A category of games for topology

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- Bourbaki (with Weil, Cartan and Dieudonné)
- topology, algebraic geometry;
- Fields medal in 1966.
The games

Emphasis on states rather than on moves:

**Def.** an *interaction structure* is given by:

- a set of *states*: \( S : \text{Set} \)
- for each state, a set of *moves*: \( A(s) : \text{Set} \)
- after each move, a set of *counter-moves*: \( D(s, a) : \text{Set} \)
- after each counter-move, a new state: \( n(s, a, d) \in S \)
Strategies, Angel’s side

For $F \subseteq S$ (final states) and $s \in S$, $A(F, s)$ is the set of $F$-winning strategies.

It is an inductive definition:

- if $s \in F$ then $\varepsilon \in A(F, s)$;
- if $a \in A(s)$ and $(\forall d : D(s, a)) f(d) \in A(F, n(s, a, d))$, then $\langle a, f \rangle \in A(F, s)$.
Strategies, Demon’s side

For $G \subseteq S$ and $s \in S$,

$\mathcal{J}(G, s)$ is the set of $G$-restrictive strategies.

It is an co-inductive definition:

$\langle f, k \rangle \in \mathcal{J}(G, s)$ if

- $s \in G$
- $\left( \forall a \in A(s) \right) f(a) \in D(s, a) \land k(a) \in \mathcal{J}(G, n(s, a, f(a)))$
Simulations

Generalization of usual simulation on automata.

Def.: if \( w_1 \) and \( w_2 \) are 2 interaction structures, and \( R \) a relation on \( S_1 \times S_2 \), \( R \) is a \textit{simulation} if:

\[
\forall a_1 \exists a_2 \ \forall d_2 \exists d_1 \quad n_1(s_1, a_1, d_1) \ R \ n_2(s_2, a_2, d_2)
\]

with \( a_1 \in A_1(s_1) \), \( a_2 \in A_2(s_2) \), \( d_2 \in D_2(s_2, a_2) \) and \( d_1 \in D_1(s_1, a_1) \).
**Refinement**

**Def.:**

A *refinement* from $w_1$ to $w_2$...

...is a simulation from $w_1$ to "$w_2^*$".

where "_*" is a reflexive / transitive closure.

**Prop.** $w_1 \rightarrow w_2^*$ is isomorphic to $w_1^* \rightarrow w_2^*$. 
Saturation (difficult!)

For a refinement $R$, we write $\overline{R}$ for the relation:

$$s_1 \overline{R} s_2 \iff s_2 \in A(R(s_1))$$

Prop. $\overline{R}$ is still a refinement.

$R$ and $T$ have the same strength if $\overline{R} = \overline{T}$

(We write $R \approx T$.)
Space, covering relation

Idea:
- points are too complex;
- open (and closed) sets are more important.

A formal topology is given by a set $S$ of basic open sets.

There is a covering relation: $\triangleleft$.

If $s \in S$ and $U \subset S$,
$s \triangleleft U$ means that “$s$ is covered by $U$”

This is enough to start doing topology!
And so...

**Th.** The category of “non-distributive” formal topologies “is” the category of interaction structures...

- topological spaces are interaction structures;
- continuous relations are (inverses) of refinements;
- equality is \( \approx \).
If you *(really)* want more...

Details for the above
- P. Hancock, P. Hyvernat: “Interaction, computer science and topology”.

Basic topologies
Giovanni Sambin’s “basic picture” articles, especially:
- G. Sambin and S. Gebellato: “Pointfree continuity and convergence”

Inductive generation of formal topologies