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Towards a Semantics for Reductive Logic
Simplifying proofs \( \equiv \) computation in functional programming

\[ [\phi] : [\Phi] \vdash [I] \]

Reasoning about proofs via propositions-as-types analogy

\[ \frac{\text{Conclusion}}{R} \]

using rules like

Proofs are constructed from premises to conclusions (top down)

Introduction
attents

use term reduction and reduction operators to describe such

Try to apply inference rules backwards to obtain proof

Start with conclusion (Theorem to be proved)

Have very different situation in Theorem Proving

Proof Search
Which formula to operate on? When to backtrack?

Need to model control aspects of searches (which reduction rule?

Sufficient premises might not be provable.

Premises but premises but

Start from Putative Conclusions and construct Sufficient

Proof is total; reduction is partial;

Issue:

Would like to have semantics to reason about search.

What about Semantics?
Can we obtain similar pictures?

Framework for Reductive Semantics
For each world \( \wp \), have set of morphisms describing effect of

\[
\begin{align*}
\wp & \vdash \phi \\
\wp & \vdash \psi \\
\end{align*}
\]

model use of arbitrary leaves \( \wp \) (indeterminates) by \( \phi \) operators:

Order: prefix-order (extension of worlds \( \equiv \) application of reduction of operators)

Worlds: sequences of reduction operators

reduction:

use Kripke-structure to model increase of information during Semantics
Treatment of control (eg. backtracking)

Next step: substitution of proofs without indefinite for all indeterminates. Intuitionistic proof iff exists morphism from corresponding to theorem 1. A reduction for the sequent \( \phi \) is complementable to an...
Need backtracking to obtain proof.

\[ \text{Tories} \]

\[ \begin{align*}
\text{H} \subseteq & \quad \text{\( \emptyset \subseteq \phi \)} \\
\text{T} \subseteq & \quad \text{\( \emptyset \subseteq \phi \)} \\
\text{\( \forall x \)} & \quad \text{\( \emptyset \subseteq \phi \)} \\
\end{align*} \]

Consider following example in intuitionistic logic:

Towards modelling control.
Consider embedding of intuitionistic in classical logic

\[ m \subset \phi \iff m \subset \phi, \phi \subset \phi \]

\[ \forall \, m \subset \phi \iff \exists \, i, m \subset \phi, \phi \subset \phi \]

\[ \exists \, \text{Exchange} \]

\[ m \subset \phi \]

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Now exploit higher semantics:

\[
\frac{\nabla B, \forall \psi : \exists \theta}{\nabla B \land \forall \theta : \exists \psi, \forall \psi} \vdash \nabla B'
\]

with IK-like disjunction:

\[
\nabla B \vdash \forall \psi
\]

Judgments:

Induces use \( \forall \)-calculus (Prawitz, Ritter, Pym, Williamson)

Semantics for classical reductions
works because everything is natural

modeling

can combine abstractions with indeterminates and abstraction for

Direct transfer from intuitionistic case:

classical reduction structures
Exchange is interpreted as continuation, i.e., as change of base

Continuation models provide a good model (Homann/Streicher)

Continuation models
Assume $\phi \land \phi$.

$
\phi \land \phi
$

Are trees for and respectively?

$\phi \land \phi$

Distinct sum of arenas for and:

Are trees for and respectively?

Have games in style of Hyland and One

Games model of intuitionsist proof search
If proponent wants start playing the unmarked link, must first play switching move:

● Each player plays as many moves as he likes;

● Opponent always starts by asking initial question;

A play for arena is a sequence of moves such that

\[ \phi \subseteq \phi \]

Games semantics continued
Worlds are finite set of strategies with oracles, ordered by inclusion.

Substitution for indeterminates works by composing strategies.

Moves in the arena correspond to oracles.

Indeterminates captured by oracles: Proponent can make arbitrary

p-moves: Left reductions

O-moves: Right reductions

Strategies is a function from set of O-moves to set of p-moves

Games Semantics; continued
Failure corresponds to abandon scratchpad

Exchange operation corresponds to read and write from

appropriate type

Starting playing arena for \( \phi \land \phi \), which stores answers of

During play, Proponent may open or close scratchpad when

Backtracking interpreted by scratchpads:

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Interpretation of Backtracking
handles paradigmatic aspect of proof search: failure and restart

semantics for \( \forall \)-calculus

First step: capture backtracking in intuitionistic logic by extension of

need highly intensional semantics

• have to consider partial, possibly non-completable proofs

search constructed bottom-up

• proofs constructed top-down

semantics of proof search different from standard semantics:
reduction rules to capture preferences

Will pursue domain-theoretic approach: consider ordering of

which sequence to work on next

and order on choices between rules

characterise further aspects of control:

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Further work