SUMMARY (lectures 1-4)

pointed posets

* elements represent "pieces of info"
* some elements convey more info than others ($\sqsubseteq$)
* one element conveys no info ($\perp$)

monotonic functions

* a function $f : A \to B$ on posets is monotonic iff

$$x \sqsubseteq y \text{ implies } f(x) \sqsubseteq f(y)$$

* monotonic = "structure preserving fn on posets"
* monotonic = "more info in $\Rightarrow$ more info out".

combining info: least upper bound

* if $X \subseteq A$ then $x \in A = \bigsqcup X$ iff

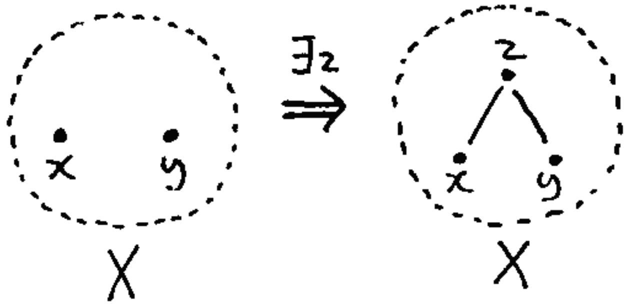
① $X \sqsubseteq x$ "upper bound"

② $X \sqsubseteq x'$ implies $x \sqsubseteq x'$ "least"

* elements can be inconsistent: $\bigsqcup X$ need not exist.
* $\bigsqcup X$ = "all the info in X, but nothing extra".

generalised chains: directed sets

* a non-empty $X \subseteq A$ is directed iff



* directed set = "set which is going somewhere".

complete partial orders

* a poset A is a cpo iff

    1. A has a $\bot$ — "pointed"
    2. $\bigsqcup X$ exists for all directed $X \subseteq A$ — "directed complete"

* cpo = "the directed sets get there"

continuous functions

* a function $f : A \to B$ on cpo's is continuous iff

    1. f is monotonic
    2. $f(\bigsqcup X) = \bigsqcup(fX)$ for all directed $X \subseteq A$

* continuous = "structure preserving fn on cpo's"

* continuous = "nothing is invented at infinity".

constructions on cpo's.

* cpo's are closed under

  $\times,\ \to,\ +,\ \otimes,\ \multimap,\ \oplus,\ (-)_\perp.$

* there are "constructors" and "destructors":

op	constructors	destructors
$\times$	$\langle -,- \rangle$	π_0 π_1
$\to$	curry	apply
$+$	inl inr	$[-,-]$
$(-)_\perp$	lift	drop

* each op on cpo's extends to an op on continuous fns.

  " $\times$, $\to$, $+$, etc are <u>functors</u>"

recursion and fixpoints

* a recursive prog $f : A \to B$ can be given a semantics as a fixpoint of a non-recursive $\Phi : (A \to B) \to (A \to B)$.

* <u>Theorem</u>: any continuous $f : A \to A$ on a cpo A has a <u>least fixpoint</u>: $fix(f) = \bigsqcup \{ f^n(\perp) \mid n \in \mathbb{N} \}$.

finite elements in cpo's

* $x \in A$ is <u>finite</u> or <u>compact</u> iff

$$x \sqsubseteq \bigsqcup X \Rightarrow \exists y \in X.\ x \sqsubseteq y \quad (\text{for all directed } X \subseteq A)$$

* finite = "approx the lub ⇒ approx an element".

algebraic cpo's

* a cpo D is <u>ω-algebraic</u> iff

$$\left.\begin{array}{ll} ① & K(D) = \{x \in D \mid x \text{ is finite}\} \text{ is countable} \\ ② & \downarrow(x) = \{a \in K(D) \mid a \sqsubseteq x\} \text{ is directed} \\ ③ & x = \bigsqcup \downarrow(x) \end{array}\right\} \forall x \in D$$

* algebraic = "elements are the lub of their finite approx".

* <u>theorem</u>: algebraic cpo's are completely determined (up to iso) by their compact elements:

$$D \cong \overline{K(D)}, \text{ the "ideal completion" of } D.$$

* <u>problem</u>: algebraic cpo's are not closed under $\rightarrow$

Scott domains

* a cpo D is a <u>Scott domain</u> iff

  ① D is ω-algebraic

  ② $\bigsqcup X$ exists for all consistent $X \subseteq A$

* <u>consistent</u> = "has an upper bound in A"

* memory aid: "Scott domain = a c^3po"

* Scott domains are closed under

  $\times, \ \to, \ +, \ \otimes, \ \multimap, \ \oplus, \ (-)_\perp.$

theses

* semantic domains are Scott domains

* computable functions are continuous functions

<u>but</u>... remember that cpo's and continuous functions are sufficient for most purposes in semantics.

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