Automatic Presentations and Classes of Semigroups

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Automatic Presentations

- *Finite* presentations of *infinite* structures
- Generic approach to deciding FO theory
- Restriction of recursive structures
- Inspired by theory of automatic groups and automatic semigroups

Automatic Presentations

$$x = x_1 x_2 \ldots x_r \qquad y = y_1 y_2 \ldots y_s \qquad |x| \leq |y|$$

Convolution of x and y, conv(x, y), is:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \ldots \begin{pmatrix} x_r \\ y_r \end{pmatrix} \begin{pmatrix} \square \\ y_{r+1} \end{pmatrix} \ldots \begin{pmatrix} \square \\ y_s \end{pmatrix}$$

An *Automatic Presentation* for
a structure $(S, R_1, \ldots, R_n)$ consists of:

- Regular language $L \subseteq \Sigma^*$
- Surjective map: $\theta : L \to S$
- $L_= = \{\text{conv}(x_1, x_2) : \theta(x_1) = \theta(x_2)\}$ is regular
- $L_{R_i} = \{\text{conv}(x_1, \ldots, x_k) : R_i(\theta(x_1), \ldots, \theta(x_k)\}$ is regular

Automatic Presentations

For a structure S with an automatic presentation:

- The FO-theory of S is decidable
- If T is FO-interpretable in S then T has an automatic presentation
- S is FO-interpretable in $(\mathbb{N}, +, |_2)$
- S is FO-interpretable in $(\{0,1\}^\star, \preceq, P_0, P_1, el)$

Semigroups

$(S, \circ)$, where $\circ$ is an associative binary function.

1. Examples:

   (a) $\Sigma = \{a, b\}$, $(\Sigma^\star \circ)$
   where e.g. $aba \circ bbb = ababbb$

   (b) $(\mathbb{N}, \circ), (\mathbb{Z}, \circ), (\mathbb{Q}, \circ), (\mathbb{R}, \circ), (\mathbb{C}, \circ)$
   where $\circ \in \{+, \times\}$

   (c) (Partial) Automorphisms

Groups

- $1 \circ s = s \circ 1 = s$

- $\exists s^{-1}, s \circ s^{-1} = 1 = s^{-1} \circ s$

A group is *virtually abelian* if it contains an abelian subgroup of finite index.

Theorem (G.O, R.Thomas STACS05):
A f.g. **group** G has an **automatic presentation**
if and only if G is **virtually abelian**

Corollary:
The class of **f.g. groups** with **automatic presentations** is **properly contained** in the class of **automatic groups**.

Commutative Semigroups

- $x \circ y = y \circ x$

Theorem (Taitslin):
All f.g. **commutative** semigroups are FO-**interpretable** in $(\mathbb{N}, +)$.

Corollary:
All f.g. **commutative** semigroups have **automatic presentations**.

Note: There exists a f.g. **commutative** semigroup that is **not automatic**. (Hoffmann, Thomas)

Cancellative Semigroups

- $a \circ x = a \circ y \Rightarrow x = y$

- $x \circ b = y \circ b \Rightarrow x = y$

Proposition (G.O, R.Thomas):
If a f.g. **semigroup** S has an automatic presentation,
then S has **polynomial growth**

Let $G_S = \{s^{-1} \circ t : s, t \in S\}$. If G_S is a group, it is called the *group of (left) quotients* of S.

Proposition (Grigorchuk):
If a f.g. **cancellative** semigroup S has **polynomial growth**,
then S has a **group of (left) quotients** G_S

Cancellative Semigroups

S - semigroup, G_S - group of (left) quotients

Proposition:
G_S is FO-**interpretable** in S

FO-interpretation:

$$\theta(x, y) := \forall z, z = z$$

$$\begin{aligned} f : S^2 &\to G \\ f(s, t) &= s^{-1} \circ t \end{aligned}$$

$$\begin{aligned} \theta_{=}(x_1, y_1; x_2, y_2) := \exists p, q(x_1 \circ p &= y_1 \circ q \\ \wedge\, x_2 \circ p &= y_2 \circ q) \end{aligned}$$

$$\begin{aligned} \theta_{\circ}(x_1, y_1; x_2, y_2; x_3, y_3) := \exists p, q(x_3 &= p \circ x_1 \\ \wedge\, y_3 &= q \circ y_2 \\ \wedge\, q \circ x_2 &= p \circ y_1) \end{aligned}$$

Cancellative Semigroups

Theorem:
Let S be a f.g. **cancellative** semigroup with an **automatic presentation**;
then, S embeds in a **virtually abelian** group.

Proof:
The group of (left) quotients of S, G_S, has an automatic presentation;
so, G_S is virtually abelian.

Conjecture:
A f.g. **cancellative** semigroup S has an **automatic presentation**
if and only if S embeds in a **virtually abelian** group.

Conclusion

Theorem:
A f.g. **group** G has an **automatic presentation**
if and only if G is **virtually abelian**

Theorem:
All f.g. **commutative** semigroups have **automatic presentations**.

Theorem:
Let S be a f.g. **cancellative** semigroup with an **automatic presentation**;
then, S embeds in a **virtually abelian** group.