An Elementary Theory of the Category of Locally Compact Locales

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Abstract

The category of locally compact locales over any elementary topos is characterised by means of the axioms of abstract Stone duality (monadicity of the topology, considered as a self-adjoint exponential $\Sigma^{(-)}$, and Scott continuity, $F\phi = \exists \ell.\ F(\lambda n.\ n \in \ell) \wedge \forall n \in \ell.\ \phi n$), together with an "underlying set" functor that is right adjoint to the inclusion of the full subcategory of overt discrete objects (those admitting equality and existential quantification). This full subcategory is then the topos.

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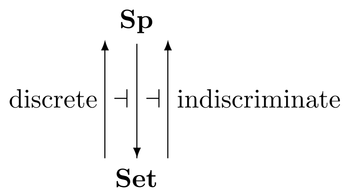
1 Introduction

Abstract Stone duality is a re-axiomatisation of general topology, in which the topology on a space is not a *set* with infinitary lattice structure, but another *space* (with finitary structure). This theory is consciously weaker than the traditional one, in order to overcome some well known conflicts with recursion theory. But in this paper we identify the additional ingredient that is needed to make ASD equivalent to standard topology, or rather to intuitionistic locale theory over an elementary topos.

Recall that, classically, the category of (not necessarily T_0) spaces is related to the category of sets by the adjunctions



where the middle functor yields the *underlying set of points* of a space and the other two equip any set with its greatest and least topologies. These are respectively the *discrete* one in which all subsets are open, and the *indiscriminate* one in which only the empty set and the whole space are open. However, the rightmost functor no longer exists if we require the spaces to be T_0, or *ā fortiori* sober.

Although abstract Stone duality is formulated without any reference to "sets" at all, the full subcategory $\mathcal{E} \subset \mathcal{S}$ of *overt discrete* objects has emerged as the replacement for the category of sets in various roles in general topology. The inclusion of this subcategory corresponds to the leftmost of the above functors, so the central idea of this paper is the supposition that this inclusion have a right adjoint. Since the "underlying set" that this right adjoint assigns to any space lies in the subcategory, it is equipped with an equality test and an existential quantifier.

Recall, however, that the reason for studying abstract Stone duality is in order to unify general topology with recursion theory, and in particular to legitimise the class of recursively enumerable subsets as a (indeed, the) topology on $\mathbb{N}$. This cannot have an "underlying set", because the equality operation on that space would solve the Halting Problem. The new axiom would, therefore, not be an acceptable addition to the principal version of the theory.

Nevertheless, we find in this paper that adding this hypothesis to the axioms of abstract Stone duality as studied elsewhere in the programme *precisely* recovers the "official" theory of locally compact locales.

That is, apart from the fact that the topos $\mathcal{E}$ is itself *constructed* from the axioms. This is not the usual situation in a mathematical discourse, where the foundational system (whether it be **Set**, another topos, traditional set theory or a type theory) is an *assumption*, albeit a silent one. However, those who have studied constructive systems and compared them with their classical counterparts have learned that the latter must be observed in their natural habitat, for example Bourbachiste topology has to be done in the context of the axiom of choice [5]. In other words we must be *holistic*, taking the mathematical structure and its foundations together.

If, therefore, you require locally compact locales over a *particular* topos $\mathcal{E}_0$ (such as the one that you choose to call "**Set**"), the theory has to be extended with base types, constants and equations that force $\mathcal{E} \simeq \mathcal{E}_0$.

More significantly, the infinitary joins that are needed to axiomatise either traditional topology or locale theory are also a consequence of the axioms, and not a part of them. This is the sense in which the new axiomatisation deserves to be called *elementary.* In fact, the logical power of the theory (powersets in the topos, and the infinitary joins) arises out of the "underlying set" assumption.

The fact that the central idea is that sets with equality form a coreflective subcategory of the category of spaces brings to mind some other points of view that have arisen in the background to this subject, which we pause to consider.

In *set theory* the axiom of extensionality, though it may appear innocuous to the naïve observer, actually carries much of the force of the theory. Dana Scott showed [10], for example, that Zermelo–Fraenkel set theory (ZF) *without* extensionality is provably consistent within Zermelo set theory (Z). (Recall that ZF is Z plus the axiom of replacement.)

When well-foundedness is presented in categorical terms using coalgebras, the extensional ones form a *reflective* subcategory, but the axiom of replacement may be needed to construct the reflection functor [12, Exercise 9.62].

In *synthetic domain theory* the category of predomains is a reflective subcategory of some topos, where here the topos is coreflective in the category of spaces. Beware that some of the remarks in [C], specifically 2.10(b), 2.13 and especially 8.9, were written with that situation in mind, rather than the present one. This paper develops Remarks 6.14–15 from there.

This paper follows on very closely (and tidies up some loose ends) from *Geometric and Higher Order Logic* [C], which you would be well advised to have to hand.

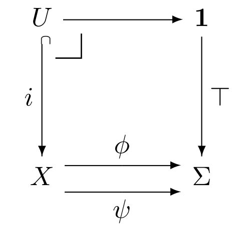
In order to develop the theory for the examples where $\mathcal{S}$ is already an elementary topos, or is $\mathbf{CCD}^{\mathsf{op}}$, in parallel to that for locally compact locales, we postpone until last the "Scott continuity" Assumption 7.4 that characterises the topological case. The other two examples may be captured by alternative axioms, but are not considered in this version of the paper.

Assumption 1.1 Throughout, let $\mathcal{S}$ be a category with finite products and a pointed object $\top : \mathbf{1} \to \Sigma$, such that the exponential Σ^X exists in $\mathcal{S}$ for every object $X \in \mathsf{ob}\mathcal{S}$, and the self-adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$ is monadic.

A λ-calculus equivalent to these categorical properties is developed in [A, B]. It is also shown in [B, Section 11] that such a category $\mathcal{S}$ has stable disjoint coproducts.

Definition 1.2 $\top : \mathbf{1} \to \Sigma$ in $\mathcal{S}$ is a ***dominance*** [9] if

(a) the pullback of $\top : \mathbf{1} \to \Sigma$ along any map $\phi : X \to \Sigma$ exists in $\mathcal{S}$



(in this case we say call i an ***open inclusion*** and say that ϕ ***classifies*** it);

(b) if $\phi, \psi : X \rightrightarrows \Sigma$ both classify the same open inclusion $i : U \to X$ then $\phi = \psi$; and

(c) if $i : U \to V$ and $j : V \to W$ are open inclusions (classified by $\phi : V \to \Sigma$ and $\psi : W \to \Sigma$ respectively) then so is $j \cdot i : U \to W$ (and we write $\exists_j \phi : X \to \Sigma$ for their classifier).

Notice that $\top \cdot !_X : X \to \mathbf{1} \to \Sigma$ classifies id_X, and $\phi \cdot f : Y \to X \to \Sigma$ classifies the pullback $f^{-1}(i)$ of i along $f : Y \to X$.

Theorem 1.3 Under Assumption 1.1, $\top : \mathbf{1} \to \Sigma$ is a dominance iff there is a (unique) binary operation $\wedge : \Sigma \times \Sigma \to \Sigma$ such that $(\Sigma, \top, \wedge)$ is an internal semilattice satisfying the *Euclidean principle*

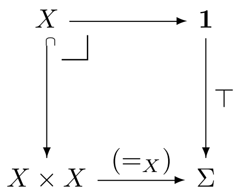
$$\sigma : \Sigma,\ F : \Sigma^\Sigma \ \vdash\ \sigma \wedge F\sigma \ =\ \sigma \wedge F\top.$$

In this case, for any open inclusion $i : U \hookrightarrow X$, there is a map $\exists_i : \Sigma^U \to \Sigma^X$ that satisfies $\exists_i \dashv \Sigma^i$ and the Frobenius and Beck–Chevalley equations.

Proof [C, Sections 2–3]; and the lattice dual of the result is used in Corollary 7.5. □

Definition 1.4 An object $X \in \mathsf{ob}\mathcal{S}$ is called ***overt*** if $\Sigma^{!_X} : \Sigma \to \Sigma^X$ has a left adjoint (written $\exists_X$) and ***compact*** if there is a right adjoint ($\forall_X$). The Beck–Chevalley laws for both connectives are automatic, whilst the Frobenius law for $\exists_X$ follows from the Euclidean principle [C, Section 8].

Definition 1.5 $X \in \mathsf{ob}\mathcal{S}$ is called ***discrete*** if the diagonal subspace is open, in which case we write $(=_X)$ for the classifying map [C, Section 6].



Definition 1.6 $i : X \rightarrowtail Y$ is a **Σ*-split inclusion*** if there is some map $I : \Sigma^X \rightarrowtail \Sigma^Y$ with $\Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}$.

Assumption 1.7 For every object X there is a Σ-split inclusion $X \rightarrowtail \Sigma^N$ with N overt discrete. We shall call this the ***basis assumption***, *cf.* [E].

The first main result that we shall prove in this paper is the

Theorem 1.8 The full subcategory $\mathcal{E} \subset \mathcal{S}$ of overt discrete objects is *coreflective* iff $\mathcal{E}$ is an elementary *topos*.

Recall that being coreflective means that the inclusion functor $\Delta : \mathcal{E} \subset \mathcal{S}$ has a right adjoint, $\Delta \dashv \mathsf{U}$, although we do not usually write the Δ. This means that for every object $X \in \mathsf{ob}\mathcal{S}$ there is a *couniversal* map $\varepsilon_X : \mathsf{U}X \to X$ with $\mathsf{U}X \in \mathsf{ob}\mathcal{E}$, *i.e.* any map $\Gamma \to X$ with $\Gamma \in \mathsf{ob}\mathcal{E}$ factors uniquely as $\Gamma \to \mathsf{U}X \to X$. Notation 3.6 expresses this type-theoretically.

Examples 1.9 (a) $\mathcal{S}$ may be the classical category of sets and functions, with $\Sigma = \{\top, \bot\}$. Then $\mathcal{E} = \mathcal{S}$ and $\Delta = \mathsf{U} = \mathsf{id}$.

(b) $\mathcal{S}$ may, more generally, be any elementary topos, with $\Sigma = \Omega$. Again, $\mathcal{E} = \mathcal{S}$ [C].

(c) $\mathcal{S}$ may be **LKSp**, the classical category of locally compact sober spaces, and Σ the Sierpiński space. (In this case the monadic property depends on the axiom of choice.) Then the topos $\mathcal{E}$ is again **Set**. The inclusion functor Δ endows any set with its discrete topology, whilst the right adjoint U yields the underlying set (of points) of any locally compact space [A, Theorem 5.12].

(d) $\mathcal{S}$ may be **LKLoc**, the category of locally compact locales (*i.e.* the opposite of the category of distributive continuous lattices and frame homomorphisms) over an elementary topos, Σ being the free frame on one generator [B, Theorem 3.11].
Then $\mathcal{E}$ is equivalent to the given topos and, for $N \in \mathsf{ob}\mathcal{E}$, ΔN is the powerset Ω^N considered as a frame. Conversely, $\mathsf{U}X$ is the set of "points" of the locale $X \in \mathsf{ob}\mathcal{S}$, in the sense of locale theory, *i.e.* homomorphisms from frame (corresponding to) X to Σ. This functor (the right adjoint to Δ) *exists* in general in intuitionistic locale theory, but it is only *faithful* if we assume the axiom of choice [5].

(e) Let $\aleph$ be a regular cardinal, *i.e.* a class of sets including **0** and **2** that is closed under isomorphism, products and quotients, and also unions indexed by members of the class. (See [6] for a categorical structure that expresses the same thing.) The finite meets in the definitions of topological spaces, locales and local compactness may be generalised to those of size $< \aleph$, and the resulting $\aleph$-locally compact locales provide another model.

(f) $\mathcal{S}$ may be $\mathbf{CCD}^{\mathsf{op}}$, the opposite of the category of (constructively) completely distributive lattices and their homomorphisms, Σ being the free such lattice on one generator. This is the case $\aleph = \infty$ of the previous example.
Again, $\mathcal{E}$ is the base topos and ΔN is the powerset of N, considered as a completely distributive lattice. $\mathsf{U}X$ is the set of complete lattice homomorphisms from X (*qua* lattice) to Ω [7].

Of these examples, we are most interested in that of **LKLoc**, which we shall characterise by the further Assumption 7.4, that for each overt discrete object N,

$$F : \Sigma\Sigma N,\ \phi : \Sigma N \ \vdash\ F\phi \ = \ \exists \ell{:}\mathsf{K}N.\ F(\lambda n.\ n \in \ell) \wedge \forall n \in \ell.\ \phi n,$$

where $\mathsf{K}N$ denotes the free semilattice on N (its finite powerset).

However, we shall find that the theory can be developed quite a long way, whilst nevertheless in a style similar to locale theory, before we introduce this final assumption. This will allow us to apply the techniques to the somewhat less familiar $\mathbf{CCD}^{\mathsf{op}}$.

On the basis of Assumptions 1.1 and 1.7, together with the equivalent conditions of Theorems 1.3 and 1.8, Theorem 4.11 shows that $\mathcal{S}^{\mathsf{op}}$ is a full subcategory of the Eilenberg–Moore category for a submonad of the double powerset $\mathcal{PP}(-)$ over $\mathcal{E}$. The *elementary* assumptions that we have stated are therefore equivalent to more familiar ones expressed in terms of infinitary lattice theory.

The structure under study for most of the paper will therefore be a hybrid of the examples above. Whilst the product of two such categories is another, and more generally the structure may resemble one example over one part of a topologically nontrivial (Grothendieck) topos $\mathcal{E}$ and some other example elsewhere, we shall find that these examples are qualitatively exhaustive. In particular, one of the generalisations of ASD that is envisaged is not consistent with the "underlying set" axiom:

Lemma 1.10 If U exists then Σ is a distributive lattice.

Proof Any coreflective subcategory is closed under (colimits, but in particular) finite coproducts, which $\mathcal{S}$ has because $\mathcal{S}^{\mathsf{op}}$ is monadic over $\mathcal{S}$, which has finite products. Thus **0** and **2** are overt, and their existential quantifiers are $\bot$ and $\vee$ respectively [C, Proposition 9.1]. We give a more detailed symbolic proof of this result in Section 5. □

Corollary 1.11 It would be possible to develop abstract Stone duality (with the monadic and Euclidean axioms) for models based on stable domains [C, Example 4.5] or other treatments of sequential functions. However, the "topology" Σ^X *must* then be considered as a "space" (an internal semilattice in $\mathcal{S}$) and does not come equipped with an "underlying set" of "opens". In this case the overt discrete objects form a regular category, rather than a pretopos, since **2** is not overt. $\square$

2 The topos of overt discrete objects

This section proves Theorem 1.8, that the full subcategory of overt discrete objects is a topos iff it is coreflective. However, the greater part of the proof was already in [C], which culminated in the blunter result that $\mathcal{S}$ itself is a topos iff *all* objects are overt and discrete, so $\mathcal{E} = \mathcal{S}$. In particular, we rely heavily on the result proved there that any mono into an overt discrete object is an open inclusion.

Lemma 2.1 Pullbacks at discrete objects exist in $\mathcal{S}$.

Proof [C, Proposition 10.1] $\square$

Lemma 2.2 Let $U \to X$ be an open inclusion. If X is overt or discrete then so too is U.

Proof [C, Propositions 6.11(e) and 8.3(c)]. $\square$

Proposition 2.3 $\mathcal{E} \subset \mathcal{S}$ has finite limits.

Proof [C, Sections 6,8].

(a) **1** is overt [C, Proposition 8.3(a)];

(b) **1** is discrete [C, Proposition 6.11(a)];

(c) if X and Y are overt then so is $X \times Y$ [C, Proposition 8.3(b)];

(d) if X and Y are discrete then so is $X \times Y$ [C, Proposition 6.11(c)];

(e) $\mathcal{E}$ has equalisers, by Lemmas 2.1 and 2.2. $\square$

Lemma 2.4 If X is overt then $p_1 : X \times Y \to Y$ has $\exists_{p_1} \dashv \Sigma^{p_1}$ satisfying Frobenius and Beck–Chevalley.

Proof [C, Propositions 8.1–2]. $\square$

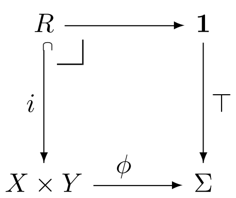
Proposition 2.5 Any mono $i : X \to D$ from an overt object a discrete one is an open inclusion.

Proof [C, Sections 8,10; Corollary 10.3]. $\square$

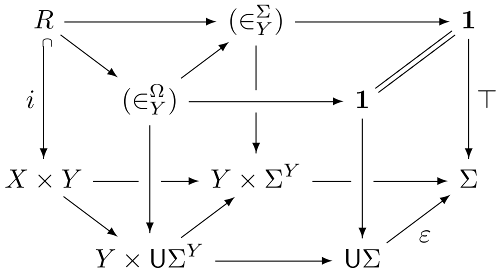
Theorem 2.6 If $\Delta \dashv \mathsf{U}$ exists then $\mathcal{E}$ is an elementary topos.

Proof We shall show that $\mathsf{U}\Sigma^Y$ is the powerset of $Y \in \mathsf{ob}\mathcal{E}$. Given that $\mathcal{E}$ also has finite limits, this is sufficient to make $\mathcal{E}$ a topos, with subobject classifier $\Omega \equiv \mathsf{U}\Sigma$,

Let $i : R \hookrightarrow X \times Y$ be any mono in $\mathcal{E}$ (a "binary relation" from X to Y). This is an open inclusion in $\mathcal{S}$, and is therefore classified by some unique $\phi : X \times Y \to \Sigma$:



Using the exponential transpose, $\tilde{\phi} : X \to \Sigma^Y$, this square factorises into the two pullback squares at the back of the diagram



Since U is right adjoint to a full inclusion, it preserves the objects of the subcategory, together with, $\mathbf{1}$, $\times$ and pullbacks. Applying it to the squares at the back of the diagram therefore says that those at the front (from R to $\mathsf{U}\Sigma$) are also pullbacks, where

$$(\in_Y^\Omega) \;\equiv\; \mathsf{U}(\in_Y^\Sigma) \hookrightarrow Y \times \mathsf{U}\Sigma^Y$$

is therefore the generic binary relation on Y, as required. □

Notation 2.7 Following Vickers [13], we write $\Omega X \equiv \mathsf{U}(\Sigma^X)$ and, for $f : X \to Y$ in $\mathcal{S}$,

$$\Omega f \equiv f^* \equiv \mathsf{U}(\Sigma^f) : \Omega Y \to \Omega X \text{ in } \mathcal{E}.$$

Although ΩY is the exponential Ω^Y within the topos $\mathcal{E}$, when we try to extend its universal property (*i.e.* the bijection between maps $\Gamma \to \Omega Y$ and $\Gamma \times Y \to \Omega$) to the larger category $\mathcal{S}$, we find that it only holds when Γ and Y are overt discrete. So the property does not extend at all.

In the type theory that we develop for $\mathcal{S}$ in Section 3, we shall therefore not use ΩY directly as an exponential, but always *via* the adjunction $\Delta \dashv \mathsf{U}$, so application and formation of terms of type ΩY will always involve ε and τ. explicitly.

Turning to the converse of Theorem 1.8, we need first to sharpen the basis assumption.

Lemma 2.8 Every object X is an equaliser $X \rightarrowtail \Sigma^N \rightrightarrows \Sigma^M$ with N and M overt discrete.

$$X \overset{i}{\rightarrowtail} \Sigma^N \underset{\psi \mapsto \lambda F.\, I(F \cdot i)\psi}{\overset{\psi \mapsto \lambda F.\, F\psi}{\rightrightarrows}} \Sigma\Sigma\Sigma N \overset{j}{\rightarrowtail} \Sigma^M$$

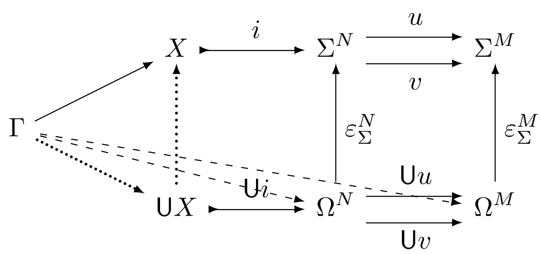
Proof Using Assumption 1.7 for X itself and again for Σ^{Σ^N}, let $i : X \rightarrowtail \Sigma^N$ with $\Sigma^i \cdot I = \mathsf{id}_{\Sigma^X}$ and $j : \Sigma\Sigma\Sigma N \rightarrowtail \Sigma^M$. Then, since X is sober by Assumption 1.1, the diagram above is an equaliser [B, Section 4]. □

Theorem 2.9 If $\mathcal{E}$ is a topos then $\Delta : \mathcal{E} \subset \mathcal{S}$ has a right adjoint.

Proof The generic subobject $\top : \mathbf{1} \to \Omega$ in the topos $\mathcal{E}$ is a mono between overt discrete objects in $\mathcal{S}$, which is therefore open and classified by some map $\varepsilon_\Sigma : \Omega \equiv \mathsf{U}\Sigma \to \Sigma$ by Proposition 2.5.

The right adjoint U must preserve exponentials. So define $\mathsf{U}\Sigma^N = \Omega^N$ and $\varepsilon_{\Sigma^N} = \varepsilon_\Sigma^N$, for any $N \in \mathsf{ob}\mathcal{E}$. Then $\Gamma \to \Sigma^N$ corresponds to $\Gamma \times N \to \Sigma$, which classifies an open subspace of the overt discrete space $\Gamma \times N$, *i.e.* a subobject of this object of the topos $\mathcal{E}$, and therefore corresponds to $\Gamma \times N \to \Omega$ and to $\Gamma \to \Omega^N$.

The right adjoint U must also preserve equalisers. Using Lemma 2.8, any object is an equaliser of the form in the top row, so define $\mathsf{U}X$ as the equaliser on the bottom row.



Now let $\Gamma \to X$ with $\Gamma \in \mathcal{E}$. Then $\Gamma \to X \to \Sigma^N$ corresponds to $\Gamma \to \Omega^N$ and similarly the common composite $\Gamma \to \Sigma^M$ corresponds to $\Gamma \to \Omega^M$. As the composites are equal, they factor through the equaliser $\mathsf{U}X$, which is therefore the coreflection of $X \in \mathsf{ob}\mathcal{S}$ into $\mathcal{E}$. $\square$

3 Type-theoretic notation

In this section we shall introduce a notation or type theory in which the assumptions of the Introduction may be manipulated, and use it to develop some of the main structure of locale theory. This shows how we can reason in general topology with a calculus whose terms denote *points*, but which is nevertheless equivalent to the intuitionistic locale theory that has hitherto required the manipulation of infinitary lattice theory.

The full set of type-theoretic rules that realises the categorical monadic Assumption 1.1 (without further lattice structure) is set out over two pages in [B, §8]. In particular, the exponential Σ^X gives rise to a simply typed λ-calculus that is *restricted* in that we may form the function type Σ^X (or $X \to \Sigma$) but not Y^X in general.

The monadic assumption says, on the one hand, that every object is *sober*, giving rise to a term-forming operation focus that we shall need in Section 4. On the other hand, it provides certain Σ-*split* subspaces, using a type-forming calculus that we largely avoid discussing except in Lemmas 5.3–5.4 and Proposition 8.13, and then only indirectly.

The (semi)lattice structure on Σ must obviously also form part of the term calculus, but its role as a dominance (Definition 1.2) gives rise to a calculus of (open) predicates that was introduced in [C, §8]. We begin with a couple of properties of the dominance.

Lemma 3.1 Let $\Gamma \vdash \alpha, \beta, \gamma : \Sigma$. Then

$$\frac{\Gamma,\ \alpha\ \vdash\ \beta\ \leq\ \gamma : \Sigma}{\Gamma\ \vdash\ \alpha \wedge \beta\ \leq\ \gamma : \Sigma}$$

Proof By Theorem 1.3, we may work in terms of the open subobjects that these terms classify. On the top line, β and γ are interpreted as

$$[\alpha] \cap [\beta] \subset [\alpha] \cap [\gamma] \hookrightarrow [\alpha] \hookrightarrow \Gamma$$

so $\Gamma \vdash \alpha \wedge \beta \leq \alpha \wedge \gamma : \Sigma$, which is just $\Gamma \vdash \alpha \wedge \beta \leq \gamma : \Sigma$, and conversely. $\square$

Lemma 3.2 If there is a proof of

$$\frac{\Gamma\ \vdash\ \alpha = \top : \Sigma}{\Gamma\ \vdash\ \beta = \top : \Sigma}$$

then $\Gamma \vdash \alpha \leq \beta : \Sigma$.

Proof Add α to the context in each line of the proof, and use Lemma 3.1 to deduce $\alpha \leq \beta$. $\square$

Now the rules for equality and the quantifiers, applicable to discrete, overt and compact objects.

Definition 3.3 N is a ***discrete*** object if it comes equipped with $n, m : N \vdash n(=_N)m : \Sigma$ such that

$$\text{for } \Gamma \vdash n, m : N, \qquad \frac{\Gamma \vdash (n =_N m) = \top : \Sigma}{\Gamma \vdash n = m : N}$$

where $(=)$ below the line denotes provable equality of terms of type N, whilst $(=_N)$ above the line denotes the new structure.

Definition 3.4 N is an ***overt*** object if it comes equipped with $\phi : \Sigma^N \vdash \exists_N \phi : \Sigma$ such that

$$\frac{\Gamma,\ x : N\ \vdash \phi x\ \leq\ \sigma : \Sigma}{\Gamma\ \vdash \exists x.\ \phi x\ \leq\ \sigma : \Sigma}$$

where we write $\exists x{:}X.\ \phi x$ for $\exists_N(\lambda x{:}N.\ \phi x)$, and the Frobenius law,

$$\sigma \wedge \exists x.\ \phi x \ = \ \exists x.\ \sigma \wedge \phi x$$

follows automatically from the Euclidean principle. (The Beck–Chevalley condition is also automatic.)

Definition 3.5 K is a ***compact*** object if it comes equipped with $\phi : \Sigma^N \vdash \forall_K \phi : \Sigma$ such that

$$\frac{\Gamma,\ x : K\ \vdash \sigma\ \leq\ \phi x : \Sigma}{\Gamma\ \vdash \sigma\ \leq\ \forall x{:}K.\ \phi x : \Sigma}$$

Again the Beck–Chevalley condition is automatic.

Now the rules that arise from the "underlying set" axiom ($\Delta \dashv \mathsf{U}$) of Theorem 1.8.

Notation 3.6 For any type (space, object of $\mathcal{S}$) X, we have

(a) another type, the ***underlying set***, $\mathsf{U}X$;

(b) as this is discrete, an ***equality***, $(=_{\mathsf{U}X}) : \mathsf{U}X \times \mathsf{U}X \to \Sigma$, satisfying the rules in Definition 3.3;

(c) as it is overt, an ***existential quantifier***, $(\exists_{\mathsf{U}X}) : \Sigma^{\mathsf{U}X} \to \Sigma$, satisfying the rules in Definition 3.4;

(d) the counit, which is a function-symbol $x : X \vdash \varepsilon x : \mathsf{U}X$,

(e) for any *overt discrete* context Γ (so $\Gamma \in \mathsf{ob}\mathcal{E}$) the transformation

$$\frac{\Gamma\ \vdash\ a : X}{\Gamma\ \vdash\ \tau.\ a : \mathsf{U}X}$$

satisfying the β- and η-rules

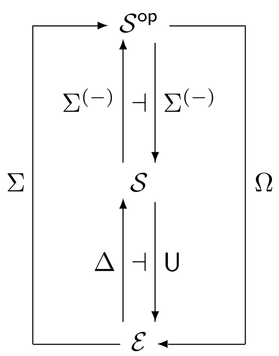
$$\Gamma \vdash \varepsilon(\tau.\ a) = a : X \qquad \text{and} \qquad x : \mathsf{U}X \vdash x = (\tau.\ \varepsilon x) : \mathsf{U}X.$$

In short, $\tau.$ may be applied to any term all of whose free variables are of overt discrete type; Proposition 8.7 relies on this restriction to ensure that only Scott-continuous functions in $\mathcal{E}$ lift to morphisms of $\mathcal{S}$ in the topological example.

This notational trick with ε and $\tau.$ is applicable to any coreflective subcategory: [12, Remark 9.5.2] and [B, Section 11] use it for comprehension.

4 Comparing the monads

We have the following composition of adjunctions,



of which the top one is monadic (over $\mathcal{S}$) by Assumption 1.1. In this section we compare $\mathcal{S}^{\mathsf{op}}$ with the category of algebras for the monad over $\mathcal{E}$.

Remark 4.1 A covariant adjunction such as $\Delta \dashv \mathsf{U}$ has a unit and a counit, but it is convenient to regard the composite adjunction between $\Sigma \equiv \Sigma^{\Delta(-)}$ and $\Omega \equiv \mathsf{U}\Sigma^{(-)}$ as between *contravariant* functors that relate $\mathcal{S}$ to $\mathcal{E}$. Then instead of a unit and counit, we have two units,

$$\begin{array}{lll} N \to \Omega\Sigma^N & \text{by} & n : N \vdash \tau.\ \lambda\phi{:}\Sigma^N.\ \phi n \\ X \to \Sigma^{\Omega X} & \text{by} & x : X \vdash \lambda\phi{:}\Omega X.\ \varepsilon\phi x \end{array}$$

where N is overt discrete. The second is the double exponential transpose $\tilde{\varepsilon}$ of $\varepsilon : \Omega X \equiv \mathsf{U}\Sigma^X \to \Sigma^X$, so the symbol $\tilde{\varepsilon}$ will occur where you would normally expect to see η.

Remark 4.2 The composite adjunction (over $\mathcal{E}$) need not be monadic; indeed, it is not in the topological example (**LKLoc**). However, using the Lemma, we shall show that it is "of descent type", *i.e.* the Eilenberg–Moore comparison functor is full and faithful. Stripped of the jargon, this means that

(a) morphisms $f : U \to X$ in $\mathcal{S}$,

(b) Eilenberg–Moore homomorphisms $\Sigma^X \to \Sigma^U$ for the monad on $\mathcal{S}$ and

(c) Eilenberg–Moore homomorphisms $\Omega X \to \Omega U$ for the monad on $\mathcal{E}$

are in bijective correspondence.

Lemma 4.3 For any object $X \in \mathsf{ob}\mathcal{E}$, the map $\Sigma\varepsilon\Sigma X$ is split mono,

$$\Sigma\Sigma X \underset{\delta}{\overset{\Sigma\varepsilon\Sigma X}{\rightleftarrows}} \Sigma\Omega X,$$

but the splitting δ need not be natural in X.

Proof First consider $X = \Sigma^N$, where $\Sigma^{\varepsilon\Sigma\Sigma N}\mathcal{F} \;=\; \lambda G{:}\Omega\Sigma N.\ \mathcal{F}(\varepsilon G)$. Define

$$\mathcal{G} : \Sigma\Omega\Sigma N \;\vdash\; \delta_{\Sigma N}\mathcal{G} \;\equiv\; \lambda F{:}\Sigma\Sigma N.\ \mathcal{G}(\tau.\ \lambda\phi{:}\Sigma N.\ F\phi).$$

Then $\mathcal{F} \mapsto \lambda G.\ \mathcal{F}(\varepsilon G) \;\mapsto\; \lambda F.\ \mathcal{F}(\varepsilon\tau.\ \lambda\phi.\ F\phi) \;=\; \lambda F.\ \mathcal{F}(\lambda\phi.\ F\phi) \;=\; \lambda F.\ \mathcal{F}F \;=\; \mathcal{F}$

Now let $i : X \rightarrowtail \Sigma^N$ and $I : \Sigma X \rightarrowtail \Sigma\Sigma N$ with $\Sigma^i \cdot I = \mathsf{id}$ as in Assumption 1.7.

$$\begin{array}{ccccc} \Sigma\Sigma\Sigma N & \xrightarrow{\Sigma\varepsilon\Sigma\Sigma N} & \Sigma\Omega\Sigma N & \xrightarrow{\delta_{\Sigma N}} & \Sigma\Sigma\Sigma N \\ \uparrow{\scriptstyle \Sigma\Sigma i} & & \uparrow{\scriptstyle \Sigma\Omega i} & & \downarrow{\scriptstyle \Sigma I} \\ \Sigma\Sigma X & \xrightarrow{\Sigma\varepsilon\Sigma X} & \Sigma\Omega X & \overset{\delta_X}{\cdots\cdots\twoheadrightarrow} & \Sigma\Sigma X \end{array}$$

Then, defining δ_X by composition, it splits $\Sigma\varepsilon\Sigma X$ as required, since $\Sigma\varepsilon\Sigma$ is natural with respect to i. $\square$

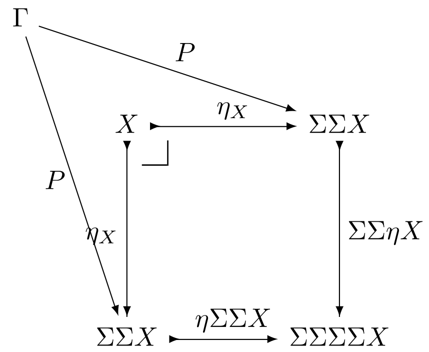
Corollary 4.4 $\tilde{\varepsilon}_{\Sigma X} : X \rightarrowtail \Sigma^{\Omega X}$ is Σ-split, replacing N by the more specific ΩX in Assumption 1.7. $\square$

Remark 4.5 A map $H : \Sigma^X \to \Sigma^U$ is an Eilenberg–Moore homomorphism iff its double exponential transpose P has equal composites

$$\Gamma \equiv U \xrightarrow{P} \Sigma\Sigma X \underset{\eta\Sigma\Sigma X}{\overset{\Sigma\Sigma\eta X}{\rightrightarrows}} \Sigma\Sigma\Sigma\Sigma X,$$

and in this case P is said to be ***prime*** [A, §4].

Part of the monadicity Assumption 1.1 is that every object X be ***sober***, *i.e.* that it be the equaliser of this parallel pair. Equivalently, the naturality square



for η with respect to ηX is a pullback, whilst P is prime iff the quadrilateral from Γ to $\Sigma\Sigma\Sigma\Sigma X$ commutes.

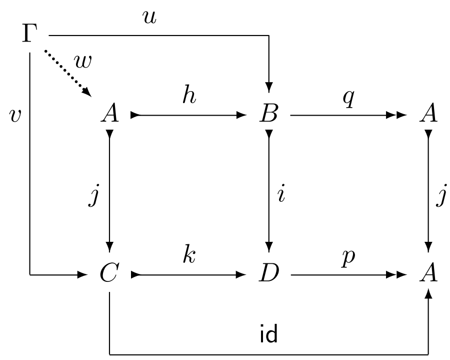
Definition 4.6 Primality can be expressed as a λ-equation, which may also be formulated for the corresponding map $\Omega X \to \Omega U$ in $\mathcal{E}$. Then $\Gamma \vdash P : \Sigma^{\Sigma^X}$ and $\Gamma \vdash Q : \Sigma\Omega X$ are called **Σ*-prime*** and **Ω*-prime*** respectively if

$$\begin{aligned}\Gamma,\ \mathcal{F} : \Sigma\Sigma\Sigma X \quad &\vdash \quad \mathcal{F}P \;=\; P\big(\lambda x{:}X.\ \mathcal{F}(\lambda\phi{:}\Sigma X.\ \phi x)\big) : \Sigma \\ \Gamma,\ \mathcal{G} : \Omega\Sigma\Omega X \quad &\vdash \quad \varepsilon\mathcal{G}Q \;=\; Q\big(\tau.\ \lambda x{:}X.\ (\varepsilon\mathcal{G})(\lambda\psi{:}\Omega X.\ \varepsilon\psi x)\big) : \Sigma\end{aligned}$$

Note that the sub-term to which Q is applied is well formed because its only free variable is $\mathcal{G}$, whose type is overt discrete, irrespectively of whether Γ is (Notation 3.6).

When P is Σ-prime it is of the form $P = \eta_X(a)$ for some unique $\Gamma \vdash a : X$, by sobriety of X, and we write $a = \mathsf{focus}P$.

Lemma 4.7 In *any* category, suppose that the two squares commute, the horizontal composites are identities and the verticals are mono. Then the left-hand square is a pullback



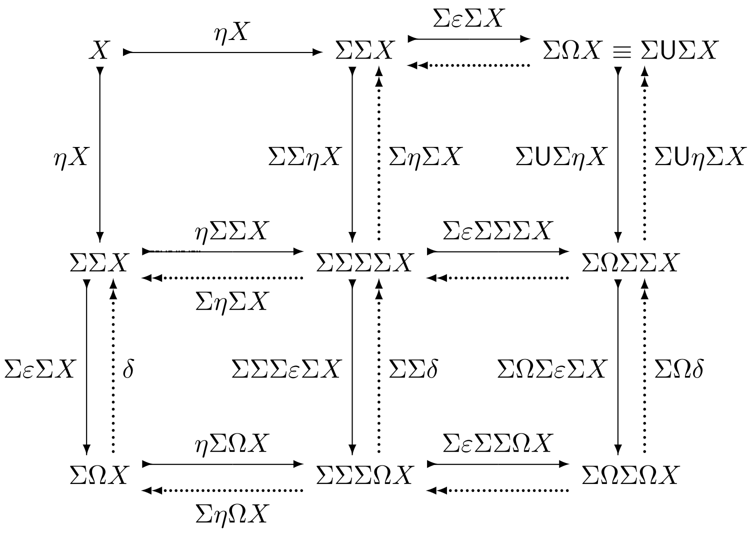
Proof Given $k \cdot u = i \cdot v$, put $w = q \cdot v$. Then

$$j \cdot w \;=\; j \cdot q \cdot v \;=\; p \cdot i \cdot v \;=\; k \cdot u \cdot p \;=\; u$$

and
$$i \cdot h \cdot w \;=\; k \cdot j \cdot w \;=\; k \cdot u \;=\; i \cdot v$$

so $h \cdot w = v$ since i is mono. Hence w is a mediator, and is unique because h is mono. □

Proposition 4.8 The four (solid) squares of monos are pullbacks:



Proof Notice that the composite along the top or the right is the unit $\tilde{\varepsilon}$ of the symmetric adjunction between Σ and Ω (Remark 4.1).

The two squares on the left state naturality of η, and those on the right that of $\Sigma\varepsilon\Sigma$, with respect to ηX at the top and $\Sigma\varepsilon\Sigma X$ at the bottom.

The top left square is a pullback because X is sober, which is part of the monadic assumption (Remark 4.5)

The other monos are split by the accompanying dotted maps, by the unit law for η and the basis assumption.

The two lower squares commute from bottom left to top right by naturality of η and $\Sigma\varepsilon\Sigma$ with respect to δ.

The top right and bottom left squares commute from the centre outwards, by naturality of $\Sigma\varepsilon$ with respect to $\eta\Sigma X$, and of $\Sigma\eta$ with respect to $\varepsilon\Sigma X$.

Hence, by the Lemma, these three squares are pullbacks. □

Corollary 4.9 If $\Gamma \vdash P : \Sigma^{\Sigma^X}$ is Σ-prime then $\Gamma \vdash Q \equiv \lambda\psi{:}\Omega X.\ P(\varepsilon\psi) : \Sigma\Omega X$ is Ω-prime, and conversely every Ω-prime Q arises from a unique Σ-prime P in this way.

Proof P and Q respectively test the pullback properties of the top left square and of the whole diagram (Remark 4.5). □

Proposition 4.10 The map $f : X \to Y$ and the Ω-prime Q are mutually defined by

$$x : X \ \vdash\ fx \equiv \mathsf{focus}\,(\delta Q) \quad \text{and} \quad Q \equiv \lambda\psi{:}\Omega X.\ (\varepsilon\psi)(fx).$$

Proof

$$\begin{aligned}
\mathsf{focus}\,(\delta Q) &= \mathsf{focus}\,(\delta\lambda\psi{:}\Omega X.\ (\varepsilon\psi)(fx)) \\
&= \mathsf{focus}\,(\lambda\psi.\ \psi(fx)) \ =\ fx \\
\lambda\psi.\ (\varepsilon\psi)(fx) &= \lambda\psi.\ (\varepsilon\psi)(\mathsf{focus}\,\delta Q) \\
&= \lambda\psi.\ (\delta Q)(\varepsilon\psi) \\
&= \lambda\psi.\ Q\psi \ =\ Q
\end{aligned}$$
□

Theorem 4.11 $\mathcal{S}^{\mathsf{op}}$ is a full subcategory of the Eilenberg–Moore category for the monad $\Omega\Sigma(-)$ over $\mathcal{E}$. □

Corollary 4.12 In order to identify the category $\mathcal{S}$, it suffices to characterise the algebras ΩX and their homomorphisms $f^* \equiv \Omega f \equiv \mathsf{U}\Sigma^f$ for the monad on $\mathcal{E}$. □

Remark 4.13 Also, $\mathcal{S}$ is $\mathcal{E}$-enriched, or locally internal to $\mathcal{E}$, the hom-object being given by the equaliser

$$\mathcal{S}(X,Y) \rightarrowtail \Omega X^{\Omega Y} \overset{\Omega\tilde{\varepsilon}_{\Sigma X}\cdot\Omega\Sigma(-)}{\underset{(-)\cdot\Omega\tilde{\varepsilon}_{\Sigma Y}}{\rightrightarrows}} \Omega X^{\Omega\Sigma\Omega Y}.$$

that internalises what it is to be an Eilenberg–Moore homomorphism for the monad on $\mathcal{E}$. Such a definition is justified because Σ and Ω define a *strong* monad, and we may define maps $\Gamma \to \mathcal{S}(X,Y)$ in $\mathcal{E}$ as $\mathcal{S}$-maps $\Gamma \times X \to Y$.

Remark 4.14 We have shown that the situation defined by the four elementary assumptions of the Introduction is equivalent to one involving powersets and infinitary lattice theory in the (more familiar) setting of an intuitionistic set theory. This leaves a problem that is suitable for a graduate student in category theory:

Let T be a subfunctor of the double powerset, *i.e.* we have a natural mono $T \to \mathcal{PP}(-)$, and let $\mathcal{A}$ be the category of algebras for the monad T (η and μ being inherited from $\mathcal{PP}(-)$). Let $\mathcal{S}$ be a full subcategory of $\mathcal{A}^{\mathsf{op}}$, and Σ the object of $\mathcal{S}$ that corresponds to $T\mathbf{1} \in \mathsf{ob}\mathcal{A}$.

Find the additional conditions under which $(\mathcal{S}, \Sigma)$ is a model of the assumptions of the Introduction. Then describe how this model is a generalisation of locally compact locales in which finite meets are replaced by certain infinitary ones, the "cardinality" of which is fixed in terms of either traditional set theory or the categorical versions of it in [12, Section 6.3] and [6, Chapter 1].

The category of locales may be defined by a monad of this form on a topos: **Frm** is the category of algebras for $T = \Upsilon(\mathsf{K}(-))$, where $\mathsf{K}(-)$ is the "finite powerset" and $\Upsilon(-)$ the set of upper subsets of a poset [5, Theorem II 1.2]. In order to answer the main question of this paper, it therefore only remains to formulate a further Assumption that forces $\Omega\Sigma N \cong \Upsilon(\mathsf{K}N)$. We shall do that in Section 7.

5 Lattice structure on Σ and Ω

The two presentations of $\mathcal{S}^{\mathsf{op}}$ as categories of algebras over $\mathcal{S}$ or over $\mathcal{E}$ illustrate the way in which abstract Stone duality captures *intrinsic* structure *within* the category $\mathcal{S}$ of "spaces", where traditional developments of topology and of other subjects in mathematics *impose* it on "sets" (objects of $\mathcal{E}$). This is inspired by Marshall Stone's dictum that one should always identify the topology on a mathematical construction, even one that arises from discrete algebra.

Structure is being *imposed* whenever we find a definition such as that "a *widget* is a set equipped with ..." in mathematical discourse, for example when a topological space is defined as a set (of "points") together with a collection of ("open") subsets. I have described this kind of topology as "chipboard" (sawdust plus glue). Imposed structure as a foundation for mathematics is set out canonically in [1].

The use of intrinsic structure was pioneered in synthetic domain theory. In that subject there is, for example, and object ϖ whose definable elements are 0, 1, 2, ..., ∞, which form an ascending sequence according to the intrinsic order. The usual arithmetic order on $\mathbb{N}$, on the other hand, is imposed [11].

Remark 5.1 Indeed, the point at which the dichotomy is most apparent is where we consider the order relations on Σ^X and ΩX. In particular

(a) *all* maps $\Sigma^Y \to \Sigma^X$ preserve the order $\leq$ that is defined by the lattice structure (that is, in the examples of **LKLoc** and $\mathbf{CCD}^{\mathsf{op}}$, but not **Set**, where $\Sigma \equiv \Omega$), whereas

(b) since ΩY and ΩX are merely "sets" (overt discrete objects), maps $\Omega Y \to \Omega X$ may preserve, reverse or ignore the order $\preccurlyeq$ as they please.

In the second case, the order on ΩX is a subobject

$$(\preccurlyeq) \rightarrowtail \Omega X \times \Omega X,$$

but as ΩX is overt discrete, this subobject is necessarily open (Proposition 2.5), and therefore classified by a map

$$(\preccurlyeq) : \Omega X \times \Omega X \longrightarrow \Sigma.$$

The order $\leq$ on Σ^X is also a subspace, but it is neither open nor closed, and therefore has no classifier. Indeed, for $\phi, \psi : \Sigma^X$, the predicate $(\phi \leq \psi)$ is order-reversing in ϕ, so such a classifier would violate the monotonicity property. In the application to recursion theory, this would solve the Halting Problem.

Notation 5.2 Established mathematical usage and the availability of suitable symbols make it impossible to be consistent in the notational distinction between imposed and intrinsic structure, but we shall employ the following correspondence:

intrinsic	Σ	$\mathcal{S}$	$\top$	$\bot$	$\wedge$	$\vee$	$\leq$	$\exists$	$\forall$	$\dashv$
imposed	Ω	$\mathcal{E}$	$\top$	$\bot$	$\curlywedge$	$\curlyvee$	$\preccurlyeq$	$\bigvee$	$\bigwedge$	$\dashv$

Notice that some of the symbols will have to be dis-ambiguated by context. Certain others, notably $=$, $\neq$, $\Rightarrow$, $\neg$ and $\downarrow$, will only occur as imposed structure.

Later in this section we develop the basic results concerning the relationship between the intrinsic structure on Σ^X and the imposed structure on ΩX. But first we re-prove Lemma 1.10, that Σ is a distributive lattice and not merely a $\wedge$-semilattice as was postulated in Theorem 1.3. This gives us some practice in using the type-theoretic notation. However, it also uses the "comprehension" calculus in [B, §8], which we did not set out in Section 3. If you do not have [B] to hand, you should skip the next two lemmas, and take the lattice structure on Σ as an additional assumption.

Lemma 5.3 Σ has a least element, and $\mathbf{0}$ is overt.

Proof We define $\mathbf{0} \overset{i}{\rightarrowtail} \Sigma$ by $E \equiv \lambda F{:}\Sigma^\Sigma.\ \lambda\sigma{:}\Sigma.\ \sigma$. Then by $\{\}E0$ of [B, §8],

$$x : \mathsf{U}\mathbf{0},\ \phi : \Sigma^\Sigma\ \vdash\ \phi(i(\varepsilon x)) = (E\phi)(i(\varepsilon x)) \equiv i(\varepsilon x).$$

Applying this with $\phi = \lambda y.\ \theta$ and $\phi = \lambda y.\ \top$,

$$x : \mathsf{U}\mathbf{0},\ \theta : \Sigma\ \vdash\ \top = (\lambda y.\ \top)(i(\varepsilon x)) = i(\varepsilon x) = (\lambda y.\ \theta)(i(\varepsilon x)) = \theta \leq \theta$$

so, by Definition 3.4, $\theta : \Sigma\ \vdash\ (\exists x{:}\mathsf{U}\mathbf{0}.\ \top)\ \leq\ \theta$. Hence $\bot \equiv (\exists x{:}\mathsf{U}\mathbf{0}.\ \top)$ is the least element of Σ, and $\exists_{\mathbf{0}} \equiv \bot : \Sigma^{\mathbf{0}} \equiv \mathbf{1} \to \Sigma$ is the quantifier that makes $\mathbf{0}$ overt. □

Lemma 5.4 Σ has binary joins, and $\mathbf{2}$ is overt.

Proof We define $\mathbf{2} \overset{i}{\rightarrowtail} \Sigma^{\Sigma\times\Sigma}$ by $E \equiv \lambda\mathcal{F}{:}\Sigma\Sigma(\Sigma\times\Sigma).\ \lambda F{:}\Sigma^{\Sigma\times\Sigma}.\ F(\mathcal{F}\pi_0, \mathcal{F}\pi_1)$ as in [B, Lemma 11.5]. Then for $x : \mathsf{U}\mathbf{2}$, $P \equiv i(\varepsilon x) : \Sigma^{\Sigma\times\Sigma}$ satisfies

$$x : \mathsf{U}\mathbf{2},\ \mathcal{F} : \Sigma\Sigma(\Sigma\times\Sigma)\ \vdash\ P(\mathcal{F}\pi_0, \mathcal{F}\pi_1)\ =\ \mathcal{F}P.$$

In this, consider $\sigma \leq \theta \geq \tau \vdash \mathcal{F} \equiv \lambda F.\ F(\sigma,\tau) \wedge \theta$, so

$$i(\varepsilon x)(\sigma,\tau) \equiv P(\sigma,\tau) \equiv P(\sigma\wedge\theta, \tau\wedge\theta) \equiv P(\mathcal{F}\pi_0, \mathcal{F}\pi_1)\ =\ \mathcal{F}P \equiv P(\sigma,\tau)\wedge\theta \leq \theta,$$

so, by Definition 3.4, $\sigma \leq \theta \geq \tau\ \vdash\ (\exists x{:}\mathsf{U}\mathbf{2}.\ i(\varepsilon x)(\sigma,\tau))\ \leq\ \theta$.

On the other hand, $0, 1 : \mathsf{U}\mathbf{2}$ are $(\tau.\ \mathsf{admit}\ \pi_0)$ and $(\tau.\ \mathsf{admit}\ \pi_1)$. Then

$$\sigma\ =\ \pi_0(\sigma,\tau)\ =\ i(\varepsilon 0)(\sigma,\tau)\ \leq\ \exists x{:}\mathsf{U}\mathbf{2}.\ i(\varepsilon x)(\sigma,\tau).$$

Hence $\sigma \vee \tau \equiv (\exists x{:}\mathsf{U}\mathbf{2}.\ i(\varepsilon x)(\sigma,\tau))$ is the join, and $\exists_{\mathbf{2}} \equiv \vee : \Sigma^{\mathbf{2}} \equiv \Sigma\times\Sigma \to \Sigma$ is the quantifier that makes $\mathbf{2}$ overt. □

Proposition 5.5 Σ is a distributive lattice, and all finitely enumerable objects are overt discrete.

Proof For $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, consider $F(\sigma) = (\sigma \wedge \beta) \vee (\sigma \wedge \gamma)$ in the Euclidean principle (Theorem 1.3). $\square$

Corresponding to this intrinsic structure on the "spaces" Σ and Σ^X, we have imposed structure on the "sets" Ω and ΩX.

Notation 5.6 Since U is a right adjoint, it preserves finite limits and equations, so in particular it takes the internal lattice structure $(\top, \bot, \wedge, \vee)$ on Σ^X to another such structure $(\top, \bot, \curlywedge, \curlyvee)$ on $\Omega X \equiv \mathsf{U}\Sigma^X$. Symbolically,

(a) $\top \equiv \mathsf{U}\top \equiv \tau.\ \lambda x.\ \top : \Omega X$, so $\varepsilon\top = \top : \Sigma^X$;

(b) $\bot \equiv \mathsf{U}\bot \equiv \tau.\ \lambda x.\ \bot : \Omega X$, so $\varepsilon\bot = \bot : \Sigma^X$;

(c) $\curlywedge \equiv \mathsf{U}\wedge : \Omega X \times \Omega X \to \Omega X$ by $\phi \curlywedge \psi \equiv \tau.\ \lambda x.\ (\varepsilon\phi)x \wedge (\varepsilon\psi)x$, so

$$\begin{array}{ccc} \Omega X \times \Omega X & \xrightarrow{\ \curlywedge\ } & \Omega X \\ {\scriptstyle \varepsilon \times \varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ \Sigma^X \times \Sigma^X & \xrightarrow{\ \wedge\ } & \Sigma^X \end{array}$$

(d) $\curlyvee \equiv \mathsf{U}\vee : \Omega X \times \Omega X \to \Omega X$ by $\phi \curlyvee \psi \equiv \tau.\ \lambda x.\ (\varepsilon\phi)x \vee (\varepsilon\psi)x$;

(e) $\preccurlyeq$ by $(\phi \preccurlyeq \psi) \equiv \big((\phi \curlywedge \psi) =_{\Omega X} \phi\big) : \Sigma$, where we recall that ΩX, being discrete, is equipped with an equality test $(=_{\Omega X}) : \Omega X \times \Omega X \to \Sigma$.

We can also define $(\phi \Rightarrow \psi)$ on Ω as $\tau.\ (\phi \preccurlyeq \psi)$, and $\neg\phi \equiv \tau.\ (\phi =_{\Omega} \bot)$; Proposition 6.2 extends these to ΩX.

The imposed order $\preccurlyeq$ on ΩX agrees with the intrinsic pointwise order on Σ^X, in the following sense.

Lemma 5.7 Let $\Gamma \vdash \phi, \theta : \Omega X$, where the context $\Gamma \in \mathsf{ob}\mathcal{E}$ is overt discrete. Then

$$\frac{\Gamma\ \vdash\ (\phi \preccurlyeq \theta) = \top : \Sigma}{\Gamma,\ x : X\ \vdash\ (\varepsilon\phi)x \leq (\varepsilon\theta)x : \Sigma}$$

Proof We may deduce in both directions,

$$\begin{array}{lcll} \Gamma & \vdash & (\phi \preccurlyeq_{\Omega X} \theta)\ =\ \top : \Sigma & \\ \Gamma & \vdash & \big((\phi \curlywedge_{\Omega X} \theta) =_{\Omega X} \phi\big)\ =\ \top : \Sigma & \text{def } \preccurlyeq_{\Omega X} \\ \Gamma & \vdash & \phi \curlywedge_{\Omega X} \theta\ =\ \phi : \Omega X & \text{def } =_{\Omega X} \\ \Gamma & \vdash & \varepsilon(\phi \curlywedge_{\Omega X} \theta)\ =\ \varepsilon\phi : \Sigma^X & \mathsf{\Delta} \dashv \mathsf{U} \\ \Gamma & \vdash & \varepsilon\phi \wedge_{\Sigma^X} \varepsilon\theta\ =\ \varepsilon\phi : \Sigma^X & \text{def } \curlywedge_{\Omega X} \\ \Gamma,\ x : X & \vdash & \varepsilon\phi x \wedge_{\Sigma} \varepsilon\theta x\ =\ \varepsilon\phi x : \Sigma & \text{def } \wedge_{\Sigma^X} \\ \Gamma,\ x : X & \vdash & \varepsilon\phi x\ \leq_{\Sigma}\ \varepsilon\theta x : \Sigma & \text{def } \leq_{\Sigma} \end{array}$$

where the deduction upwards from the line marked "$\mathsf{\Delta} \dashv \mathsf{U}$" relies on the assumption that the context Γ is overt discrete. $\square$

Remark 5.8 The lattice Σ^X also has intrinsic M-indexed joins, for any overt object M. These are given by $\exists m{:}M.\ \phi^m$. The corresponding imposed structure on ΩX is

$$\bigvee \equiv \mathsf{U}\exists_M : (\Omega X)^M \to \Omega X \quad \text{where} \quad \bigvee_{m:M} \psi^m\ \equiv\ \tau.\ \lambda x.\ \exists m{:}M.\ \varepsilon\psi^m x.$$

However, when we use this, we shall need M to be a *dependent type*, given, in traditional comprehension notation (*not* that of [B]), by

$$M \;\equiv\; \{n : N \mid \alpha_n\} \;\subset\; N,$$

where α_n selects the subset of indices n for which $\phi^n : \Sigma^X$ or $\psi^n : \Omega X$ is to contribute to the join. Since N is overt discrete, this subset is necessarily open, and $\alpha_n : \Sigma$. We shall use the sub- and super-script notation here (and in [E]) to indicate that ϕ^n typically varies covariantly and α_n contravariantly with respect to an imposed order on N.

This means that, when using the existential quantifier, we can avoid introducing dependent types by defining

$$\exists n{:}\{n : N \mid \alpha_n\}.\ \phi^n \quad \text{as} \quad \exists n{:}N.\ \alpha_n \wedge \phi^n$$

and, for ΩX,

$$\bigvee_{n:\alpha_n} \psi^n \;\equiv\; \tau.\ \exists n{:}N.\ \alpha_n \wedge \varepsilon\psi^n.$$

Then, when Propositions 7.10 and 7.11 say that $Q : \Sigma\Omega X$ *preserves* this join, they mean that

$$Q(\bigvee_{n:\alpha_n} \psi^n) \;\equiv\; Q(\tau.\ \exists n{:}N.\ \alpha_n \wedge \varepsilon\psi^n) \;=\; \exists n{:}N.\ \alpha_n \wedge Q(\psi^n) \;\equiv\; \exists n{:}\{n : N \mid \alpha_n\}.\ Q(\psi^n).$$

We cannot use the same trick for the universal quantifier, which we shall need for the study of $\mathbf{CCD}^{\mathsf{op}}$, so we shall have to introduce dependent types before considering that case.

6 Some familiar adjoints

In the following constructions, we shall use joins over the dependent subtypes of ΩX classified by

$$\alpha_\theta \;=\; (\phi \curlywedge \theta \preccurlyeq \psi), \qquad (F\theta), \quad \text{and} \quad (f^*\theta \preccurlyeq \phi).$$

These are essentially applications of the adjoint function theorem for the complete lattices ΩX and ΩY. Unfortunately, the cost of the trick that we used in Remark 5.8 to eliminate dependent subtypes is that the proofs of the adjunctions are more difficult.

Notation 6.1 Let $X \in \mathsf{ob}\mathcal{S}$. Define

(a) the ***Heyting implication***, $(\Rightarrow) : \Omega X \times \Omega X \to \Omega X$ by

$$\phi, \psi : \mathsf{U}\Sigma^X \;\vdash\; (\phi \Rightarrow \psi) \;\equiv\; \tau.\ \lambda x.\ \exists \theta{:}\mathsf{U}\Sigma^X.\ \varepsilon\theta x \wedge (\phi \curlywedge \theta \preccurlyeq \psi) : \mathsf{U}\Sigma^X$$

(b) the ***Heyting negation***, $(\neg) : \Omega X \to \Omega X$ by $\neg\phi \equiv (\phi \Rightarrow \bot)$

(c) the ***lower sets***, $\downarrow : \Omega X \to \Omega\Omega X$ by

$$\phi : \mathsf{U}\Sigma^X \;\vdash\; \downarrow\phi \;\equiv\; \tau.\ \lambda\psi{:}\mathsf{U}\Sigma^X.\ (\psi \preccurlyeq \phi) : \mathsf{U}\Sigma^{\mathsf{U}\Sigma^X}$$

(d) the ***join***, $\bigvee : \Sigma^{\Omega X} \to \Sigma^X$ by

$$F : \Sigma^{\mathsf{U}\Sigma^X} \;\vdash\; \bigvee F \;\equiv\; \lambda x{:}X.\ \exists\theta{:}\mathsf{U}\Sigma^X.\ F\theta \wedge (\varepsilon\theta)x : \Sigma^X$$

As we typically form the join of a *lower* subset, F cannot be monotone, or act on Σ^X: in the typical situation, it takes the imposed order $\preccurlyeq$ on ΩX to $\geq$ on Σ.

Proposition 6.2 $(-) \curlywedge \phi \dashv \phi \Rightarrow (-)$ in the sense that

$$\phi, \psi, \theta : \Omega X \;\vdash\; \big((\theta \curlywedge \phi) \preccurlyeq \psi\big) \;=\; \big(\theta \preccurlyeq (\phi \Rightarrow \psi)\big) : \Sigma$$

Proof $\big((\theta \curlywedge \phi) \preccurlyeq \psi\big) \;\le\; \big(\theta \preccurlyeq (\phi \Rightarrow \psi)\big)$ because (using Lemma 3.2)

$$
\begin{array}{lcll}
\Gamma & \vdash & (\theta \curlywedge \phi \preccurlyeq \psi) \;=\; \top & \\
\Gamma,\ x,\ \varepsilon\theta x & \vdash & \varepsilon\theta x \wedge (\theta \curlywedge \phi \preccurlyeq \psi) \;=\; \top & \text{weakening} \\
\Gamma,\ x,\ \varepsilon\theta x & \vdash & \exists\xi.\ \varepsilon\xi x \wedge (\xi \curlywedge \phi \preccurlyeq \psi) \;=\; \top & \text{Def 3.4} \\
\Gamma,\ x & \vdash & \varepsilon\theta x \;\le\; \exists\xi.\ \varepsilon\xi x \wedge (\xi \curlywedge \phi \preccurlyeq \psi) & \text{Lemma 3.1 (*)} \\
\Gamma,\ x & \vdash & \varepsilon\theta x \;\le\; \varepsilon(\phi \Rightarrow \psi)x & \text{def} \Rightarrow \\
\Gamma & \vdash & \big(\theta \preccurlyeq (\phi \Rightarrow \psi)\big) \;=\; \top & \text{Lemma 5.7}
\end{array}
$$

where the last three deductions are reversible, $\theta, \phi, \psi : \Omega X \equiv \mathsf{U}\Sigma^X$ and $x : X$.

For the converse we first need

$$
\begin{array}{lcll}
\Gamma,\ \xi,\ x,\ (\phi \curlywedge \xi \preccurlyeq \psi) & \vdash & \varepsilon\phi x \wedge \varepsilon\xi x \;\le\; \varepsilon\psi x & \text{Notation 5.6, Lemma 5.7} \\
\Gamma,\ \xi,\ x,\ \varepsilon\phi x & \vdash & \varepsilon\xi x \wedge (\phi \curlywedge \xi \preccurlyeq \psi) \;\le\; \varepsilon\psi x & \text{Lemma 3.1 twice} \\
\Gamma,\ x,\ \varepsilon\phi x & \vdash & \big(\exists\xi.\ \varepsilon\xi x \wedge (\xi \curlywedge \phi \preccurlyeq \psi)\big) \;\le\; \varepsilon\psi x & \text{Def 3.4}
\end{array}
$$

Combining this with (*), which is equivalent to $\theta \preccurlyeq (\phi \Rightarrow \psi)$, we deduce

$$
\begin{array}{lcll}
\Gamma,\ x,\ \varepsilon\phi x & \vdash & \varepsilon\theta x \;\le\; \exists\xi.\ \varepsilon\xi x \wedge (\xi \curlywedge \phi \preccurlyeq \psi) \;\le\; \varepsilon\psi x & \\
\Gamma,\ x & \vdash & \varepsilon\phi x \wedge \varepsilon\theta x \;\le\; \varepsilon\psi x & \text{Lemma 3.1} \\
\Gamma & \vdash & (\phi \curlywedge \theta \preccurlyeq \psi) \;=\; \top & \text{Notation 5.6, Lemma 5.7} \;\square
\end{array}
$$

Proposition 6.3 $\mathsf{U}\bigvee \dashv {\downarrow}$ in the sense that

$$\phi : \Omega X,\ F : \Omega\Omega X \;\vdash\; (\tau.\ \textstyle\bigvee \varepsilon F \preccurlyeq \phi) \;=\; (F \preccurlyeq {\downarrow}\phi) : \Sigma$$

Proof We may deduce in either direction as follows:

$$
\begin{array}{lcll}
\Gamma & \vdash & (\tau.\ \bigvee \varepsilon F \preccurlyeq \phi) \;=\; \top : \Sigma & \\
\Gamma,\ x & \vdash & \exists\theta.\ (\varepsilon F)\theta \wedge (\varepsilon\theta)x \;\le\; (\varepsilon\phi)x : \Sigma & \text{Lemma 5.7} \\
\Gamma,\ x & \vdash & (\varepsilon F)\theta \wedge (\varepsilon\theta)x \;\le\; (\varepsilon\phi)x : \Sigma & \text{Def 3.4} \\
\Gamma,\ (\varepsilon F\theta),\ x & \vdash & (\varepsilon\theta)x \;\le\; (\varepsilon\phi)x : \Sigma & \text{Lemma 3.1} \\
\Gamma,\ (\varepsilon F\theta) & \vdash & (\theta \preccurlyeq \phi) \;=\; \top : \Sigma & \text{Lemma 5.7} \\
\Gamma & \vdash & (\varepsilon F\theta) \;\le (\theta \preccurlyeq \phi) : \Sigma & \text{Lemma 3.1} \\
\Gamma & \vdash & (F \preccurlyeq {\downarrow}\phi\theta) \;=\; \top : \Sigma & \text{Lemma 5.7}
\end{array}
$$

where $\phi, \theta : \Omega X \equiv \mathsf{U}\Sigma^X$ and $F : \Omega\Omega X$. The second and last lines use the definitions of $\bigvee$ and $\downarrow$. $\square$

Theorem 6.4 For each space $X \in \mathsf{ob}\mathcal{S}$, ΩX carries the imposed structure of a complete Heyting algebra internally to the topos $\mathcal{E}$. $\square$

Notation 6.5 Let $f : X \to Y$ in $\mathcal{S}$. Define

(a) the ***inverse image*** $f^* \equiv \Omega f \equiv \mathsf{U}\Sigma^f : \Omega Y \to \Omega X$ by

$$\psi : \mathsf{U}\Sigma^Y \;\vdash\; f^*\psi \;\equiv\; \tau.\ \lambda x{:}X.\ (\varepsilon\psi)(fx) : \mathsf{U}\Sigma^X$$

(b) and the ***direct image*** $f_* : \Omega X \to \Omega Y$ by

$$\phi : \mathsf{U}\Sigma^X \;\vdash\; f_*\phi \;\equiv\; \tau.\ \lambda y{:}Y.\ \exists\theta{:}\mathsf{U}\Sigma^Y.\ (\varepsilon\theta)y \wedge (f^*\theta \preccurlyeq \phi) : \mathsf{U}\Sigma^Y.$$

Proposition 6.6 $f^* \dashv f_*$ in the sense of $\preccurlyeq$.

Proof To show $(f^*\psi \preccurlyeq \phi) \le (\psi \preccurlyeq f_*\phi)$,

$$\begin{array}{lll}
\Gamma & \vdash (f^*\psi \preccurlyeq \phi) \;=\; \top : \Sigma & \\
\Gamma,\ y,\ (\varepsilon\psi y) & \vdash (\varepsilon\psi y) \wedge (f^*\psi \preccurlyeq \phi) \;=\; \top : \Sigma & \text{weakening} \\
\Gamma,\ y,\ (\varepsilon\psi y) & \vdash \exists\theta.\ (\varepsilon\theta y) \wedge (f^*\theta \preccurlyeq \phi) \;=\; \top : \Sigma & \text{Def 3.4} \\
\Gamma,\ y & \vdash (\varepsilon\psi y) \le \exists\theta.\ (\varepsilon\theta y) \wedge (f^*\theta \preccurlyeq \phi) : \Sigma & \text{Lemma 3.1} \\
\Gamma & \vdash \psi \preccurlyeq \tau.\ \lambda y.\ \exists\theta{:}\mathsf{U}\Sigma^Y.\ (\varepsilon\theta y) \wedge (f^*\theta \preccurlyeq \phi) & \text{Lemma 5.7} \\
\Gamma & \vdash \psi \preccurlyeq f_*\phi & \text{def } f_*
\end{array}$$

where $\phi : \mathsf{U}\Sigma^X$ and $\psi, \theta : \mathsf{U}\Sigma^Y$. Conversely, $\psi \preccurlyeq f_*\phi$ means

$$\begin{array}{lll}
\Gamma,\ y & \vdash \varepsilon\psi y \le \varepsilon(f_*\phi)y : \Sigma & \text{Lemma 5.7} \\
\Gamma,\ x & \vdash (\varepsilon\psi)(fx) \le \varepsilon(f_*\phi)(fx) : \Sigma, & \text{substitution}
\end{array}$$

where $x : X$ and $y : Y$, so we need

$$\begin{array}{lll}
\Gamma,\ \theta,\ (f^*\theta \preccurlyeq \phi),\ x & \vdash \varepsilon(f^*\theta)x \;\equiv\; (\varepsilon\theta)(fx) \;\le\; \varepsilon\phi x : \Sigma & \text{Lemma 5.7} \\
\Gamma,\ \theta,\ x & \vdash (f^*\theta \preccurlyeq \phi) \wedge (\varepsilon\theta)(fx) \;\le\; \varepsilon\phi x : \Sigma & \text{Lemma 3.1} \\
\Gamma,\ x & \vdash f_*\phi x \;\equiv\; \exists\theta.\ (f^*\theta \preccurlyeq \phi) \wedge (\varepsilon\theta)(fx) \;\le\; \varepsilon\phi x & \text{Def 3.4}
\end{array}$$

Hence

$$\begin{array}{lll}
\Gamma,\ x & \vdash (\varepsilon\psi)(fx) \;\le\; f_*\phi x \;\le\; \varepsilon\phi x : \Sigma & \\
\Gamma,\ x & \vdash (\varepsilon\psi)(fx) \;\le\; \varepsilon\phi x : \Sigma & \\
\Gamma & \vdash (f^*\psi \equiv \tau.\ \lambda x.\ (\varepsilon\psi)(fx) \preccurlyeq \phi) \;=\; \top : \Sigma & \text{Lemma 5.7 } \square
\end{array}$$

The direct and inverse image maps for $\varepsilon : \mathsf{U}X \to X$ relate this abstract theory to more traditional accounts of general topology.

Definition 6.7 We say that $X \in \mathsf{ob}\mathcal{S}$ is ***spatial*** (in a different usage from [B, Definition 3.1]) if the following rule is valid for $\phi, \theta : \Sigma^X$:

$$\frac{p : \mathsf{U}X \;\vdash\; \theta(\varepsilon p) \le \phi(\varepsilon p) : \Sigma}{x : X \;\vdash\; \theta x \le \phi x : \Sigma}$$

This means that there are *enough points* p in the underlying set $\mathsf{U}X$ to distinguish ϕ from θ as open subsets of X. Equivalently, $\Sigma^\varepsilon : \Sigma^X \to \Sigma^{\mathsf{U}X}$ is mono; by sobriety, this makes ε itself epi (or Σ-epi in the language of synthetic domain theory).

As all spaces are spatial in traditional topology, we see (using [C, Lemma 7.2]) why the concept of overtness was not recognised:

Corollary 6.8 All spatial objects are overt, with $\exists x{:}X.\ \phi x \equiv \exists p{:}\mathsf{U}X.\ \phi(\varepsilon p)$. $\square$

Lemma 6.9 X is spatial iff $\varepsilon_* \cdot \varepsilon^* = \mathsf{id}_{\Omega X}$.

$$\begin{array}{ccc}
\mathsf{U}X & \qquad & \Omega\mathsf{U}X \\
\varepsilon \big\downarrow & & \varepsilon^* \big\uparrow \dashv \big\downarrow \varepsilon_* \\
X & & \Omega X
\end{array}$$

Proof Σ^ε is mono iff ε^* is, and this happens iff $\varepsilon_* \cdot \varepsilon^* = \mathsf{id}$, as $\varepsilon_* \cdot \varepsilon^* \cdot \varepsilon_* = \varepsilon_*$ for any adjunction. Alternatively, Lemma 5.7 says that the top and bottom of the spatiality rule (with $\varepsilon\theta$ and $\varepsilon\phi$ instead of θ and ϕ) are $\varepsilon^*\theta \preccurlyeq \varepsilon^*\phi$ and $\theta \preccurlyeq \phi$ respectively. So

$$\varepsilon_*(\varepsilon^*\phi) \;\equiv\; \tau.\ \lambda x{:}X.\ \exists\theta{:}\Omega X.\ \varepsilon\theta x \wedge (\varepsilon^*\theta \preccurlyeq \varepsilon^*\phi)$$

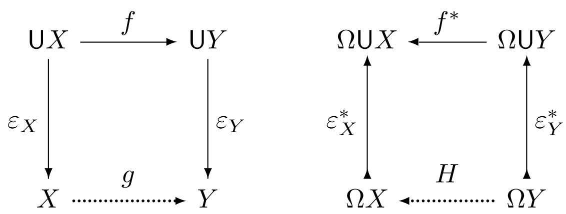
may be simplified to $\tau.\ \lambda x.\ \varepsilon\psi x = \psi$. $\square$

Lemma 6.10 Σ^Y is spatial.

Proof By Lemma 4.3, $\Sigma\varepsilon\Sigma Y$ is split by δ, so $\mathsf{U}\delta\cdot\varepsilon^* = \mathsf{U}(\delta\cdot\Sigma^\varepsilon) = \mathsf{id}$. Hence $\varepsilon_*\cdot\varepsilon^* = \mathsf{U}\delta\cdot\varepsilon^*\cdot\varepsilon_*\cdot\varepsilon^* = \mathsf{U}\delta\cdot\varepsilon^* = \mathsf{id}$. □

Remark 6.11 The traditional explanation of all this is that $\Omega\mathsf{U}X$ is the powerset (the collection of *all* subsets of the points of X), whilst ΩX is the topology (the sub-collection of *open* subsets). The map ε^* includes one lattice in the other, whilst its right adjoint ε_* yields the open ***interior*** of an arbitrary subset.

A ***continuous map*** $g : X \to Y$ in $\mathcal{S}$ is a function $f \equiv \mathsf{U}g : \mathsf{U}X \to \mathsf{U}Y$ between the underlying sets whose inverse image f^* preserves openness. The latter means that there is a (unique) map H making the right hand square commute:



It is not difficult to show that, as ε^*_X, ε^*_Y and f^* are Eilenberg–Moore homomorphisms for the monad on $\mathcal{E}$, so is H, and therefore $H = g^*$ for some unique $g : X \to Y$. □

7 Continuous maps in topology

We now specialise to the topological example, **LKLoc**, treating the morphisms in this section and the objects in the next. By Theorem 4.11, we just have to find a convenient way of stating that the monad on $\mathcal{E}$ has $\Omega\Sigma N \cong \Upsilon(\mathsf{K}N, \preccurlyeq)$, or that the homomorphisms preserve *finite* meets and arbitrary joins. But, to do this, we have to employ a more formal definition of finiteness than the fact that it is generated from 0 and 2.

Definition 7.1 For $N \in \mathsf{ob}\mathcal{E}$, $\mathsf{K}N \in \mathsf{ob}\mathcal{E}$ is the free semilattice on N in $\mathcal{E}$.

This may be constructed in any elementary topos (in fact, without even the need for a natural numbers object) as the sub-$\curlywedge$-semilattice of ΩN generated by the singletons [8, Appendix 2] [4, Theorem 9.16] [12, Proposition 6.6.11]. The semilattice structure, for which we write $\curlyvee$ and $\preccurlyeq$, is imposed, not intrinsic.

Proposition 7.2 $\mathsf{K}N$ is overt discrete and is the free (imposed) semilattice on N in $\mathcal{S}$.

Proof It was defined as a subobject of ΩN, so is overt discrete.

If $(S, 0, +)$ is a semilattice in $\mathcal{S}$ (imposed or intrinsic) and $f : N \to S$ then $(\mathsf{U}S, \mathsf{U}0, \mathsf{U}+)$ is an (imposed) semilattice in $\mathcal{E}$ with $\tau.\ f : N \to \mathsf{U}S$. This extends to a semilattice homomorphism $\mathsf{K}N \to \mathsf{U}S \to S$. It is unique because if $g : \mathsf{K}N \to S$ is a semilattice homomorphism then so is $\tau.\ g : \mathsf{K}N \to \mathsf{U}S$, and this must agree with the extension of $\tau.\ f$. □

Notation 7.3 For $\ell : \mathsf{K}N$, the expressions

$$\lambda n.\ n \in \ell : \Sigma^N, \qquad \phi : \Sigma^N \vdash \exists n \in \ell.\ \phi n : \Sigma \quad \text{and} \quad \phi : \Sigma^N \vdash \forall n \in \ell.\ \phi n : \Sigma$$

are defined as the unique semilattice homomorphisms from $\mathsf{K}N$ to $(\Sigma^N, \bot, \vee)$, $(\Sigma^{\Sigma^N}, \bot, \vee)$ and $(\Sigma^{\Sigma^N}, \top, \wedge)$ that extend $m \mapsto \lambda n.\ (n =_N m)$ and $m \mapsto \lambda\phi.\ \phi m$ (twice). In particular,

$$(\ell \preccurlyeq \ell') = (\forall n \in \ell.\ n \in \ell').$$

We are now ready to state the axiom that characterises the topological case in the context of the assumptions in the Introduction.

Assumption 7.4 For $N \in \mathsf{ob}\mathcal{E}$, $F : \Sigma\Sigma N$ and $\phi : \Sigma N$,

$$F\phi \;=\; \exists \ell{:}\mathsf{K}N.\ F(\lambda n.\ n \in \ell) \wedge \forall n \in \ell.\ \phi n$$

or, in dependent type notation (Remark 5.8),

$$F\phi \;=\; \exists \ell{:}\{\ell : \mathsf{K}N \mid F(\lambda n.\ n \in \ell)\}.\ \forall n \in \ell.\ \phi n.$$

Corollary 7.5 With $N = \mathbf{1}$, so $\mathsf{K}N = \{\mathbf{0}, \mathbf{1}\}$, we obtain the ***Phoa principle*** [C, §5],

$$F : \Sigma^{\Sigma},\ \sigma : \Sigma \ \vdash\ F\sigma \;=\; F\bot \vee \sigma \wedge F\top.$$

In particular, F is monotone, and the dual Euclidean principle,

$$\sigma \vee F\sigma \;=\; \sigma \vee F\bot,$$

also holds. By Theorem 1.3, this means that $\bot : \mathbf{1} \to \Sigma$ is a dominance, the subspaces that it classifies being called ***closed***. □

Lemma 7.6 $$\Sigma\Sigma N \underset{\lambda\phi.\ \exists \ell:\mathsf{K}N.\ G\ell \wedge \forall n \in \ell.\ \phi n \;\leftarrow\!\shortmid\; G}{\overset{F \;\mapsto\; \lambda\ell.\ F(\lambda n.\ n \in \ell)}{\rightleftarrows}} \Sigma\mathsf{K}N.$$

Proof We recover $F \mapsto G \mapsto F$ by Assumption 7.4. □

Notation 7.7 For any overt discrete object S equipped with an imposed order relation $\preccurlyeq$, we write $\Upsilon(S, \preccurlyeq) \subset \Omega S$ for the $\mathcal{E}$-object of upper subsets, or the functions $S \to \Omega$ that take $\preccurlyeq$ to the intrinsic order $\leq$ on Σ.

Proposition 7.8 $$\Omega\Sigma N \underset{\tau.\ \lambda\phi.\ \exists \ell:\mathsf{K}N.\ \varepsilon G\ell \wedge \forall n \in \ell.\ \phi n \;\leftarrow\!\shortmid\; G}{\overset{F \;\mapsto\; \tau.\ \lambda\ell.\ \varepsilon F(\lambda n.\ n \in \ell)}{\cong}} \Upsilon(\mathsf{K}N, \preccurlyeq).$$

Proof Since $\ell \preccurlyeq \ell' \Rightarrow (\lambda n.\ n \in \ell) \leq (\lambda n.\ n \in \ell'))$ and εF is monotone with respect to the intrinsic order, G takes the imposed order $\preccurlyeq$ to $\leq$ in Σ, so $G \in \Upsilon(\mathsf{K}N, \preccurlyeq)$. Then we recover $F \mapsto G \mapsto F$ by Assumption 7.4 as in the Lemma. Conversely,

$$G \mapsto F \mapsto \tau.\ \lambda\ell'.\ \exists\ell.\ \varepsilon G\ell \wedge \forall n \in \ell.\ n \in \ell',$$

which is G because only those $\varepsilon G\ell$ with $\ell \preccurlyeq \ell'$ contribute to the union, and $\varepsilon G\ell'$ is the greatest term by (imposed) monotonicity of G. □

Theorem 7.9 The Eilenberg–Moore category for the monad on $\mathcal{E}$ is the category of frames and their homomorphisms over $\mathcal{E}$.

Proof $\Upsilon(\mathsf{K}N, \preccurlyeq)$ is the free frame on N as constructed in [5, Theorem II 1.2]. □

Alternatively, we may use Assumption 7.4 directly to recover f from Q satisfying the last condition.

Proposition 7.10 $x : X \vdash Q \equiv \lambda\psi{:}\Omega Y.\ (\varepsilon\psi)(fx)$ preserves finite meets and arbitrary joins, in the following sense:

Proof Let $n : N \vdash \alpha_n : \Sigma$, $\psi^n : \Omega X$, where N is overt; then by Remark 5.8,

$$Q(\tau.\ \exists n{:}N.\ \alpha_n \wedge \varepsilon\psi^n) \;=\; \exists n{:}N.\ \alpha_n \wedge (\varepsilon\psi^n)(fx) \;=\; \exists n{:}N.\ \alpha_n \wedge Q(\psi^n).$$

Also $Q(\top) = (\varepsilon\top)(fx) = \top(fx) = \top$, and, using Notation 5.6,

$$Q(\phi \curlywedge \psi) \;=\; \varepsilon(\phi \curlywedge \psi)(fx) \;=\; (\varepsilon\phi \wedge \varepsilon\psi)(fx) \;=\; (\varepsilon\phi)(fx) \wedge (\varepsilon\psi)(fx) \;=\; Q\phi \wedge Q\psi. \qquad \square$$

Proposition 7.11 $\Gamma \vdash Q : \Sigma\Omega Y$ is Ω-prime, *i.e.*

$$\Gamma,\ \mathcal{G} : \Omega\Sigma\Omega Y \ \vdash\ \varepsilon\mathcal{G}Q \ = \ Q\big(\tau.\ \lambda y{:}Y.\ (\varepsilon\mathcal{G})(\lambda\psi{:}\Omega Y.\ \varepsilon\psi y)\big)$$

iff it preserves $\top$, $\curlywedge$ and $\bigvee$ in this sense.

Proof Put $N \equiv \Omega Y \in \mathsf{ob}\mathcal{E}$ and expand $\varepsilon\mathcal{G} : \Sigma\Sigma\Omega Y \equiv \Sigma^{\Sigma^N}$ using Assumption 7.4:

$$F : \Sigma^N \ \vdash\ \varepsilon\mathcal{G}F \ = \ \exists\ell.\ (\varepsilon\mathcal{G})(\lambda\psi.\ \psi \in \ell) \wedge \forall\psi \in \ell.\ F\psi.$$

Then

$$(\varepsilon\mathcal{G})(\lambda\psi{:}\Omega Y.\ \varepsilon\psi y) \ = \ \exists\ell{:}\mathsf{K}N.\ \varepsilon\mathcal{G}(\lambda\psi.\ \psi \in \ell) \wedge \forall\psi \in \ell.\ \varepsilon\psi y$$

so with

$$\beta^\ell \equiv \tau.\ \lambda y.\ \forall\psi \in \ell.\ \varepsilon\psi y = \bigwedge_{\psi\in\ell} \psi : \Omega Y$$

and $\alpha_\ell \equiv \varepsilon\mathcal{G}(\lambda\psi.\ \psi \in \ell) : \Sigma$, we have

$$\begin{aligned}
RHS &= Q\big(\tau.\ \lambda y.\ \exists\ell{:}\mathsf{K}N.\ \varepsilon\mathcal{G}(\lambda\psi.\ \psi \in \ell) \wedge \forall\psi \in \ell.\ \varepsilon\psi y\big) \\
&= Q(\tau.\ \exists\ell.\ \alpha_\ell \wedge \varepsilon\beta^\ell) \\
&= \exists\ell.\ \alpha_\ell \wedge Q(\beta^\ell) && Q \text{ preserves joins} \\
&= \exists\ell.\ \alpha_\ell \wedge Q(\bigwedge_{\psi\in\ell} \psi) \\
&= \exists\ell.\ \alpha_\ell \wedge \forall\psi \in \ell.\ Q(\psi) && Q \text{ preserves meets} \\
&= \exists\ell.\ \varepsilon\mathcal{G}(\lambda\psi.\ \psi \in \ell) \wedge \forall\psi \in \ell.\ Q(\psi) \\
&= \varepsilon\mathcal{G}Q \ = \ LHS && \square
\end{aligned}$$

Corollary 7.12 The morphisms of $\mathcal{S}$ coincide with continuous functions as variously formulated in traditional topology, locale theory and abstract Stone duality, being in natural bijection with

(a) Eilenberg–Moore homomorphisms $\Sigma^f : \Sigma^Y \to \Sigma^X$ for the monad on $\mathcal{S}$ arising from the adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$;

(b) Σ-primes $x : X \vdash \lambda\psi.\ \psi(fx) : \Sigma\Sigma Y$;

(c) Eilenberg–Moore homomorphisms $f^* : \Omega Y \to \Omega X$ for the monad on $\mathcal{E}$ arising from the adjunction $\Sigma \dashv \Omega$;

(d) Ω-primes $x : X \vdash \lambda\psi.\ (\varepsilon\psi)(fx) : \Sigma\Omega Y$;

(e) adjoint pairs $f^* \dashv f_*$ where f^* also preserves $\top$ and $\curlywedge$ in the imposed lattice structure;

(f) frame homomorphisms f^*, where f^* preserves $\top$, $\curlywedge$ and $\bigvee$;

(g) $x : X \vdash P \equiv \lambda\phi.\ \phi(fx) : \Sigma\Sigma Y$ preserving $\top$, $\bot$, $\wedge$ and $\vee$ (Proposition 8.5);

(h) lattice homomorphisms $H : \Sigma^Y \to \Sigma^X$;

(i) $x : X \vdash Q \equiv \lambda\psi.\ (\varepsilon\psi)(fx) : \Sigma\Omega Y$ preserving $\top$, $\curlywedge$ and $\bigvee$;

(j) functions $\mathsf{U}X \to \mathsf{U}Y$ for which the inverse image of any open subset of Y is open in X, in the case where X is spatial (Remark 6.11) $\square$

We therefore turn to the objects of $\mathcal{S}$.

8 Locally compact locales

It remains to characterise the spaces X as locally compact locales, or the algebras ΩX as continuous distributive lattices, *i.e.* to give yet another proof of the monadic property for **LKLoc** [A, Theorem 5.12] [B, Theorem 3.11]. The underlying set functor U is barely used in this section, and the arguments are much more like those in [E], so will probably be moved to that paper. In particular, we need to extend the basis expansion of $F : \Sigma\Sigma N$ in Assumption 7.4 to other types, using the string of retractions

$$\Sigma\Sigma X \lhd \Sigma\Sigma\Sigma N \lhd \Sigma\Sigma\mathsf{K}N \lhd \Sigma\mathsf{KK}N.$$

Several of the proofs have been omitted for the APPSEM proceedings, but will be found in full in the draft on the ASD Web page.

Lemma 8.1 For $\mathcal{F} : \Sigma\Sigma\Sigma N$ and $F : \Sigma\Sigma N$,

$$\mathcal{F}F \;=\; \exists L{:}\mathsf{KK}N.\;\; \mathcal{F}A_L \land \forall \ell \in L.\; F\beta^\ell$$

where $A_L \equiv \lambda\phi{:}\Sigma^N.\; \exists \ell \in L.\; \forall n \in \ell.\; \phi n$ and $\beta^\ell \equiv \lambda n.\; n \in \ell$. □

Proposition 8.2 Let $i : X \rightarrowtail \Sigma^N$ with Σ-splitting I, $G : \Sigma\Sigma X$ and $\psi : \Sigma X$. Then

$$G\psi \;=\; \exists L{:}\mathsf{KK}N.\;\; G\gamma^L \land D_L\psi,$$

where $\gamma^L = \lambda x.\; \exists \ell \in L.\; \forall n \in \ell.\; ixn$ and $D_L = \lambda\psi.\; \forall \ell \in L.\; I\psi(\lambda n.\; n \in \ell)$. □

Definition 8.3 $\Gamma,\; s : S \vdash \phi^s : \Sigma^N$ is called a ***directed diagram*** if S is overt discrete with an imposed *semilattice* structure $(S, 0, +)$ with respect to which ϕ^s is covariant:

$$\Gamma,\; s, t : S \;\vdash\; \phi^s \leq \phi^{s+t} : \Sigma^N.$$

As in Remark 5.8, we need to consider, more generally,

$$\exists s{:}\{s : S \mid \alpha_s\}.\; \phi^s \;\equiv\; \exists s{:}S.\; \alpha_s \land \phi^s$$

where $\alpha_s : \Sigma$ is contravariant. The subtype remains a semilattice so long as

$$\alpha_0 = \top \quad \text{and} \quad \alpha_{s+t} = \alpha_s \land \alpha_t.$$

Example 8.4 In particular, $\alpha_L \equiv D_L\phi$ satisfies this, where $S \equiv \mathsf{KK}N$ and $\phi : \Sigma^X$, so

$$\begin{aligned} \phi x &= \exists L.\; D_L\phi \land \gamma^L x \\ G\phi &= \exists L.\; D_L\phi \land G\gamma^L \end{aligned}$$

are directed joins. □

Lemma 8.5 All $\Gamma \vdash G : \Sigma\Sigma X$ preserve directed joins. □

Corollary 8.6 $\Gamma \vdash P : \Sigma\Sigma\Omega Y$ is prime iff it preserves the *finitary* lattice operations. □

Proposition 8.7 $F : \Omega Y \to \Omega X$ in $\mathcal{E}$ is $\mathsf{U}G$ for some $G : \Sigma^Y \to \Sigma^X$ iff F preserves directed joins, and in this case G is unique.

Proof If $F = \mathsf{U}G$ then, for $s : S \vdash \theta^s : \Omega Y$,

$$(\mathsf{U}G) \bigvee_{s:\alpha_s}^{\uparrow} \theta^s \;=\; \tau.\; G(\exists s.\; \alpha_s \land \varepsilon\theta^s) \;=\; \tau.\; \exists s.\; \alpha_s \land G(\varepsilon\theta^s) \;=\; \bigvee_{s:\alpha_s}^{\uparrow} (\mathsf{U}G)(\theta^s).$$

Conversely, given F, we obtain

$$G \equiv \lambda\phi{:}\Sigma^Y.\ \exists L.\ \varepsilon(F(\tau.\ \gamma^L)) \wedge D_L\phi,$$

with which, so long as F preserves the directed join in Examples 8.4, for $\psi : \Omega Y$:

$$\begin{aligned}
\varepsilon\psi x &= \exists L.\ D_L(\varepsilon\psi) \wedge \gamma^L x \\
\psi &= \bigvee^{\uparrow}_{L:D_L(\varepsilon\psi)} \tau.\ \gamma^L \\
F\psi &= \bigvee^{\uparrow}_{L:D_L(\varepsilon\psi)} F(\tau.\ \gamma^L) \\
&= \tau.\ \exists L.\ D_L(\varepsilon\psi) \wedge \varepsilon(F(\tau.\ \gamma^L)x) \\
&= \tau.\ G(\varepsilon\psi) \ = \ \mathsf{U}G\psi
\end{aligned}$$

Notation 3.6 allows us to write $\tau.\ \gamma^L$ because $L : \mathsf{KK}N \in \mathsf{ob}\mathcal{E}$ is the only free variable[1] in γ^L, whereas $G = \lambda\phi{:}\Sigma^Y.\ F(\tau.\ \phi)$ would not be well formed because the type of the variable ϕ is not overt discrete. □

Corollary 8.8 K is compact iff $\Sigma^{!_K}$ has a right adjoint [C, Definition 7.7].

Proof For any (locally compact) locale K, the $\mathcal{E}$-map $!_K^* \equiv \Omega !_K \equiv \mathsf{U}\Sigma^{!_K} : \Omega \to \Omega K$ has a right adjoint $!_*$ with respect to the imposed order (Proposition 6.6). This preserves directed joins iff K is compact in the sense of locale theory [5, §III 1]. In this case, $!_* = \mathsf{U}A$ for some unique $A : \Sigma^K \to \Sigma$, so

$$\mathsf{id} \preccurlyeq\ !^* \cdot !_* = \mathsf{U}(\Sigma^i \cdot A) \quad \text{and} \quad \mathsf{U}(A \cdot \Sigma^i) = !_* \cdot !^* \preccurlyeq \mathsf{id}.$$

These imposed inequalities may be lifted to intrinsic ones as Σ^K is spatial, so $\Sigma^i \dashv A$ in the intrinsic order. □

Now we return to the world of infinitary lattices to obtain a suitable characterisation of distributive continuous lattices, which are the topologies on locally compact spaces.

Lemma 8.9 Let $(A, \preccurlyeq)$ be an (imposed) lattice in any topos $\mathcal{E}$. Then it is a continuous [3] frame iff there are functions

$$A \underset{H}{\overset{J}{\rightleftarrows}} \Omega N$$

with $N \in \mathsf{ob}\mathcal{E}$ such that $H \cdot J = \mathsf{id}_A$, J preserves directed joins and H preserves finite meets and arbitrary joins. □

Corollary 8.10 ΩX is a continuous frame.

Proof Let $i : X \rightarrowtail \Sigma^N$ with Σ-splitting I by the basis Assumption 1.7. Then $H \equiv i^* \equiv \mathsf{U}\Sigma^i$ preserves finite meets and arbitrary joins (Proposition 7.10), whilst $J \equiv \mathsf{U}I$ preserves directed joins (Proposition 8.7). □

[1]The inclusion $i : X \rightarrowtail \Sigma^N$ is also involved in the definition of γ^L, but i is a closed term as we haven't allowed dependent types.

For the converse, we use the idempotent $E = I \cdot \Sigma^i$ on $\Sigma\Sigma N$ or $E' = \mathsf{U}I = J \cdot H$ on $\Omega\Sigma N$, for which the term *nucleus* was appropriated in [B, §2]. We have to show that the properties of H and J in Lemma 8.9 provide the hypotheses of Beck's theorem characterising monads.

Lemma 8.11 Let H and J be functions between semilattices such that $H \cdot J = \mathsf{id}$ and H preserves $\curlywedge$. Then $E \equiv J \cdot H$ satisfies $E(\phi \curlywedge \psi) = E(E\phi \curlywedge E\psi)$. □

Lemma 8.12 If $E : \Sigma^X \to \Sigma^X$ satisfies

$$\phi, \psi : \Sigma^X \ \vdash\ E(\phi \wedge \psi) \ = \ E(E\phi \wedge E\psi) \quad \text{and} \quad E(\phi \vee \psi) = E(E\phi \vee E\psi)$$

then it is a nucleus in the sense of [B, Definition 4.3], *i.e.*

$$\mathcal{F} : \Sigma\Sigma\Sigma X \ \vdash\ E\big(\lambda x.\ \mathcal{F}(\lambda\phi.\ E\phi x)\big) \ = \ E\big(\lambda x.\ \mathcal{F}(\lambda\phi.\ \phi x)\big).$$

Proof Note first that $E\phi = E(E\phi)$ and $E(E\phi \vee E\psi \vee \theta) = E(\phi \vee \psi \vee \theta)$ (*sic*). Although we needn't have $E\top = \top$ or $E\bot = \bot$, we may extend the binary $\vee$-formula to finite (possibly empty) sets $\ell : \mathsf{K}N$:

$$E(\exists n \in \ell.\ \alpha_n \wedge \phi^n) \ = \ E(\exists n \in \ell.\ \alpha_n \wedge E\phi^n),$$

where $n : N \vdash \alpha_n : \Sigma$ and $\phi^n : \Sigma^X$. Similarly but more simply, from the $\wedge$-equation,

$$E(\forall n \in \ell.\ \phi^n) \ = \ E(\forall n \in \ell.\ E\phi^n).$$

The $\exists$ equation extends by Scott continuity, (Proposition 8.5), Now we can expand $\mathcal{F}$ using Proposition 8.2.

$$\begin{array}{rcll}
E\big(\lambda x.\ \mathcal{F}(\lambda\phi.\ E\phi x)\big) & = & E\big(\exists L.\ \mathcal{F}A_L \wedge \forall \ell \in L.\ E\beta^\ell\big) & \text{Proposition 8.2} \\
& = & E\big(\exists L.\ \mathcal{F}A_L \wedge E(\forall \ell \in L.\ E\beta^\ell)\big) & \exists \text{ equation} \\
& = & E\big(\exists L.\ \mathcal{F}A_L \wedge E(\forall \ell \in L.\ \beta^\ell)\big) & \forall \text{ equation} \\
& = & E\big(\exists L.\ \mathcal{F}A_L \wedge \forall \ell \in L.\ \beta^\ell\big) & \exists \text{ equation} \\
& = & E\big(\lambda x.\ \mathcal{F}(\lambda\phi.\ \phi x)\big) & \text{Proposition 8.2}
\end{array}$$ □

Proposition 8.13 Every distributive continuous lattice A arises as some ΩX.

Proof By Lemma 8.9, A gives H and J and so $E' = J \cdot H$ on $\Omega\Sigma A$. As this preserves directed joins, Proposition 8.7 gives E on $\Sigma\Sigma A$ with $E' = \mathsf{U}E$.

$$\Sigma\Sigma A \xrightarrow{\ \Sigma^i\ } \Sigma^X \xrightarrow{\ I\ } \Sigma\Sigma A$$

$$\Omega\Sigma A \xrightarrow{\ H\ } A \xrightarrow{\ J\ } \Omega\Sigma A$$

Using Lemma 6.10, E inherits the property of Lemma 8.11 from E', and then Lemma 8.12 applies. Hence E defines a Σ-split subspace $X \rightarrowtail \Sigma^N$ such that $A \cong \Omega X$. □

Theorem 8.14 $\mathcal{S}$ is the category of locally compact locales over $\mathcal{E}$. □

Remark 8.15 If the topos $\mathcal{E}$ satisfies the axiom of choice (for example in the form that all epis split, [4, §5.2]) then all objects of $\mathcal{S}$ are spatial [5, Theorem VII 4.3].

This result does not generalise to $\aleph$-locally compact locales. The real unit interval, which is not spatial in $\mathbf{CCD}^{\mathsf{op}}$ [2, Example 9] [B, Example 3.12], also provides a counterexample in the intermediate $\aleph$-case, except when $\aleph = \aleph_0$. □

References

[1] Nicolas Bourbaki. *Eléments de Mathématique XXII: Théories des Ensembles, Livre I, Structures.* Number 1258 in Actualités scientifiques et industrielles. Hermann, 1957. English translation, "Theory of Sets," 1968.

[2] Barry Fawcett and Richard Wood. Completely distributive lattices I. *Mathematical Proceedings of the Cambridge Philosophical Society*, 107:81–9, 1990.

[3] Gerhard Gierz, Karl Heinrich Hoffmann, Klaus Keimel, Jimmie Lawson, Michael Mislove, and Dana Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, 1980.

[4] Peter Johnstone. *Topos Theory.* Number 10 in London Mathematical Society Monographs. Academic Press, 1977.

[5] Peter Johnstone. *Stone Spaces.* Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982.

[6] André Joyal and Ieke Moerdijk. *Algebraic Set Theory.* Number 220 in London Mathematical Society Lecture Notes in Mathematics. Cambridge University Press, 1995.

[7] Francisco Marmolejo, Robert Rosebrugh, and Richard Wood. A basic distributive law. *Journal of Pure and Applied Algebra*, 168:209–226, 2002.

[8] Christian Mikkelsen. *Lattice-theoretic and Logical Aspects of Elementary Topoi.* PhD thesis, Århus Universitet, 1976. Various publications, number 25.

[9] Giuseppe Rosolini. *Continuity and Effectiveness in Topoi.* D. Phil. thesis, University of Oxford, 1986.

[10] Dana Scott. More on the axiom of extensionality. In Yehoshua Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and Abraham Robinson, editors, *Essays of the Foundations of Mathematics*, pages 115–139. Magnes Press, Hebrew University, 1966. Distributed by Oxford University Press.

[11] Paul Taylor. The fixed point property in synthetic domain theory. In Gilles Kahn, editor, *Logic in Computer Science 6*, pages 152–160. IEEE, 1991.

[12] Paul Taylor. *Practical Foundations of Mathematics.* Number 59 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[13] Steven Vickers. The double powerlocale and exponentiation. Submitted to *Theory and Applications of Categories*, 2002.

The papers on abstract Stone duality may be obtained from

`http://www.di.unito.it/~pt/ASD`

[A] Paul Taylor. Sober spaces and continuations. *Theory and Applications of Categories*, 10(12):248–299, 2002.

[B] Paul Taylor. Subspaces in abstract Stone duality. *Theory and Applications of Categories*, 10(13):300–366, 2002.

[C] Paul Taylor. Geometric and higher order logic using abstract Stone duality. *Theory and Applications of Categories*, 7(15):284–338, 2000.

[D] Paul Taylor. Non-Artin gluing in recursion theory and lifting in abstract Stone duality, 2000.

[E] Paul Taylor. Local compactness and the Baire category theorem in abstract Sone duality, *Category Theory and Computer Science*, Ottawa, 2002.

[F] Paul Taylor. Scott domains in abstract Stone duality, 2002.

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