots \times H_{s_n}(|\mathcal{A}|_{s_n}) \to H_s(|\mathcal{A}|_s)}$$

Since H's sort index is equal to the sort/sort-index of its argument, let us drop it and rewrite the rule as follows:

$$\frac{v \in |\mathcal{A}|_s}{H(v) \in |\mathcal{B}|_s} \qquad \frac{\sigma^{\mathcal{A}} \ : \ |\mathcal{A}|_{s_1} \times \ldots \times |\mathcal{A}|_{s_n} \to |\mathcal{A}|_s \qquad n \geq 0}{H(\sigma^{\mathcal{A}}) \ : \ H(|\mathcal{A}|_{s_1}) \times \ldots \times H(|\mathcal{A}|_{s_n}) \to H(|\mathcal{A}|_s)}$$

H must preserve the algebraic structure, *i.e.*, the equations satisfied by $\sigma^{\mathcal{A}}$ are satisfied by $H(\sigma^{\mathcal{A}})$ in $\mathcal{B}$. This is captured by the following rule:

$$\frac{v_1 \in |\mathcal{A}|_{s_1} \ \cdots \ v_n \in |\mathcal{A}|_{s_n} \qquad \sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \ldots s_n s} \qquad n \geq 0}{H(\sigma^{\mathcal{A}}(v_1, \ldots, v_n)) \ = \ (H(\sigma^{\mathcal{A}}))(H(v_1), \ldots, H(v_n))} \ \text{HOM}$$

□

There can be Σ-homomorphisms from algebras that satisfy fewer equations to algebras that satisfy more equations but not the opposite; otherwise, different values in $|\mathcal{B}|_s$ would be images of the same value in $|\mathcal{A}|_s$, for some s, and therefore $H : |\mathcal{A}| \to |\mathcal{B}|$ would not be an S-function. We are interested in models that only satisfy equations syntactically provable from axioms (no confusion) and where every value is symbolised by at least one term (no junk).

Algebraic models can be ordered by a 'less junk and less confusion than' pre-order relation which, like any pre-order, has a unique least (or initial) element and there is a unique Σ-homomorphism from it to any other model [LEW96].

With non-empty carriers, the initial model always exists. It is trivial to check (and we do so in a moment) that in the absence of variables and equations, $\mathit{Term}(\Sigma, \emptyset)$ considered as an algebra is *by construction* the initial model, where the S-set of carriers is $\mathit{Term}(\Sigma, \emptyset)$ itself and the S-set of algebraic operators is just Ω. In the presence

of variables and equations, Γ is not empty and we assume the existence of a value-assignment ρ that conforms to Γ. However, recall from Definition A.1.6 that under repeated substitution $Term(\Sigma, \Gamma)$ becomes $Term(\Sigma, \emptyset)$. Since substitution is compatible with syntactic equality, the proof system induces a least congruence S-relation $\equiv_L$ between closed terms that partitions the set of closed terms into equivalence classes yielding the S-quotient $Term(\Sigma, \emptyset)/\equiv_L$, which is the initial model.

A.1.7 Initial models

In order to keep the distinction between syntax and semantics, and for consistency with the use of cartesian products in carrier-signatures, we define the notion of open term algebra as different from the set of open terms.

DEFINITION A.1.12 The ***open term algebra*** is $T_{\Sigma\Gamma} \stackrel{\text{def}}{=} (|T_{\Sigma\Gamma}|, \Omega^{T_{\Sigma\Gamma}})$ where $|T_{\Sigma\Gamma}|$ is an S-set of carriers and $\Omega^{T_{\Sigma\Gamma}}$ is an S^+-set of algebraic operators such that the carrier $|T_{\Sigma\Gamma}|_s$ of *open term values* of sort s is defined inductively as follows:

$$\frac{\Gamma(x) = s}{x \in |T_{\Sigma\Gamma}|_s} \qquad \frac{\sigma \in \Omega_s}{\sigma \in |T_{\Sigma\Gamma}|_s} \qquad \frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in |T_{\Sigma\Gamma}|_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{\sigma(t_1, \ldots, t_n) \in |T_{\Sigma\Gamma}|_s}$$

□

The change of notation from $Term(\Sigma, \Gamma)$ to $T_{\Sigma\Gamma}$ and from Ω to $\Omega^{T_{\Sigma\Gamma}}$ is similar to a type cast where the same entity is considered as having a different meaning. Notationally, Ω is the set of syntactic symbols whereas $\Omega^{T_{\Sigma\Gamma}}$ has the same elements as Ω now considered as algebraic operators:

$$\frac{\sigma : s_1 \ldots s_n \to s \qquad \sigma \in \Omega_{s_1 \ldots s_n s} \qquad n \geq 0}{\sigma : |T_{\Sigma\Gamma}|_{s_1} \times \ldots \times |T_{\Sigma\Gamma}|_{s_n} \to |T_{\Sigma\Gamma}|_s \qquad \sigma \in \Omega^{T_{\Sigma\Gamma}}_{s_1 \ldots s_n s}}$$

Sort-signatures have changed to carrier-signatures with cartesian products. The grammar of open terms is now:

$$t \ ::= \ x \mid \sigma \mid \sigma(t_1, \ldots, t_n)$$

With closed terms, $\Gamma = \emptyset$ and there is no need for value assignments: the ***closed term algebra*** is $T_\Sigma \stackrel{\text{def}}{=} T_{\Sigma\emptyset}$.

Now, it is not difficult to glean from Definition A.1.7 that the symbol-mapping and the semantic S-functions *both* make up a Σ-homomorphism. The overloaded symbol-mapping S-function has types $I : S \to |T_\Sigma|$ and $I : \Omega \to \Omega^{T_\Sigma}$ such that:

$$\frac{s \in S}{I(s) \stackrel{\text{def}}{=} |T_\Sigma|_s} \qquad \frac{\sigma \ : \ s_1 \ldots s_n \to s \qquad n \geq 0}{I(\sigma) \ : \ |T_\Sigma|_{s_1} \times \ldots \times |T_\Sigma|_{s_n} \to |T_\Sigma|_s}$$

The rightmost consequent follows from the fact that:

$$\frac{\sigma \in \Omega_s}{I(\sigma) \stackrel{\text{def}}{=} \sigma}\ \text{INJ}$$

where I is an injection, not an identity. In Definition A.1.7, there is already a rule for I in similar format as that of Definition A.1.11:

$$\frac{\sigma \ : \ s_1 \ldots s_n \to s \qquad n \geq 0}{I(\sigma) \ : \ I(s_1) \times \ldots \times I(s_n) \to I(s)}$$

Let us overload I even more and use it to name the semantic S-function in Definition A.1.7, that is, for all s let us write I for $\mathcal{A}_s[\![\cdot]\!]$. In Definition A.1.7, Rule SEM1 has to be removed because there are no variables. Rule SEM2 holds by Rule INJ. Rule SEM3 can be rewritten as:

$$\frac{v_1 \in |T_\Sigma|_{s_1} \ \cdots \ v_n \in |T_\Sigma|_{s_n} \qquad \sigma \in \Omega^{T_\Sigma}_{s_1 \ldots s_n s} \qquad n \geq 0}{I(\sigma(v_1, \ldots, v_n)) \stackrel{\text{def}}{=} (I(\sigma))(I(v_1), \ldots, I(v_n))}$$

Finally, a definition induces an equality (Chapter 2), making the previous rule an instance of Rule HOM in Definition A.1.11.

In the presence of equations the proof system induces a least congruence S-relation $\equiv_L$ between closed terms that partitions the set of closed terms into equivalence classes yielding the S-quotient $Q \stackrel{\text{def}}{=} T_\Sigma/\equiv_L$ which is defined inductively as follows.

Let $[t]$ denote the equivalence class of term t:

$$\frac{\sigma \in |T_\Sigma|_s}{[\sigma] \in Q_s} \qquad \frac{\sigma \in \Omega^{T_\Sigma}_{s_1 \ldots s_n s} \qquad t_i \in Q_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{[\sigma(t_1, \ldots, t_n)] \in Q_s}$$

Because $\equiv_L$ is a congruence, $[\sigma(t_1 \dots t_n)] = \sigma([t_1], \dots, [t_n])$. Furthermore, because the substitution lemma holds, the meaning of an open term of $T_{\Sigma\Gamma}/\equiv_L$ can be determined from the meaning of its free variables, which is given by the equivalence class of value-assignments $[\rho]$, where $[\rho](x) \stackrel{\text{def}}{=} [\rho(x)]$.

By construction, Q is an initial model: all the values of the carrier are represented by at least one term and Q satisfies all the equations of L closed under syntactic provability. T_Σ constitutes a free algebra that is an initial model of the theory with no equations.

A.2 Our partial formalism, set-theoretically

We now provide a set-theoretical formalisation of partial specifications and their semantics. Only the theory part must change; the definitions of Σ, $\text{Term}(\Sigma, \Gamma)$, T_Σ and $T_{\Sigma\Gamma}$ remain. The difference now is that undefined terms are junk and do not symbolise any algebraic value.

DEFINITION A.2.1 A ***Partial* Σ*-Algebra*** $\mathcal{A}$ is a pair $(|\mathcal{A}|, \Omega^{\mathcal{A}})$ where for $\sigma \in \Omega_{ws}$:

- $I(\sigma) \stackrel{\text{def}}{=} \sigma^{\mathcal{A}}$ may be *undefined* when $w = \epsilon$, that is, $\sigma^{\mathcal{A}} \notin |\mathcal{A}|_s$.
- $I(\sigma) \stackrel{\text{def}}{=} \sigma^{\mathcal{A}}$ may be *partial* when $w = s_1 \dots s_n$, that is, for $n > 0$ and $i \in \{1 \dots n\}$, it may be the case that $v_i \in |\mathcal{A}|_{s_i}$ and $\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \dots s_n s}$ but $\sigma^{\mathcal{A}}(v_1, \dots, v_n) \notin |\mathcal{A}|_s$.

□

A Σ-homomorphism between partial algebras is a *total* S-function and can only map defined values in carriers to defined values.

DEFINITION A.2.2 A Σ-homomorphism $H : \mathcal{A} \to \mathcal{B}$ between two partial algebras $\mathcal{A}$ and $\mathcal{B}$ is a total S-function that satisfies the following:

$$\frac{\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \dots s_n s} \qquad \sigma^{\mathcal{A}}(v_1, \dots, v_n) \in |\mathcal{A}|_s \qquad n \geq 0}{(H(\sigma^{\mathcal{A}}))(H(v_1), \dots, H(v_n)) \in |\mathcal{B}|_s}$$

$$\frac{\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \dots s_n s} \qquad \sigma^{\mathcal{A}}(v_1, \dots, v_n) \in |\mathcal{A}|_s \qquad n \geq 0}{H(\sigma^{\mathcal{A}}(v_1, \dots, v_n)) \;=\; (H(\sigma^{\mathcal{A}}))(H(v_1), \dots, H(v_n))}$$

□

The fact that the symbol-mapping S-function I may be undefined for certain constants means the semantic S-function may also be undefined for some terms. It can also be undefined for terms involving proper partial operators. Consequently, I and $\mathcal{A}[\![\cdot]\!]$ no longer make up a Σ-homomorphism, which must be a total S-function (Section A.1.7). T_Σ is no longer an initial algebra, for it contains junk.

By definition, given a Σ-homomorphism $H : \mathcal{A} \to \mathcal{B}$ between two partial algebras, $|\mathcal{B}|$ may have more defined terms than $|\mathcal{A}|$ but not the opposite; otherwise, different values in $|\mathcal{B}|_s$ would be images of the same value in $|\mathcal{A}|_s$, for some s, and therefore $H : |\mathcal{A}| \to |\mathcal{B}|$ would not be an S-function.

There is a 'less partial than' relation between partial algebras. The ***initial partial algebra*** has a unique Σ-homomorphism from it to any other partial algebra. For a closed term algebra to be an initial model it must have ***least junk*** (contain only those terms that are defined in *all* models), ***least confusion*** (two defined terms have the same value only if they do in *all* models), and there is at least a defined term symbolising every intended value.

We first revise our definition of theory to account for conditional equations.

DEFINITION A.2.3 A ***partial theory*** is a pair (Σ, L) where Σ is a signature and L is a set of laws involving terms of $Term(\Sigma, \Gamma)$ in a context Γ. More precisely, equations in L_s have the form $\langle P \Rightarrow E,\ \Gamma\rangle$ where P is a possibly empty conjunction of preconditions and E is an equation $t = t'$ or a partial equation $t \simeq t'$. Preconditions consist of equations. A conditional equation $P \Rightarrow t \simeq t'$ is syntactic sugar for:

$$P \wedge DEF(t) \wedge DEF(t') \Rightarrow t = t'$$

and predicate $DEF(t)$ is syntactic sugar for $t = t$. □

The proof system does not have to be enlarged with introduction and elimination rules for conditional equations. Conditional equations can be considered object-level representations of meta-level inference rules. Every conditional equation $\langle E_1 \wedge \ldots \wedge E_n \Rightarrow E,\ \Gamma\rangle$ where $n > 0$ introduces an inference rule:

$$\frac{\vdash\ \langle E_1, \Gamma\rangle\ \cdots\ \vdash\ \langle E_n, \Gamma\rangle}{\vdash\ \langle E, \Gamma\rangle}$$

An equation between two terms must be satisfied in all models (their interpretation must produce the same value). For partial equations, if both terms have values (are defined in the model) then their interpretation must be the same. If one or both terms are undefined the equation is vacuously satisifed. Consequently, we only have to add definedness premises in the definition of model.

DEFINITION A.2.4 An algebra $\mathcal{A}$ satisfies an equation $E \stackrel{\text{def}}{=} \langle t_1 = t_2,\ \Gamma\rangle$ or a partial equation $E \stackrel{\text{def}}{=} \langle t_1 \simeq t_2,\ \Gamma\rangle$ in the value-assignment ρ when:

$$\frac{\mathcal{A}_s[\![t_1]\!]\rho \in |\mathcal{A}|_s \qquad \mathcal{A}_s[\![t_2]\!]\rho \in |\mathcal{A}|_s \qquad \mathcal{A}_s[\![t_1]\!]\rho = \mathcal{A}_s[\![t_2]\!]\rho}{\mathcal{A} \models_\rho E}$$

An algebra $\mathcal{A}$ satisfies a conditional equation $\langle P \Rightarrow E,\ \Gamma\rangle$ in the value assignment ρ when:

$$\frac{P \stackrel{\text{def}}{=} E_1 \wedge \ldots \wedge E_n \qquad \mathcal{A} \models_\rho E_1\ \cdots\ \mathcal{A} \models_\rho E_n \qquad n \geq 0}{\mathcal{A} \models_\rho E}$$

An algebra *satisfies the laws in a value-assignment* when it satisfies *all* the laws in L. The algebra is a ***model*** if it satisfies *all* equations in *all* possible value-assignments. □

A partial theory has an initial model that is the least defined: it has least junk (a term is defined if and only if it is defined in all models), it has least confusion (two terms have the same meaning only if they do in all models), and every value is symbolised by at least one term. Such a model is defined in terms of the subset of T_Σ that contains all defined terms, which we denote by Dom_L, and a congruence relation between those terms induced by the proof system of conditional equations, which we denote by $\equiv_L$.

In order to characterise the initial algebra precisely we need to define the notion of immediate subterm of a term.

DEFINITION A.2.5 The set of ***immediate subterms*** $Sub(t)$ of a term t is inductively defined as follows:

$$\frac{\sigma \in \Omega}{Sub(\sigma) \stackrel{\text{def}}{=} \emptyset} \qquad \frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i \in Term(\Sigma,\Gamma)_{s_i} \qquad i \in \{1 \ldots n\} \qquad n > 0}{Sub(\sigma\ t_1 \ldots t_n) \stackrel{\text{def}}{=} \{t_1, \ldots, t_n\}}$$

With this set in hand we can define an immediate subterm relation $\preccurlyeq$ between terms: $t' \preccurlyeq t \Leftrightarrow t' \in Sub(t)$. □

DEFINITION A.2.6 Given a theory (Σ, L), let the set $Dom_L \subseteq T_\Sigma$, and the congruence $\equiv_L \subseteq Dom_L \times Dom_L$ be the smallest set and congruence that satisfy the following properties:

1. For every equation $\langle t_1 = t_2,\ \Gamma\rangle$, every immediate closed term t' of sort s in $Sub(t_1)$ and $Sub(t_2)$ is defined, that is, $\mathcal{A}_s[\![t']\!]\rho \in |Dom_L|_s$.

2. Dom_L is closed with respect to the immediate subterm relation:

$$\frac{\mathcal{A}_s[\![t]\!]\rho \in |Dom_L|_s \qquad t' \in Term(\Sigma,\Gamma)_{s'} \qquad t' \preccurlyeq t}{\mathcal{A}_{s'}[\![t']\!]\rho \in |Dom_L|_{s'}}$$

3. The congruence is reflexive, symmetric and transitive:

$$\frac{t \in |Dom_L|_s}{t \equiv_L t} \qquad \frac{t_1 \equiv_L t_2}{t_2 \equiv_L t_1} \qquad \frac{t_1 \equiv_L t_2 \qquad t_2 \equiv_L t_3}{t_1 \equiv_L t_3}$$

   Notice that reflexivity is conditional on definedness.

4. The congruence is compatible with substitution of variables for values: for every conditional equation $\langle E_1 \dots E_n \Rightarrow E,\ \Gamma\rangle$ in L such that $E_i \stackrel{\text{def}}{=} t_i = t_i'$ and $E \stackrel{\text{def}}{=} t = t'$:

$$\frac{t_i, t_i' \in Term(\Sigma,\Gamma)_{s_i} \qquad \mathcal{A}_{s_i}[\![t_i]\!]\rho \equiv_L \mathcal{A}_{s_i}[\![t_i']\!]\rho \qquad i \in \{1 \dots n\} \qquad n \geq 0}{\mathcal{A}_s[\![t]\!]\rho \in |Dom_L|_s \qquad \mathcal{A}_s[\![t']\!]\rho \in |Dom_L|_s \qquad \mathcal{A}_s[\![t]\!]\rho \equiv_L \mathcal{A}_s[\![t']\!]\rho}$$

   In particular, if $n = 0$ the consequent becomes an axiom.

5. The congruence is compatible with operators: either applications of operators to defined and congruent term arguments are defined and congruent, or are undefined. More precisely, either:

$$\frac{\sigma \in \Omega_{s_1 \dots s_n s} \qquad t_i, t_i' \in |Dom_L|_{s_i} \qquad t_i \equiv_L t_i' \qquad i \in \{1 \dots n\} \qquad n > 0}{\begin{array}{c}(I(\sigma))(t_1, \dots t_n) \in |Dom_L|_s \\ (I(\sigma))(t_1', \dots t_n') \in |Dom_L|_s \\ (I(\sigma))(t_1, \dots t_n) \equiv_L (I(\sigma))(t_1', \dots t_n')\end{array}}$$

or:

$$\frac{\sigma \in \Omega_{s_1 \ldots s_n s} \qquad t_i, t_i' \in |Dom_L|_{s_i} \qquad t_i \equiv_L t_i' \qquad i \in \{1 \ldots n\} \qquad n > 0}{(I(\sigma))(t_1, \ldots t_n) \notin |Dom_L|_s \qquad (I(\sigma))(t_1', \ldots t_n') \notin |Dom_L|_s}$$

□

THEOREM A.2.1 The initial algebra of a theory (Σ, L) with partial conditional equations is the quotient $Dom_L/\equiv_L$. The set of carriers is $|Dom_L|/\equiv_L$. Let us denote by $[t]$ the equivalence class of the term t of sort s in $|Dom_L|_s$. For every operator symbol $\sigma \in \Omega_{s_1 \ldots s_n s}$ where $n \geq 0$:

$$(I(\sigma))([t_1], \ldots, [t_n]) \stackrel{\text{def}}{=} \begin{cases} [(I(\sigma))(t_1, \ldots, t_n)] & \text{, if } (I(\sigma))(t_1, \ldots, t_n) \in |Dom_L|_s \\ \text{undefined} & \text{, otherwise} \end{cases}$$

□

A.3 Our formalism, categorially

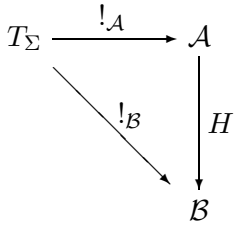
In the set-theoretical formulation, the same concepts appear again and again, namely, the existence of objects and structure-preserving arrows between them. The categorial study of specifications can focus on the syntax or the semantics as long as the categorial axioms are satisfied by the particular objects and arrows at hand. Furthermore, in some categories there are distinguished initial objects (*e.g.*, least model, least defined model).

For example, we have the category of sorts, where objects are sorts, arrows are operator symbols, and signature morphisms are functors (recall Definition A.1.8). We also have the category of algebras, where objects are carriers, arrows are algebraic operators, and every H_s is a functor (recall Definition A.1.11).

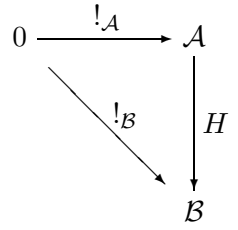
Σ-Algebras and partial Σ-Algebras with their respective Σ-homomorphisms both constitute categories. In both cases, the identity arrow is the identity Σ-homomorphism and composition is S-composition of Σ-homomorphisms. The reader can check that the categorial axioms are satisfied. In both cases, the object of interest is the initial object of the category.

Recall from Section A.1 that the unique-arrow property of the initial object is satisfied.

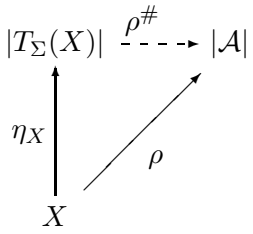
In the category of algebras this property can be depicted as a diagram:



A similar diagram can be written for $T_{\Sigma\Gamma}$. Let us denote the initial object as 0. The following diagram depicts the relationship in any category of algebras:



Recall from page 269 that the semantic S-function can be seen as an extension of value-assignments considered as S-functions $\rho : X \to |\mathcal{A}|$. Let us write $T_\Sigma(X)$ instead of $T_{\Sigma\Gamma}$ to underline the role of the, now, S-set of variables X. $T_\Sigma(X)$ is the initial model and there is a unique Σ-homomorphism from it to any other algebra. At the carrier level (category **Set** where objects are carrier sets and arrows are total functions on sets), it is witnessed by the existence of $\rho^\#$ which makes the following diagram commute [BG82]:



That is, $\rho^\# \circ \eta_X = \rho$. Function η_X maps variables to themselves as values (again think of a cast). The notation gives away the fact that η_X is a natural transformation and that $T_\Sigma(\cdot)$ is a functor; more precisely, $\eta : Id \dot{\to} T_\Sigma$, where Id is the identity functor.

A.3.1 F-Algebras

There is a categorial construction which provides an elegant definition of algebras in terms of functors. Let us first illustrate the idea by example. Take specifications NAT and STRING from Figures 5.1 and 5.2. Let us assume the existence of an implicitly defined extra sort **1** with only a constant operator *unit*. Recalling Section 3.6, we can view all other constant operators $\sigma :\rightarrow s$ as proper operators $\sigma : unit \rightarrow s$, such that where before σ was a valid term in $Term(\Sigma, \emptyset)$ now it is written $\sigma\ unit$, and consequently $\sigma(unit) \in T_\Sigma$.

Suppose $\mathcal{A}$ is a model of NAT. The set $|\mathcal{A}|_{\mathbf{1}}$ is now the carrier of nullary products. Definitions involving sort- and carrier-signatures have to be adapted to account for the existence of this carrier:

$$\frac{w \ = \ \epsilon}{|\mathcal{A}|_w \ \stackrel{\text{def}}{=} \ |\mathcal{A}|_{\mathbf{1}}} \qquad \frac{w \ = \ s_1 \ldots s_n \qquad n > 0}{|\mathcal{A}|_w \ \stackrel{\text{def}}{=} \ |\mathcal{A}|_{s_1} \times \ldots \times |\mathcal{A}|_{s_n}}$$

$$\frac{\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_s}{\sigma^{\mathcal{A}} \ : \ |\mathcal{A}|_{\mathbf{1}} \rightarrow |\mathcal{A}|_s} \qquad \frac{\sigma^{\mathcal{A}} \in \Omega^{\mathcal{A}}_{s_1 \ldots s_n s} \qquad n > 0}{\sigma^{\mathcal{A}} \ : \ |\mathcal{A}|_{s_1} \times \ldots \times |\mathcal{A}|_{s_n} \rightarrow |\mathcal{A}|_s}$$

NAT's algebraic operators are described by a diagram with objects and arrows in the category **Set** . More precisely, let

$$I(\text{zero}) \ \stackrel{\text{def}}{=} \ \text{zero}^{\mathcal{A}}$$
$$I(\text{succ}) \ \stackrel{\text{def}}{=} \ \text{succ}^{\mathcal{A}}$$

The diagram in Figure A.3(left) characterises operator names, their sources, and their targets. The diagram has a limit as shown in Figure A.3(right), *i.e.*, a coproduct with arrows inl and inr. By universality there is a unique mediating arrow denoted by α_{Nat}.

In **Set**, coproducts are disjoint sums whose internal structure is described in terms of labelled pairs. Let us use operator symbols as labels:

$$|\mathcal{A}|_{\mathbf{1}} + |\mathcal{A}|_{Nat} \ \stackrel{\text{def}}{=} \ \{ \ (\text{zero}, v) \mid v \in |\mathcal{A}|_{\mathbf{1}} \ \} \ \cup \ \{ \ (\text{succ}, v) \mid v \in |\mathcal{A}|_{Nat} \ \}$$

By universality α_{Nat} is unique: $\alpha_{Nat} \ = \ (\text{zero}^{\mathcal{A}} \ \triangledown \ \text{succ}^{\mathcal{A}})$

Recall that coproducts are also functors. Let us denote the coproduct functor by F.

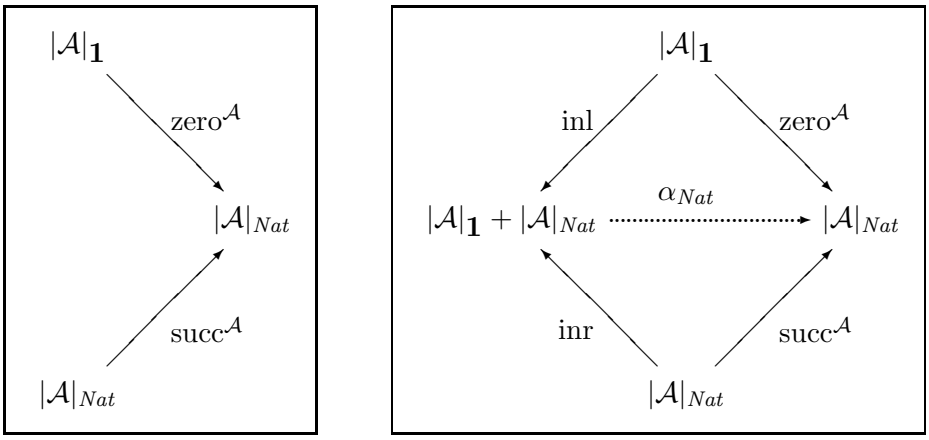


Figure A.3: Diagram describing operator names, sources, and targets. The coproduct is the limit.

At the object level it is defined as follows:

$$F(X) \stackrel{\text{def}}{=} \sum_{\sigma \in \Omega_{ws}} X_w$$

For example, in the algebra of natural numbers we have operators:

$$\begin{aligned} \text{zero}^{\mathcal{A}} &: |\mathcal{A}|_{\mathbf{1}} \to |\mathcal{A}|_{Nat} \\ \text{succ}^{\mathcal{A}} &: |\mathcal{A}|_{Nat} \to |\mathcal{A}|_{Nat} \end{aligned}$$

therefore:

$$F(X) = X_{\mathbf{1}} + X_{Nat}$$

The type of α_{Nat} is:

$$\alpha_{Nat} \ : \ |\mathcal{A}|_{\mathbf{1}} + |\mathcal{A}|_{Nat} \to |\mathcal{A}|_{Nat}$$

which can be described more succinctly using F:

$$\alpha_{Nat} \ : \ F(|\mathcal{A}|_{Nat}) \to |\mathcal{A}|_{Nat}$$

At the arrow level F is defined as follows:

$$\frac{h \ : \ |\mathcal{A}| \to |\mathcal{B}|}{F(h) \ : \ F(|\mathcal{A}|) \to F(|\mathcal{B}|)} \qquad \frac{h_s \ : \ |\mathcal{A}|_s \to |\mathcal{B}|_s \qquad s \in \{\mathbf{1}, Nat\}}{F_s(h_s) \ : \ F_s(|\mathcal{A}|_s) \to F_s(|\mathcal{B}|_s)}$$

$$\frac{v \in |\mathcal{A}|_{\mathbf{1}}}{(F_{\mathbf{1}}(h_{\mathbf{1}}))(\text{zero}, v) \stackrel{\text{def}}{=} (\text{zero}, h_{\mathbf{1}}(v))} \qquad \frac{v \in |\mathcal{A}|_{Nat}}{(F_{Nat}(h_{Nat}))(\text{succ}, v) \stackrel{\text{def}}{=} (\text{succ}, h_{Nat}(v))}$$

If we carry out the same construction for `STRING`, assuming $\mathcal{A}$ is now its model, we obtain two functions, one for each sort:

$$\begin{array}{rcl} \alpha_{Char} & : & |\mathcal{A}|_{\mathbf{1}} + \ldots + |\mathcal{A}|_{\mathbf{1}} \to |\mathcal{A}|_{Char} \\ \alpha_{Char} & = & (\text{ch0}^{\mathcal{A}} \triangledown \ldots \triangledown \text{ch255}^{\mathcal{A}}) \\ \alpha_{String} & : & |\mathcal{A}|_{\mathbf{1}} + |\mathcal{A}|_{CharString} \to |\mathcal{A}|_{String} \\ \alpha_{String} & = & (\text{empty}^{\mathcal{A}} \triangledown \text{pre}^{\mathcal{A}}) \end{array}$$

Clearly, $\alpha : F(|\mathcal{A}|) \to |\mathcal{A}|$ is an S-function. In general a Σ-Algebra $\mathcal{A}$ can be characterised in terms of the S-set carrier $|\mathcal{A}|$ and the S-function α. This construction receives the name of F-Algebra. F is the ***S-functor*** $\{F_s : \mathbf{Set} \to \mathbf{Set} \mid s \in S\}$ where objects in **Set** are the carriers $|\mathcal{A}|_s$.

We provide a definition of F-Algebras for algebras with one carrier. For many-sorted algebras, the definition below can be adapted by replacing **Set** with **S-Set** (the category of S-Sets). In that setting, $|\mathcal{A}|$ is an S-Set, α is an S-function, and F is an S-functor.

Definition A.3.1 An ***F-Algebra*** in **Set** is a pair $(\mathcal{A}, \alpha)$ where $|\mathcal{A}|$ is an object in **Set**, $\alpha : F(|\mathcal{A}|) \to |\mathcal{A}|$ is an arrow in **Set**, and $F : \mathbf{Set} \to \mathbf{Set}$ is a functor. An ***F-homomorphism*** $h : (|\mathcal{A}|, \alpha) \to (|\mathcal{B}|, \beta)$ is an arrow that makes the following diagram commute:

$$\begin{array}{ccc} F(|\mathcal{A}|) & \xrightarrow{F(h)} & F(|\mathcal{B}|) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ |\mathcal{A}| & \xrightarrow{\quad h \quad} & |\mathcal{B}| \end{array}$$

□

The reader may have realised that α and β are particular instances of the natural transformation $\eta : F \dot{\rightarrow} Id$ (Section 3.12).

The simplicity and conciseness of this definition is due to what it does not say. It only involves carriers, and the mapping from operators in $\mathcal{A}$ to operators in $\mathcal{B}$ that is part of a Σ-homomorphism is not mentioned. Such mapping maps operators to operators respecting carrier-signatures and constitutes another functor. Finally, mediating arrows α are not informative about the name of operators nor their carrier-signatures if the diagram of which the coproduct is a limit is not shown. Providing the diagram amounts to providing the carrier-signatures of algebraic operators explicitly.

One of the goodies of category theory is that, as expected, the definition of F-Algebra can be generalised to any category $\mathbf{C}$:

Definition A.3.2 Let $\mathbf{C}$ be a category and $F : \mathbf{C} \rightarrow \mathbf{C}$ a functor. An F-Algebra is a pair (X, α) where X is an object and $\alpha : F(X) \rightarrow X$ an arrow. An F-homomorphism $h : (X, \alpha) \rightarrow (Y, \beta)$ is an arrow that makes the following diagram commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(h)} & F(Y) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow{h} & Y \end{array}$$

□

It is important to notice that different algebras may be described by the same F, *e.g.*, $\alpha : F(|\mathcal{A}|) \rightarrow |\mathcal{A}|$ and $\beta : F(|\mathcal{B}|) \rightarrow |\mathcal{B}|$ where $\alpha \neq \beta$. In particular, $\mathcal{A}$ could be the algebra giving meaning to an algebraic specification *without* equations and $\mathcal{B}$ the algebra giving meaning to an algebraic specification *with* equations. Informally speaking, the same F may capture the signature of theories with wildly dissimilar equations. We come back to this point when introducing F-views (Chapter 9).

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