Permutative Conversions in Generalised Multiary λ-calculus

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$\lambda\mathbf{J^m}$: the generalised multiary λ-calculus

$$
\begin{array}{llcl}
(\lambda\mathbf{J^m} - terms) & t, u, v & ::= & x \mid \lambda x.t \mid \underbrace{t(u, l, (x)v)}_{\textit{gm-application}} \\
(\lambda\mathbf{J^m} - lists) & l & ::= & t :: l \mid [] \\
(types) & A, B & ::= & p \mid A \supset B \\
(term\ \ sequents) & & & \Gamma; - \vdash t : A \\
(list\ \ sequents) & & & \Gamma; B \vdash l : A
\end{array}
$$

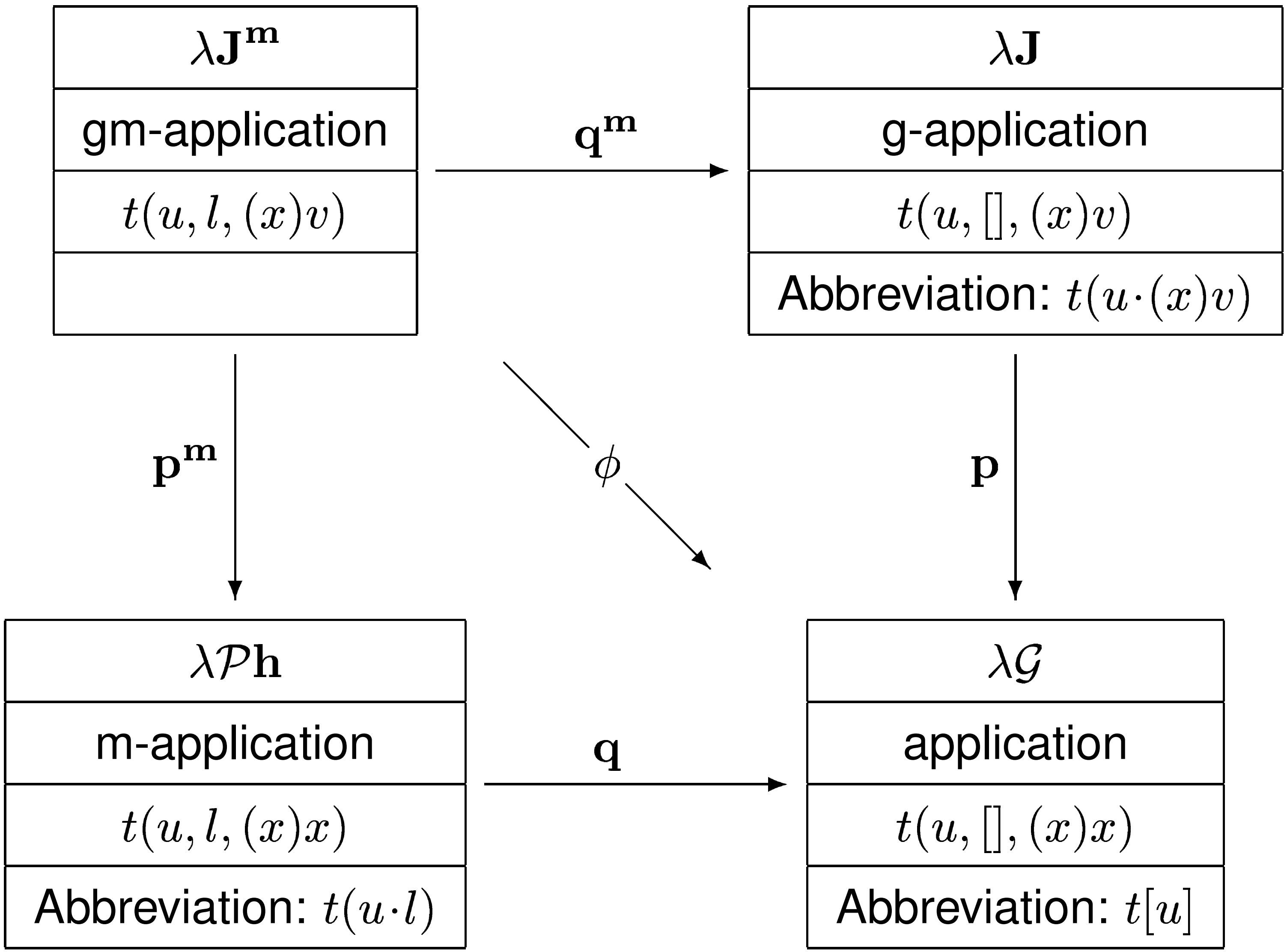
$$Typing\ \ Rules$$

$$\frac{}{\Gamma; C \vdash [] : C}\ \textit{Ax} \qquad \frac{\Gamma; - \vdash u : A \quad \Gamma; B \vdash l : C}{\Gamma; A \supset B \vdash u :: l : C}\ \textit{Lft}$$

$$\frac{}{x : A, \Gamma; - \vdash x : A}\ \textit{Axiom} \qquad \frac{x : A, \Gamma; - \vdash t : B}{\Gamma; - \vdash \lambda x.t : A \supset B}\ \textit{Right}$$

$$\frac{\Gamma; - \vdash t : A \supset B \qquad \Gamma; - \vdash u : A \quad \Gamma; B \vdash l : C \quad x : C, \Gamma; - \vdash v : D}{\Gamma; - \vdash t(u, l, (x)v) : D}\ \textit{gm-Elim}$$

Subsystems of $\lambda\mathbf{J^m}$



Elimination rules for the subsystems of $\lambda\mathbf{J}^{\mathbf{m}}$

$$\frac{\Gamma;-\vdash t:A\supset B \quad \Gamma;-\vdash u:A \quad \Gamma;B\vdash l:C \quad x:C,\Gamma;-\vdash v:D}{\Gamma;-\vdash t(u,l,(x)v):D}\ \textit{gm-Elim}\ (\lambda\mathbf{J}^{\mathbf{m}})$$

$$\frac{\Gamma;-\vdash t:A\supset B \quad \Gamma;-\vdash u:A \quad x:B,\Gamma;-\vdash v:C}{\Gamma;-\vdash t(u\cdot(x)v):C}\ \text{g-Elim}\ (\lambda\mathbf{J})$$

$$\frac{\Gamma;-\vdash t:A\supset B \quad \Gamma;-\vdash u:A \quad \Gamma;B\vdash l:C}{\Gamma;-\vdash t(u\cdot l):C}\ \text{m-Elim}\ (\lambda\mathcal{P}\mathbf{h})$$

$$\frac{\Gamma;-\vdash t:A\supset B \quad \Gamma;-\vdash u:A}{\Gamma;-\vdash t[u]:B}\ \text{Elim}\ (\lambda\mathcal{G})$$

Reduction rules for $\lambda\mathbf{J}^{\mathbf{m}}$

$$\begin{array}{rrcl}
(\beta_1) & (\lambda x.t)(u,[],(y)v) & \rightarrow & \mathbf{s}(\mathbf{s}(u,x,t),y,v) \\
(\beta_2) & (\lambda x.t)(u,v::l,(y)v) & \rightarrow & \mathbf{s}(u,x,t)(v,l,(y)v) \\
(\pi) & t(u,l,(x)v)(u',l',(y)v') & \rightarrow & t(u,l,(x)v(u',l',(y)v')) \\
(\mu) & t(u,l,(x)x(u',l',(y)v')) & \rightarrow & t(u,\mathbf{append}(l,u',l'),(y)v'), \quad x \notin u',l',v'
\end{array}$$

$(\beta_1),(\beta_2),(\pi)$-normal forms:

$$\begin{array}{rcl}
t,u & ::= & x \mid \lambda x.t \mid x(u,l,(y)v) \\
l & ::= & u::l \mid []
\end{array}$$

$(\beta_1),(\beta_2),(\pi),(\mu)$-normal forms: as above, with proviso

if $v = y(u',l',(y')v')$, y must occur either in u', l' or v'

Permutative conversions

$$\boxed{t(u, l, (x)x)}$$

$$
\begin{array}{lrcl}
(p_1) & t(u, l, (x)y) & \to & y, \quad x \neq y \\
(p_2) & t(u, l, (x)\lambda y.v) & \to & \lambda y.t(u, l, (x)v) \\
(p_3) & t_1(u_1, l_1, (x)t_2(u_2, l_2, (y)v)) & \to & \\
& \multicolumn{3}{l}{t_1(u_1, l_1, (x)t_2)(t_1(u_1, l_1, (x)u_2), \mathbf{p'_3}(t_1, u_1, l_1, x, l_2), (y)v) \text{ if } x \notin v,}
\end{array}
$$

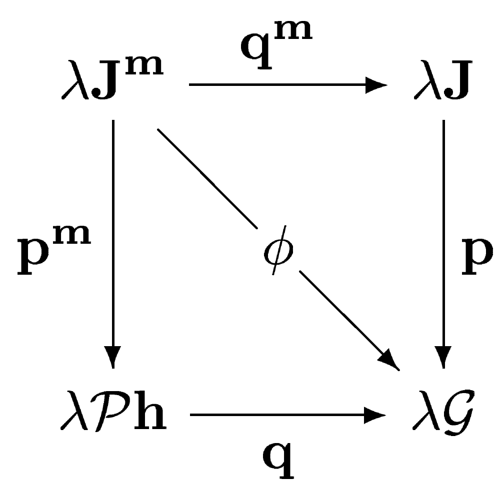
where

$$
\begin{array}{rcl}
\mathbf{p'_3}(t, u, l, x, []) & = & [] \\
\mathbf{p'_3}(t, u, l, x, u' :: l') & = & t(u, l, (x)u') :: \mathbf{p'_3}(t, u, l, x, l')
\end{array}
$$

$$\boxed{t(u, [], (x)v)}$$

$$
\begin{array}{lrcl}
(q) & t(u, v :: l, (x)v') & \to & t[u](v, l, (x)v')
\end{array}
$$

Main results



Permutability Thms.

$\phi(t_1) = \phi(t_2)$ iff $t_1 \leftrightarrow^*_{p,q} t_2$, $\forall t_1, t_2 \in \lambda\mathbf{J^m}$.

$\mathbf{p}(t_1) = \mathbf{p}(t_2)$ iff $t_1 \leftrightarrow^*_{p} t_2$, $\forall t_1, t_2 \in \lambda\mathbf{J}$.

$\mathbf{q}(t_1) = \mathbf{q}(t_2)$ iff $t_1 \leftrightarrow^*_{q} t_2$, $\forall t_1, t_2 \in \lambda\mathcal{P}\mathbf{h}$.

Representation Thms.

$\phi(t) = \downarrow^{\lambda\mathbf{J^m}}_{p,q}(t)$, $\forall t \in \lambda\mathbf{J^m}$.

$\mathbf{p}(t) = \downarrow^{\lambda\mathbf{J}}_{p}(t)$, $\forall t \in \lambda\mathbf{J}$.

$\mathbf{q}(t) = \downarrow^{\lambda\mathcal{P}\mathbf{h}}_{q}(t)$, $\forall t \in \lambda\mathcal{P}\mathbf{h}$.

Conclusion

- Permutability study on a multiary sequent calculus with cuts
- Defined the calculus $\lambda\mathbf{J}^{\mathbf{m}}$ and the notion of generalised multiary application
- Computational interpretation for fragments of sequent calculus obtained via their correspondence to extended λ-calculi ($\lambda\mathbf{J}^{\mathbf{m}}$, $\lambda\mathbf{J}$, $\lambda\mathcal{P}\mathbf{h}$, $\lambda\mathcal{G}$)