A category of games for topology

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Cultural bit...

Alexander Grothendieck

28 of March in Berlin (Germany).

- Bourbaki (with Weil, Cartan and Dieudonné)
- topology, algebraic geometry;
- Fields medal in 1966.

The games

Emphasis on *states* rather than on *moves*:

Def. an *interaction structure* is given by:

- a set of *states*: $\boxed{S : \mathsf{Set}}$
- for each state, a set of *moves*: $\boxed{A(s) : \mathsf{Set}}$
- after each move, a set of *counter-moves*: $\boxed{D(s, a) : \mathsf{Set}}$
- after each counter-move, a new state: $\boxed{n(s, a, d) \in S}$

Strategies, Angel's side

For $F \subseteq S$ (final states) and $s \in S$,

$\boxed{\mathcal{A}(F, s)}$ is the *set of F-winning strategies*.

for the Angel

It is an inductive definition:

- if $s \in F$ then $\varepsilon \in \mathcal{A}(F, s)$;
- if $a \in A(s)$ and $\big(\forall d : D(s, a)\big)\ f(d) \in \mathcal{A}\big(F, n(s, a, d)\big)$, then $\langle a, f\rangle \in \mathcal{A}(F, s)$.

Strategies, Demon's side

For $G \subseteq$ and $s \in S$,

$\boxed{\mathcal{J}(G, s)}$ is the set of G-*restrictive strategies*.

for the Demon

It is an co-inductive definition:

$\langle f, k \rangle \in \mathcal{J}(G, s)$ if

- $s \in G$
- $\big(\forall a \in A(s)\big)\ f(a) \in D(s, a)\ \wedge\ k(a) \in \mathcal{J}\big(G, n\big(s, a, f(a)\big)\big)$

Simulations

Generalization of usual simulation on automata.

Def.: if w_1 and w_2 are 2 interaction structures,
and R a relation on $S_1 \times S_2$,
R is a *simulation* if:

$$s_1 \; R \; s_2$$

$$\Longrightarrow$$

$$\forall a_1 \; \exists a_2 \; \forall d_2 \; \exists d_1 \quad n_1(s_1, a_1, d_1) \; R \; n_2(s_2, a_2, d_2)$$

with $a_1 \in A_1(s_1)$, $a_2 \in A_2(s_2)$, $d_2 \in D_2(s_2, a_2)$ and $d_1 \in D_1(s_1, a_1)$.

Refinement

Def.:
a *refinement* from w_1 to w_2...
...is a simulation from w_1 to "w_2^*".

where "_*" is a reflexive / transitive closure.

Prop. $w_1 \to w_2^*$ is isomorphic to $w_1^* \to w_2^*$.

Saturation *(difficult!)*

For a refinement R, we write $\overline{R}$ for the relation:

$$s_1 \overline{R} s_2 \iff s_2 \varepsilon \mathcal{A}\big(R(s_1)\big)$$

Prop. $\overline{R}$ is still a refinement.

R and T have the same *strength* if $\overline{R} = \overline{T}$

(We write $R \approx T$.)

Space, covering relation

Idea:

- points are too complex;
- open (and closed) sets are more important.

A formal topology is given by a set S of *basic open sets*.

(a base)

There is a *covering relation*: $\lhd$.

If $s \in S$ and $U \subset S$,

$s \lhd U$ means that "s is covered by U" ("s is smaller than the $\bigcup U$"...)

This is enough to start doing topology!

And so...

Th. The category of “non-distributive” formal topologies “is” the category of interaction structures...

- topological spaces are interaction structures;
- continuous relations are (inverses) of refinements;
- equality is $\approx$.

If you (really) want more...

Details for the above

- P. Hancock, P. Hyvernat: “Interaction, computer science and topology”.

Basic topologies

Giovanni Sambin’s “basic picture” articles, especially:

- G. Sambin and S. Gebellato: “Pointfree continuity and convergence”

Inductive generation of formal topologies

- T. Coquand, G. Sambin, J. Smith, S. Valentini: “Inductively generated topologies.”