Automatic Presentations and Classes of Semigroups

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Joint work with Prof. Rick Thomas
Automatic Presentations

- Finite presentations of infinite structures
- Generic approach to deciding FO theory
- Restriction of recursive structures
- Inspired by theory of automatic groups and automatic semigroups
Automatic Presentations

\[ x = x_1 x_2 \ldots x_r \quad y = y_1 y_2 \ldots y_s \quad |x| \leq |y| \]

Convolution of \( x \) and \( y \), \( \text{conv}(x,y) \), is:

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \ldots \begin{pmatrix} x_r \\ y_r \end{pmatrix} \begin{pmatrix} \square \\ y_{r+1} \end{pmatrix} \ldots \begin{pmatrix} \square \\ y_s \end{pmatrix}
\]

An Automatic Presentation for a structure \((S, R_1, \ldots, R_n)\) consists of:

- Regular language \( L \subseteq \Sigma^* \)
- Surjective map: \( \theta : L \rightarrow S \)
- \( L_- = \{ \text{conv}(x_1, x_2) : \theta(x_1) = \theta(x_2) \} \) is regular
- \( L_{R_i} = \{ \text{conv}(x_1, \ldots, x_k) : R_i(\theta(x_1), \ldots, \theta(x_k)) \} \) is regular
Automatic Presentations

For a structure $S$ with an automatic presentation:

- The $FO$-theory of $S$ is decidable

- If $T$ is $FO$-interpretable in $S$ then $T$ has an automatic presentation

- $S$ is $FO$-interpretable in $(\mathbb{N}, +, |_2)$

- $S$ is $FO$-interpretable in $(\{0, 1\}^*, \leq, P_0, P_1, el)$
Semigroups

\((S, \circ)\), where \(\circ\) is an associative binary function.

1. Examples:

(a) \(\Sigma = \{a, b\}, (\Sigma^* \circ)\)
   where e.g. \(aba \circ bbb = ababbb\)

(b) \((\mathbb{N}, \circ), (\mathbb{Z}, \circ), (\mathbb{Q}, \circ), (\mathbb{R}, \circ), (\mathbb{C}, \circ)\)
   where \(\circ \in \{+, \times\}\)

(c) (Partial) Automorphisms
Groups

• $1 \circ s = s \circ 1 = s$

• $\exists s^{-1}, s \circ s^{-1} = 1 = s^{-1} \circ s$

A group is virtully abelian if it contains an abelian subgroup of finite index.

Theorem (G.O, R.Thomas STACS05):
A f.g. group $G$ has an automatic presentation if and only if $G$ is virtually abelian

Corollary:
The class of f.g. groups with automatic presentations is properly contained in the class of automatic groups.
Commutative Semigroups

• $x \circ y = y \circ x$

Theorem (Taitslin):
All f.g. commutative semigroups are $FO$-interpretable in $(\mathbb{N}, +)$.

Corollary:
All f.g. commutative semigroups have automatic presentations.

Note: There exists a f.g. commutative semigroup that is not automatic.
(Hoffmann, Thomas)
Cancellative Semigroups

- \(a \circ x = a \circ y \Rightarrow x = y\)

- \(x \circ b = y \circ b \Rightarrow x = y\)

Proposition (G.O, R.Thomas):
If a f.g. semigroup \(S\) has an automatic presentation,
then \(S\) has polynomial growth

\[G_S = \{s^{-1} \circ t : s, t \in S\}\]. If \(G_S\) is a group,
it is called the group of (left) quotients of \(S\).

Proposition (Grigorchuk):
If a f.g. cancellative semigroup \(S\) has polynomial growth,
then \(S\) has a group of (left) quotients \(G_S\)
Cancellative Semigroups

$S$ - semigroup, $G_S$ - group of (left) quotients

Proposition:
$G_S$ is \textbf{FO-interpretable} in $S$

\textit{FO-interpretation:}

$$\theta(x, y) := \forall z, z = z$$

$$f : S^2 \rightarrow G$$

$$f(s, t) = s^{-1} \circ t$$

$$\theta = (x_1, y_1; x_2, y_2) := \exists p, q \left( x_1 \circ p = y_1 \circ q \right.$$ 

$$\land x_2 \circ p = y_2 \circ q \bigg)$$

$$\theta_o(x_1, y_1; x_2, y_2; x_3, y_3) := \exists p, q \left( x_3 = p \circ x_1 \right.$$ 

$$\land y_3 = q \circ y_2 \land q \circ x_2 = p \circ y_1 \bigg)$$
Cancellative Semigroups

Theorem:
Let $S$ be a f.g. cancellative semigroup with an automatic presentation;
then, $S$ embeds in a virtually abelian group.

Proof:
The group of (left) quotients of $S$, $G_S$, has an automatic presentation;
so, $G_S$ is virtually abelian.

Conjecture:
A f.g. cancellative semigroup $S$ has an automatic presentation
if and only if $S$ embeds in a virtually abelian group.
Conclusion

Theorem:
A f.g. group $G$ has an automatic presentation if and only if $G$ is virtually abelian.

Theorem:
All f.g. commutative semigroups have automatic presentations.

Theorem:
Let $S$ be a f.g. cancellative semigroup with an automatic presentation; then, $S$ embeds in a virtually abelian group.