Towards a Semantics for Reductive Logic

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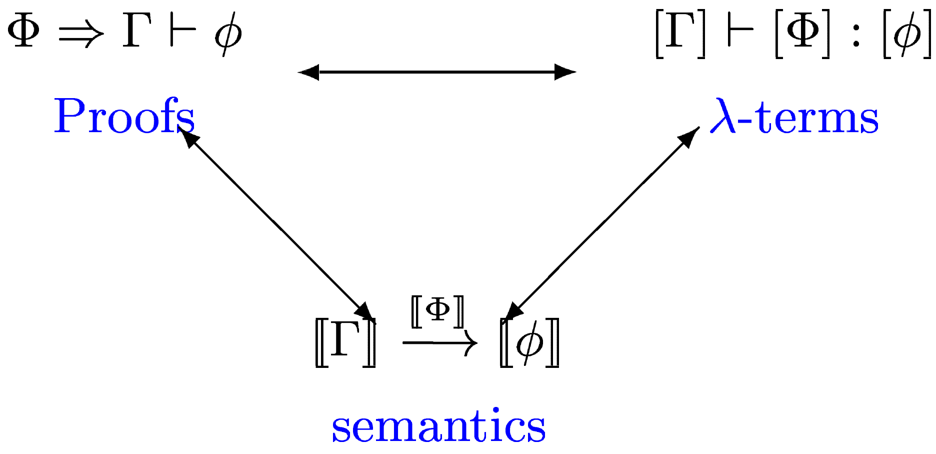
Introduction

Proofs are constructed from premisses to conclusions (top down) using rules like

$$\Downarrow \quad \frac{\textbf{Premiss}_1 \ldots \textbf{Premiss}_m}{\textbf{Conclusion}} \quad R.$$

Reasoning about proofs via propositions-as-types analogy



$\Rightarrow$ Simplifying proofs $\cong$ computation in functional programming

Proof Search

Have very different situation in Theorem Proving:

- Start with conclusion (Theorem to be proved)
- Try to apply inference rules backwards to obtain proof
  use term Reduction and reduction operators to describe such attempts

What about Semantics?

Would like to have semantics to reason about search.

Issues:

- Proof is total; reduction is partial:
  Start from Putative Conclusions and construct Sufficient Premisses but
  Sufficient Premisses might not be provable!

- Need to model control aspects of searches (which reduction rule; which formula to operate on; when to backtrack)

Framework for Reductive Semantics

Can we obtain similar picture?

$\Phi \Rightarrow \Gamma$?- ϕ $\longleftrightarrow$ $[\Gamma] \mathrel{|\!\!\sim} [\Phi] : [\phi]$

Reduction — Realisers for reductions

$[\![\Gamma]\!] \overset{[\![\Phi]\!]}{\rightsquigarrow} [\![\phi]\!]$

Semantics

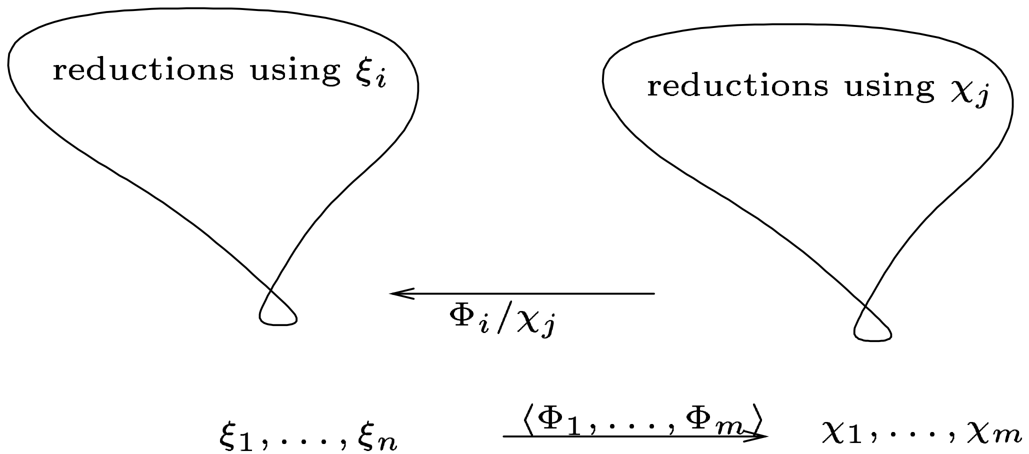
Semantics

use Kripke-structure to model increase of information during reduction:

Worlds: sequences of reduction operators

Order: prefix-order (extension of worlds $\cong$ application of reduction operators)

model use of arbitrary leaves Γ ?- ϕ (indeterminates) by fibrations:



For each world w, have set of morphisms describing effect of reduction operators in w

Soundness and Completeness

Theorem 1. *A reduction for the sequent* Γ *?-* ϕ *is completeable to an intuitionistic proof iff exists morphism* f *corresponding to substitution of proofs without indeterminates for all indeterminates.*

Next step:

Treatment of control (eg backtracking)

Towards modelling control

Consider following example in intuitionistic logic:

$$
\dfrac{
  \dfrac{???}{\phi \supset \psi, \psi \supset \omega \text{ ?- } \sigma}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\vdots}{\phi \supset \psi, \tau, \phi, \text{ ?- } \psi}
      \quad
      \dfrac{}{\phi \supset \psi, \tau, \phi, \omega \text{ ?- } \omega}\,Ax
    }{\phi \supset \psi, \psi \supset \omega, \tau, \phi \text{ ?- } \omega}\supset L
  }{\phi \supset \psi, \psi \supset \omega, \tau \text{ ?- } \phi \supset \omega}\supset R
}{\sigma \supset \tau, \phi \supset \psi, \psi \supset \omega \text{ ?- } \phi \supset \omega}\supset L
$$

Need backtracking to obtain proof.

Corresponding classical proof

$$\dfrac{\dfrac{\dfrac{\vdots}{\phi \supset \psi, \psi \supset \omega \text{ ?- } \phi \supset \omega, \sigma}}{\phi \supset \psi, \psi \supset \omega \text{ ?- } \sigma, \color{red}{\phi \supset \omega}}\,ExchangeR \qquad \dfrac{\text{essentially as above}}{\phi \supset \psi, \psi \supset \omega, \tau \text{ ?- } \phi \supset \omega}\supset R}{\sigma \supset \tau, \phi \supset \psi, \psi \supset \omega \text{ ?- } \phi \supset \omega}\supset L$$

This exchange models backtracking in intuitionistic logic!

$\Rightarrow$ consider embedding of intuitionistic in classical logic

Semantics for classical reductions

Idea: use $\lambda\mu\nu$-calculus (Parigot, Ritter/Pym/Wallen)

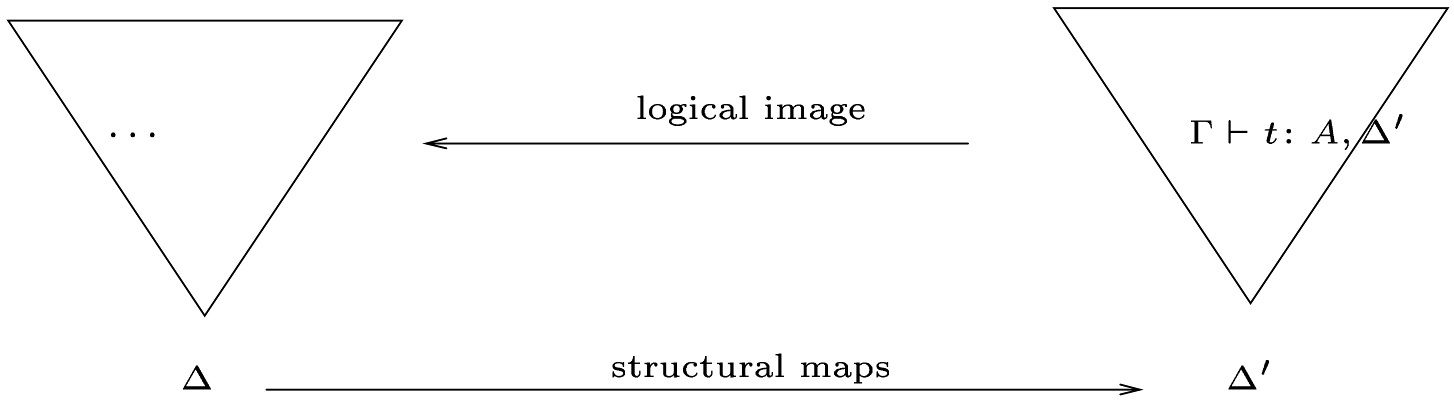
Judgements:

$$\Gamma \vdash t\colon A, \Delta$$

with LK-like disjunction:

$$\frac{\Gamma \vdash t\colon A, B^{\beta}, \Delta}{\Gamma \vdash \nu\beta.t\colon A \vee B, \Delta}$$

Now exploit fibred semantics:



Classical reduction structures

Direct transfer from intuitionistic case:

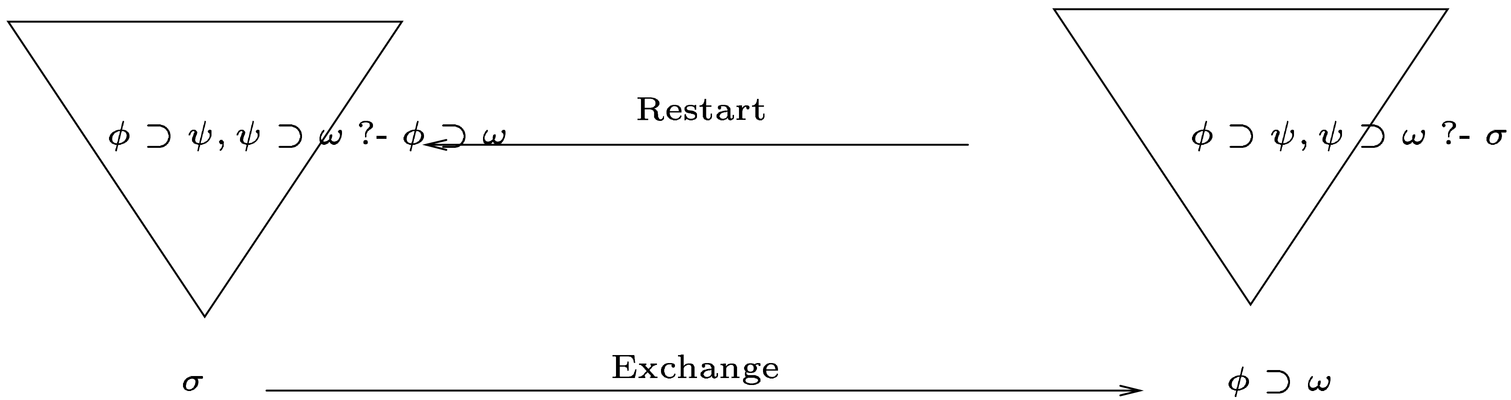
can combine fibration with indeterminates and fibration for modelling $\lambda\mu\nu$

works because everything is natural

Continuation models

Continuation provide a good model (Hofmann/Streicher, Pym/Ritter)

Exchange is interpreted as continuation, *i.e.*, as change of base

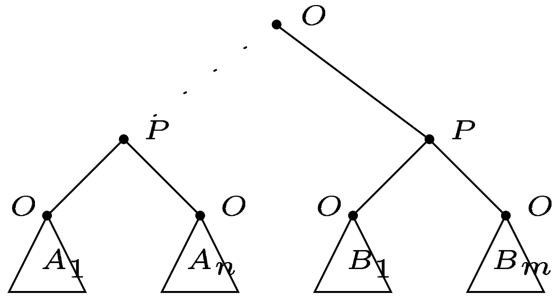


Games model of intuitionistic proof search

Have games in style of Hyland and Ong

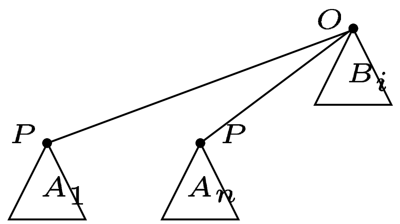
Arenas are given by

- $\top$: empty arena
- $\bot$: one node only
- $\phi \wedge \psi$: disjoint sum of arenas for ϕ and ψ;
- $\phi \vee \psi$: Assume $\mathcal{A}_i$ and $\mathcal{B}_j$ are trees for ϕ and ψ respectively;



Games semantics, continued

- $\phi \supset \psi$:



A play for arena is a sequence of moves such that

- Opponent always starts by asking initial question;
- Each player plays as many moves as he likes;
- If Proponent wants start playing in $\phi \vee \psi$ by playing the unmarked link, must first play switching move;

Games Semantics, continued

Strategies is a function from set of O-moves to set of P-moves
O-moves: right reductions
P-moves: left reductions

Indeterminates captured by oracles: Proponent can make arbitrary moves in the arena corresponding to oracle

Substitution for indeterminates works by composing strategies

Worlds are finite set of strategies with oracles, ordered by inclusion

Interpretation of Backtracking

Backtracking interpreted by scratchpads:

- During play, Proponent may open or close scratchpad when starting playing arena for $\phi \vee \psi$, which stores answers of appropriate type

- Exchange operation corresponds to read and write from scratchpad
- Failure corresponds to abandon scratchpad

Conclusions

Semantics of proof search different from standard semantics:

- proofs constructed top-down,
  search constructed bottom-up
- have to consider partial, possibly non-completeable proofs
- need highly intensional semantics

First step: capture backtracking in intuitionistic logic by extension of semantics for $\lambda\mu\nu$-calculus
handles paradigmatic aspect of proof search: failure and restart

Further work

characterise further aspects of control:

- order on choices between rules
- which sequence to work on next

Will pursue domain-theoretic approach: consider ordering of reduction rules to capture preferences