Combined normal forms in sequent calculus

J. Espírito Santo, M.J. Frade, L. Pinto

Universidade do Minho

Braga, Portugal

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Two views of the work

1. Study of the relationship between natural deduction and sequent calculus.
2. Study of extensions of λ-calculus and of ways to extend Curry-Howard to sequent calculus.

Ground ideas

1. Folklore view regards β-normal deductions as counterparts to cut-free derivations.

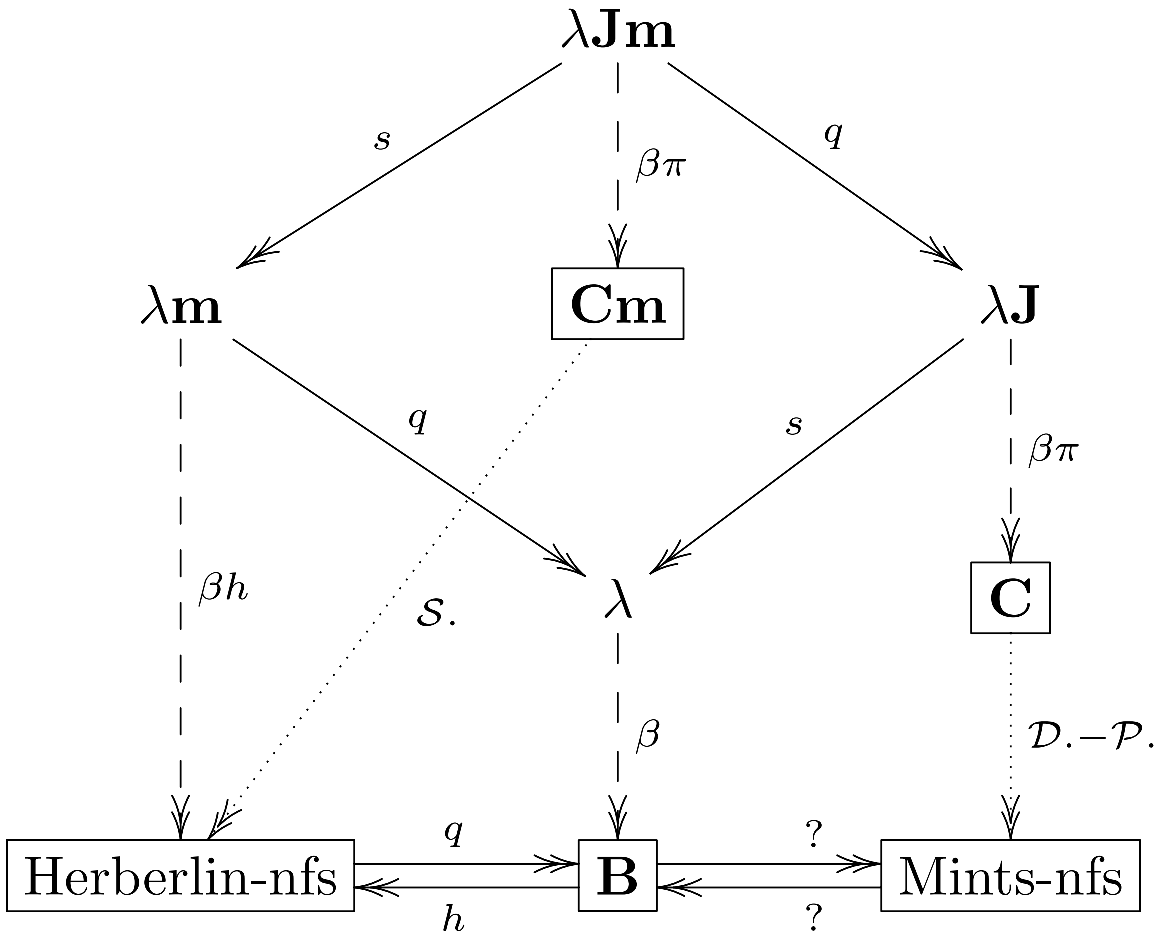
2. Various works refine this view isolating classes of cut-free derivations in 1-1 correspondence to β-normal deductions.

3. Permutation of logical inferences account for redundancy of sequent calculus as compared to natural deduction.

Works in the area

	cuts	**permutations** (logical inferences)	**term calculus**
Kleene, 1952	no	yes	no
Zucker, 1974	yes	yes	no
Pottinger, 1977	yes	yes	yes
Herbelin, 1994	yes	no	yes
Mints, 1994	no	yes	no
Dyckhoff&Pinto, 1997	no	yes	yes
Schwichtenberg, 1999	no	yes	yes
Espírito Santo&Pinto, 2003	yes	yes	yes

THE GENERALISED MULTIARY λ-CALCULUS $\lambda\mathbf{Jm}$ AND OTHER WORKS



$\mathbf{Cm}$ = cut-free multiary sequent terms

$\mathbf{C}$ = cut-free (unary) sequent terms

$\mathbf{B}$ = β-normal λ-terms

λ**Jm**: THE GENERALISED MULTIARY λ-CALCULUS

Expressions

$$t, u, v \quad ::= \quad x \mid \lambda x.t \mid \underbrace{t(u, l, (x)v)}_{gm\text{-}application}$$

$$l \quad ::= \quad [] \mid u :: l$$

Sequents $\quad \Gamma \vdash t : A \qquad \Gamma; B \vdash l : C$

Typing rules

$$\frac{}{x : A, \Gamma \vdash x : A}\ Axiom \qquad \frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\ Right$$

$$\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A \quad \Gamma; B \vdash l : C \quad x : C, \Gamma \vdash v : D}{\Gamma \vdash t(u, l, (x)v) : D}\ gm - Elim$$

$$\frac{}{\Gamma; C \vdash [] : C}\ Ax \qquad \frac{\Gamma \vdash u : A \quad \Gamma; B \vdash l : C}{\Gamma; A \supset B \vdash u :: l : C}\ Lft$$

Remark: In $\Gamma; B \vdash l : C$, i) B is "main and linear" and
ii) $B = B_1 \supset ... \supset B_k \supset C$, for some $k \geq 0$.

Reduction rules

$$
\begin{array}{rcl}
(\lambda x.t)(u, [], (y)v) & \rightarrow_{\beta_1} & \mathbf{s}(\mathbf{s}(u, x, t), y, v) \\
(\lambda x.t)(u, v :: l, (y)v) & \rightarrow_{\beta_2} & \mathbf{s}(u, x, t)(v, l, (y)v) \\
t(u, l, (x)v)(u', l', (y)v') & \rightarrow_{\pi} & t(u, l, (x)v(u', l', (y)v'))
\end{array}
$$

- s stands for *gm-substitution*
- $\beta = \beta_1 \cup \beta_2$

$\beta\pi$-normal forms:

$$
\begin{array}{rcl}
t, u, v & ::= & x \mid \lambda x.t \mid x(u, l, (y)v) \\
l & ::= & u :: l \mid []
\end{array}
$$

Result: $\rightarrow_{\beta\pi}$ is confluent and SN for typable terms.

Some definitions

(1) Particular cases of gm-application $t(u, l, (x)v)$:

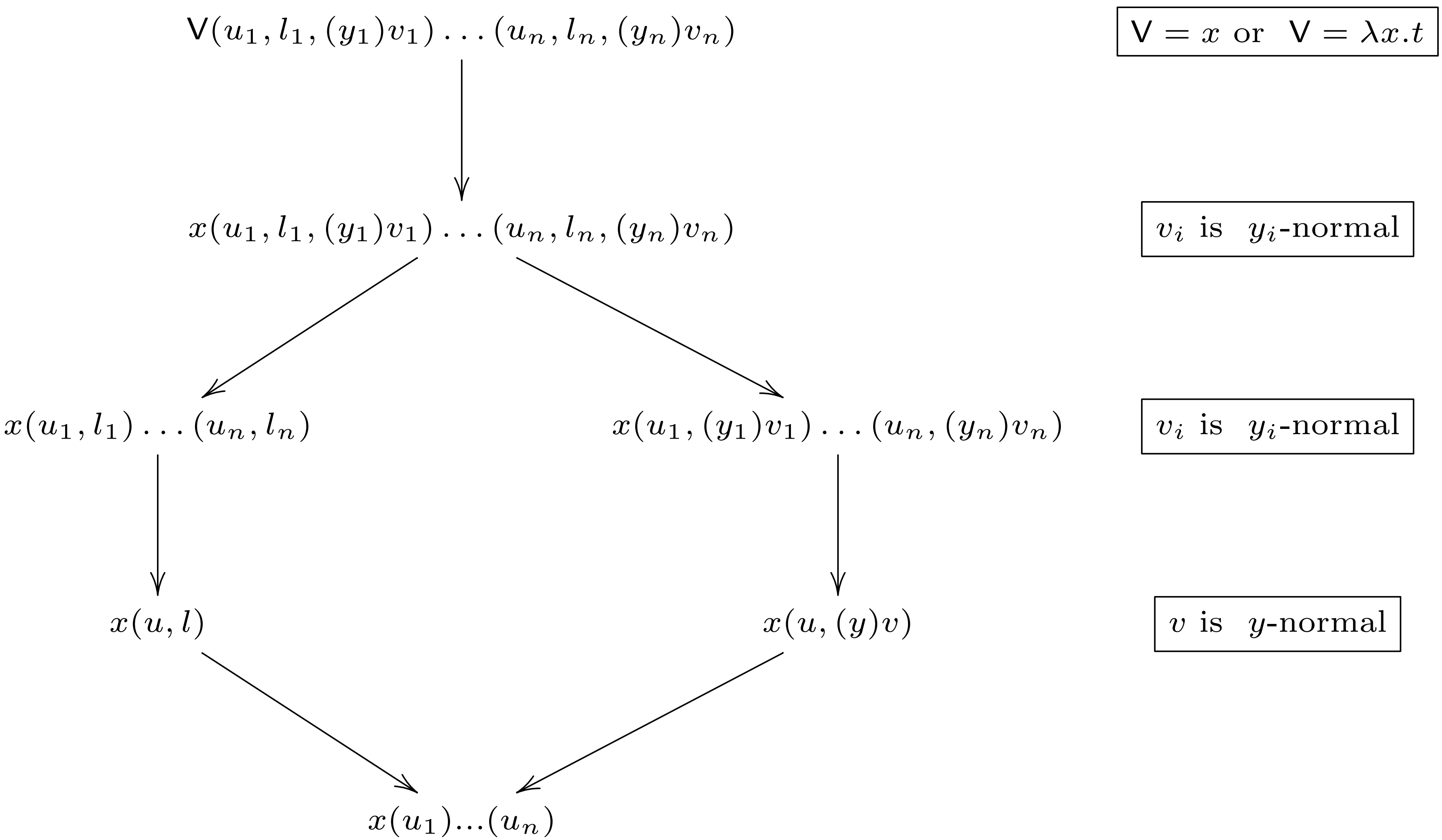
	expression	abbreviation	subsystem
generalised application	$t(u, [], (x)v)$	$t(u, (x)v)$	$\lambda\mathbf{J}$
multiary application	$t(u, l, (x)x)$	$t(u, l)$	$\lambda\mathbf{m}$
simple application	$t(u, [], (x)x)$	$t(u)$	λ

(2) v is x-*normal* if $v = x$ or $v = x(u, l, (y)v')$ v' is y-normal and $x \notin u, l, v'$.

Example: $x(u_0, l_0, (y)y(u_1, l_1, (z)z))$ is x-normal

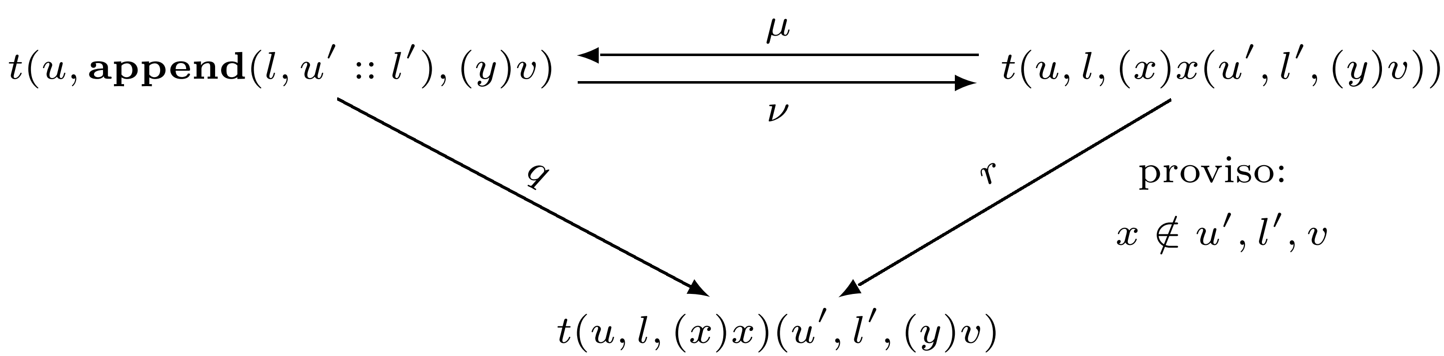
iff $x, y \notin u_0, l_0, u_1, l_1$

Classes of gm-applications



Overlaps and permutations

Three ways of expressing multiple application: (1) multiary application. (2) **normal** generality. (3) iterated application.

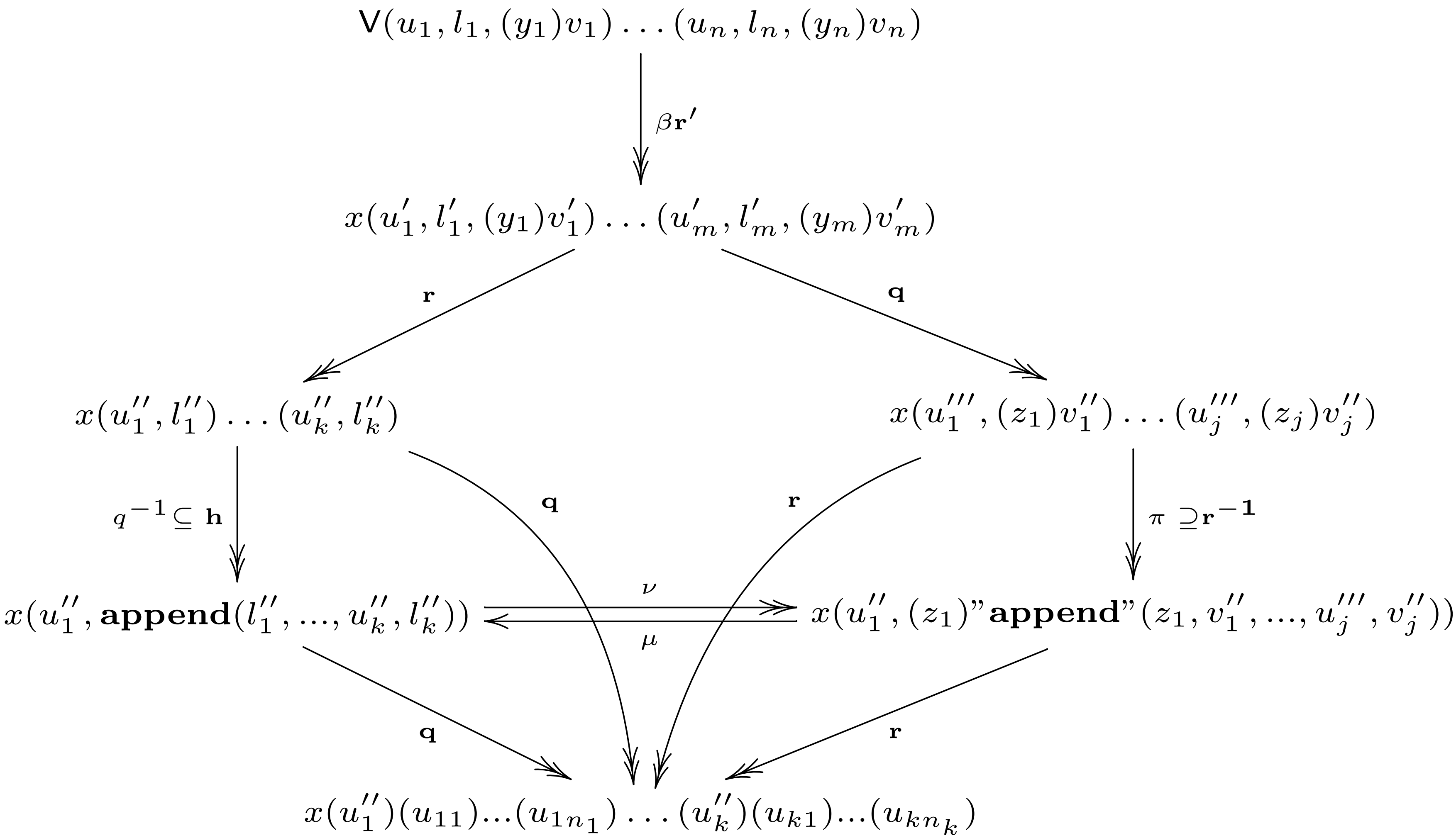


<u>Other rules:</u>

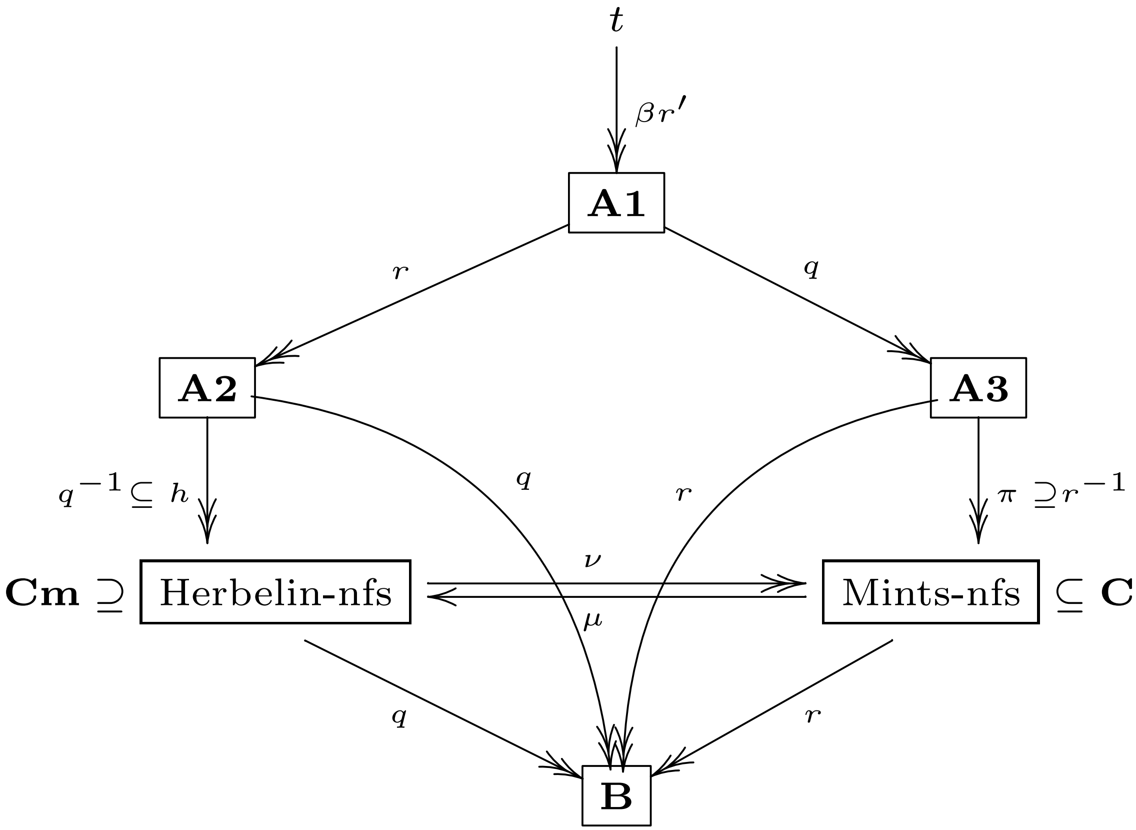
(h)	$t(u,l,(x)x)(u',l',(y)v)$	$\rightarrow_h$	$t(u,\mathbf{append}(l,u' :: l'),(y)v)$	
(s)	$t(u,l,(x)v)$	$\rightarrow_s$	$\mathbf{s}(t(u,l),x,v)$	if $v \neq x$
(r)	$t(u,l,(x)v)$	$\rightarrow_r$	$\mathbf{s}(t(u,l),x,v)$	if v is x-main-linear-appl.
(r')	$t(u,l,(x)v)$	$\rightarrow_{r'}$	$\mathbf{s}(t(u,l),x,v)$	if $v \neq x$ & is not x-main-linear-appl.

<u>Remarks:</u> $q \subseteq h^{-1}$; $r \subseteq \pi^{-1}$; $r \cup r' = s$;

Combining reduction and permutation



COMBINED NORMAL FORMS



A1	=	$\beta r'$-nfs
A2	=	$\beta r r'$-nfs
A3	=	$\beta r' q$-nfs
B	=	$\beta r r' q$-nfs
	=	β-normal λ-terms
C	=	cut-free (unary) sequent terms
Cm	=	cut-free multiary sequent terms

<u>Some results</u>:

(1) $\to_{\beta r r'}$, $\to_{\beta r r' q}$, $\to_{\beta r r' h}$ are confluent

(2) $\to_{\beta r r'}$, $\to_{\beta r' q}$, $\to_{\beta r' h}$ are SN for typable terms

(3) q and h postpone over β and $s = r r'$

Final remarks

1. λ**Jm** is a handy tool for systematic studies in structural proof theory.

2. Permutations capture overlaps between constructors of λ**Jm** and are related to alternative ways of expressing multiple application.

3. Future work includes:

   (a) the missing confluence and termination results;

   (b) postponement of those rules related to organization of multiple application (two-stages computation);

   (c) a new classification of rules (β, r' vs q, r, π, h, μ, ν).