Win, Lose and Stalemate in Impartial Games

Roland Backhouse [1] and Diethard Michaelis [2]

[1] School of Computer Science and Information Technology,
University of Nottingham, Nottingham NG8 1BB, England
[2] Beethovenstr. 55, 90513 Zirndorf, Germany

Abstract. We use fixed point calculus to characterise the winning, losing an d stalemate positions in an impartial, two-person game.

1 Introduction

Game theory [BCG82] is becoming a popular and active area of research. Our interest in the area is as a test case for the use of formalisms for the constructive derivation of algorithms. Game theory is well-suited to our goals because it is about constructing winning strategies. Moreover, examples of games are easy to explain to students, they carry no theoretical overhead, and motivation is for free.

In the study of games, as in the book "Winning Ways" [BCG82], a basic assumption is that all games are terminating. Only limited attention has been paid to games where non-termination is possible (so-called "loopy" games [BCG82, chapter 12]). In this paper, we study impartial two-person games, in which non-termination (caused by loops or infinite game graphs) is a possibility. We show how to characterise winning, losing and stalemate positions in terms of least and greatest fixed points of conjugate predicate transformers.

An *impartial, two-person game* is defined by a binary relation, denoted here by the infix operator "$\mapsto$". Elements of the domain of $\mapsto$ are called *positions*; pairs of positions related by $\mapsto$ are called *moves*. We make no restrictions on $\mapsto$; in particular, the set of positions may be infinite, and $\mapsto$ may be non-well-founded. We use variables s, t, u to denote positions. We use p to denote a predicate on positions.

A game is started in a given position. Each player takes it in turn to move to another position. That is, in position s, a move is to a position t such that $s \mapsto t$. The game *ends* when a player cannot move; the player whose turn it is to move then loses.

Allowing the move relation to be non-well-founded introduces additional difficulties in the development of the theory. For example, in traditional game theory, a fundamental element is the definition of an equivalence relation on games; that this relation is reflexive is established by a "tit-for-tat" winning strategy (the "Tweedledum and Tweedledee Argument" in [BCG82]). But, as is well-known, tit-for-tat is invalid in the case of non-well-founded game relations. One of our concerns is to identify precisely where the property of well-foundedness is vital to the development of the theory.

Throughout this paper, we use the Dijkstra-Scholten [DS90] notation for predicates and predicate transformers. In particular, we use square brackets to indiate that a predicate is true at all positions.

2 Win-Lose Equations

2.1 Fixed Point Properties of Winning and Losing

A *winning* position in a game is one from which the first player has a strategy to choose moves that guarantee that the game ends on the second player's turn to move. A *losing* position is one from which there are only moves into winning positions. All other positions are *stalemate* positions.

From the definition of winning and losing positions, we can identify two properties of positions that they must satisfy. First, winning positions have the property that there is always a move into a losing position:

(1) $$win.s \;\Rightarrow\; \langle \exists t : s \mapsto t : lose.t \rangle \quad .$$

Second, losing positions have the property that every move is a move into a winning position:

(2) $$lose.s \;\Rightarrow\; \langle \forall t : s \mapsto t : win.t \rangle \quad .$$

From (1), we abstract the predicate transformer $Some$, defined by

(3) $$Some.p.s \;\equiv\; \langle \exists t : s \mapsto t : p.t \rangle \quad ,$$

and, from (2), we abstract the predicate transformer All, defined by

(4) $$All.p.s \;\equiv\; \langle \forall t : s \mapsto t : p.t \rangle \quad .$$

A crucial observation is that $Some$ and All are *conjugate* predicate transformers. That is,

(5) $$All \;=\; \neg \bullet Some \bullet \neg \quad .$$

(This is just De Morgan's rule.) A simple consequence is that the predicate transformers $All \bullet Some$ and $Some \bullet All$ are also conjugate.

We show that the winning positions are the positions where the least fixed point of the predicate transformer $Some \bullet All$ is $true$, and the losing positions are the positions where the least fixed point of the predicate transformer $All \bullet Some$ is $true$. First, we give a brief summary of fixed-point theory applied to conjugate predicate transformers.

2.2 Conjugate Predicate Transformers

Suppose f and g are conjugate predicate transformers. Then

(6) $\neg \bullet g = f \bullet \neg \quad \wedge \quad \neg \bullet f = g \bullet \neg$.

Negation is a monotonic function from predicates ordered by "only-if" to predicates ordered by "if". So, by the rolling rule for fixed points (See eg. [Mat95]),

(7) $[\neg \mu f \equiv \nu g]$.

Lemma 8 The predicates μf, μg and $\nu f \wedge \nu g$ are mutually distinct and together cover all positions. That is,

$$
\begin{aligned}
& \mathsf{false} \\
= \quad & \\
& [\mu f \wedge \mu g] \\
= \quad & \\
& [\mu f \wedge (\nu f \wedge \nu g)] \\
= \quad & \\
& [\mu g \wedge (\nu f \wedge \nu g)] \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathsf{true} \\
= \quad & \\
& [\mu f \vee \mu g \vee (\nu f \wedge \nu g)] \quad .
\end{aligned}
$$

Proof Easy consequence of (7).
□

2.3 Application to Win-Lose Equations

Since $Some \bullet All$ and $All \bullet Some$ are conjugate predicate transformers, lemma 8 suggests that the winning positions are given by the predicate

$$\mu(Some \bullet All) \quad ,$$

the losing positions are given by

$$\mu(All \bullet Some) \quad ,$$

and the stalemate positions are given by

$$\nu(Some \bullet All) \wedge \nu(All \bullet Some) \quad .$$

Note that arbitrary fixed points of $Some \bullet All$ and $All \bullet Some$ do not necessarily solve the (mutually recursive) equations (1) and (2). Extremal fixed points do. In particular, both the pair

$$(\mu(Some \bullet All)\ ,\ \mu(All \bullet Some))\quad ,$$

and the pair

$$(\nu(Some \bullet All)\ ,\ \nu(All \bullet Some))\quad ,$$

solve the stronger equations in predicates X and Y:

(9) $$[(X \equiv \langle s :: \langle \exists t : s \mapsto t : Y.t \rangle \rangle) \wedge (Y \equiv \langle s :: \langle \forall t : s \mapsto t : X.t \rangle \rangle)]\quad .$$

We introduce the predicate U (short for *unique*) to characterise the positions in which the equation (9) has a unique solution:

(10) $$[U \equiv \mu(Some \bullet All) \vee \mu(All \bullet Some)]\quad .$$

U has the property:

(11) $$[U \equiv \mu(Some \bullet All) \equiv \nu(Some \bullet All)]\quad .$$

Note the scope of the "everywhere" brackets. This is stronger than the statement

(12) $$[U] \equiv [\mu(Some \bullet All) \equiv \nu(Some \bullet All)]\quad ,$$

which states that the solution to a fixed point equation is unique if the least and greatest fixed points coincide.

Note also that

(13) $$[\neg U \equiv \nu(Some \bullet All) \wedge \nu(All \bullet Some)]\quad .$$

We use the abbreviations W for $\mu(Some \bullet All)$, and L for $\mu(All \bullet Some)$.

We derive properties of the move relation with respect to U.

Lemma 14 From any position in $\neg U$, there is always a move to a position in $\neg U$ (it is this fact that suggests that $\neg U$ characterises the stalemate positions), and every move from a position in $\neg U$ to a position in U is to a position in W. Furthermore, every move from a position in L is to a position in W (thus, not to $\neg U$). In other words, moves from U to $\neg U$ must start at W.

□

Theorem 15 W characterises the winning positions, L characterises the losing positions, and $\neg U$ characterises the stalemate positions, in the sense that there is a winning strategy from all positions in W, and from no other positions.

Proof (Outline) We reduce the problem to the case that equation (9) has a unique solution. Formally, we consider the (truncated) move relation defined by, for all positions s and t,

$$s \mapsto_{tr} t \quad \equiv \quad s \mapsto t \wedge (\mathcal{U}.s \equiv \mathcal{U}.t) \quad ,$$

which disallows the moves between $\mathcal{U}$ and $\neg\mathcal{U}$. Consider the corresponding predicate transformers $Some_{tr}$ and All_{tr}. Then,

$$[\mu(All \bullet Some) \;=\; \mu(All_{tr} \bullet Some_{tr})] \quad ,$$

and

$$[\mu(Some \bullet All) \;=\; \mu(Some_{tr} \bullet All_{tr})] \quad .$$

□

3 Semi-Decision Procedure

Iterative algorithms for determining $\mu_{\preceq}$ for monotonic endofunction f on a set ordered by the relation $\preceq$ are based on the following algorithm. ($\perp\!\!\perp$ denotes the least element of set.)

$$\{ \quad f \in \preceq \leftarrow \preceq \quad \}$$

$$x := \perp\!\!\perp$$

$$; \quad \{ \;\; \textbf{Invariant:} \;\; x \preceq f.x \;\; \wedge \;\; \langle \forall z :: x \preceq z \Leftarrow f.z \preceq z \rangle \quad \}$$

$$\text{do} \quad \neg(f.x \preceq x) \rightarrow x := f.x$$

$$\text{od}$$

$$\{ \quad x = \mu_{\preceq} f \quad \} \quad .$$

Termination of the algorithm is not guaranteed in general (of course). But, termination is guaranteed in the case that the ordered set is finite (and evaluation of f terminates).

We systematically derive the following algorithm for determining the winning, losing and stalemate positions in a game with a finite number of positions. The procedure can also be used as a semi-decision procedure in the case that the game is not finite.

Within the algorithm, it is convenient to use predicate M to denote the move relation. That is, $M.(s,t) \equiv s \mapsto t$. The algorithm as shown does not record the winning strategy, but this can be incorporated easily into the algorithm.

{ l , w and b are predicates on positions,

m is a binary predicate on positions (i.e. a move relation) }

$$
\begin{array}{ll}
 & l,w,b \;:=\; false\,,false\,,All.false \\
; & m \;:=\; M \\
; & \{\;\; \textbf{Invariant:} \\
 & \qquad [l \Rightarrow b] \;\wedge\; [w \equiv Some.l] \;\wedge\; [b \equiv All.w] \\
 & \quad \wedge \quad \langle \forall p \;:\; [All.(Some.p) \Rightarrow p] \;:\; [l \Rightarrow p] \rangle \\
 & \quad \wedge \quad [m \Rightarrow M] \\
 & \quad \wedge \quad \langle \forall\, s,t : m.(s,t) : \neg(w.t) \rangle \\
 & \quad \wedge \quad \langle \forall\, s,t : \neg(m.(s,t)) \wedge M.(s,t) : w.t \rangle \quad \} \\
 & \mathsf{do} \quad \langle \exists s :: (\neg l \wedge b).s \rangle \quad \rightarrow \\
 & \qquad\quad [\![\quad \{ \quad x \text{ is a local variable } \} \\
 & \qquad\qquad\quad x \;=\; Some.(\neg l \wedge b) \\
 & \qquad\qquad ; \;\; w,b \;:=\; (w \vee x)\;,\; \langle s :: \langle \forall t \;:\; m.(s,t) \;:\; x.t \rangle \rangle \\
 & \qquad\qquad ; \;\; m \;:=\; \langle s,t :: m.(s,t) \wedge \neg(x.t) \rangle \\
 & \qquad\quad]\!] \\
 & \mathsf{od} \\
\{ & [l \equiv L] \;\wedge\; [w \equiv W] \quad \} \quad .
\end{array}
$$

4 Further Work

We have characterised losing, winning and stalemate positions in impartial games in terms of least and greatest fixed points. We are currently extending this work to partizan games. We are also developing systematic, calculational derivations of properties of the sum of games, as well as the theory of MEX numbers, and Nim sum. The current work indicates where unicity of solution of the fixed point equations is sufficient, and where the stronger property of well-foundedness of the move relation is required.

References

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