On the density of types with decidable lambda definability problem

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Simple typed λ calculus with one ground type O is considered.

$$\mathbb{T} := O \mid \mathbb{T} \to \mathbb{T}$$

We consider lambda definability problem limited to fourth order types

A *full type hierarchy* $\{D_\tau\}_{\tau \in \mathbb{T}}$ is a collection of finite domains, one for each type.

The whole hierarchy is determined by D_O.

$D_{\tau \to \mu} = {D_\mu}^{D_\tau}$.

All D_τ are finite.

Lambda definability problem

For the particular type τ the τ*-lambda definability problem* is the decision problem:

GIVEN: Finite domain D_O and object $f \in D_\tau$.

PROBLEM: Decide if f is lambda definable in D_τ.

Up to rank 3 types the *lambda definability problem* is decidable.

Definition 1. *Type τ is called regular if $rank(\tau) \leqslant 4$ and every component of τ has $arg \leqslant 1$. This implies that only components allowed for regular types are O, $O \to O$ and $(O^k \to O) \to O$ for any k.*

Theorem 2. *λ definability problem is decidable for all rank 1, 2, 3 types and for regular rank 4 types.*

$$((O \to O \to O) \to O) \to (O \to O)$$

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(example of Thierry Joly)

$$\mathbb{M} = (((O \to O \to O) \to O) \to O) \to (O \to O) \ .$$

We consider probability of the fact that randomly chosen 4 order type has decidable lambda definability problem.

Definition 3. *By* $\|\tau\|$ *we mean the length of type* τ *which we define as the total number of occurrences of atomic type* O *in the given type.*

Definition 4. *We associate the density $\mu(\mathcal{A})$ with a subset $\mathcal{A} \subset \mathbb{T}$ of types as:*

$$\mu(\mathcal{A}) = \lim_{n\to\infty} \frac{\#\{\tau \in \mathcal{A} : \|\tau\| = n\}}{\#\{\tau \in \mathbb{T} : \|\tau\| = n\}} \tag{1}$$

if the limit exists.

Theorem 5. *The density of rank* 4 *types with decidable* λ *definability problem among all rank 4 types is* 0*.*

Theorem 6. *The density of types of rank* $\leqslant 4$ *with the decidable* λ *definability problem among all types of rank* $\leqslant 4$ *is again* 0.