SUMMARY (lectures 1-4)

pointed posets
* elements represent "pieces of info"
* some elements convey more info than others (≤)
* one element conveys no info (⊥)

monotonic functions
* a function \( f : A \rightarrow B \) on posets is monotonic iff
  \[ x \leq y \implies f(x) \leq f(y) \]
* monotonic = "structure preserving fn on posets"
* monotonic = "more info in ⇒ more info out".

combining info: least upper bound
* if \( X \subseteq A \) then \( x \in A = \bigcup X \) iff
  1. \( X \subseteq x \) "upper bound"
  2. \( X \subseteq x' \) implies \( x \subseteq x' \) "least"

* elements can be inconsistent: \( \bigcup X \) need not exist.
* \( \bigcup X \) = "all the info in \( X \), but nothing extra".
generalised chains : directed sets

* a non-empty \( X \subseteq A \) is directed iff

* directed set = "set which is going somewhere".

complete partial orders

* a poset \( A \) is a cpo iff

1. \( A \) has a \( \bot \) "pointed"
2. \( \bigcup X \) exists for all directed \( X \subseteq A \) "directed complete"

* cpo = "the directed sets get there"

continuous functions

* a function \( f : A \to B \) on cpo's is continuous iff

1. \( f \) is monotonic
2. \( f(\bigcup X) = \bigcup(fX) \) for all directed \( X \subseteq A \)

* continuous = "structure preserving \( f \) on cpo's"

* continuous = "nothing is invented at infinity".
**constructions on cpo's.**

* cpo's are closed under

\[ X, \to, +, \otimes, \circ, \oplus, \lnot. \]

* there are "constructors" and "destructors":

<table>
<thead>
<tr>
<th>op</th>
<th>constructors</th>
<th>destructors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>((-,-))</td>
<td>( \Pi_0, \Pi_1 )</td>
</tr>
<tr>
<td>( \to )</td>
<td>curry</td>
<td>apply</td>
</tr>
<tr>
<td>(+)</td>
<td>( \text{inl, inr} )</td>
<td>([-,-])</td>
</tr>
<tr>
<td>((\sim))</td>
<td>lift</td>
<td>drop</td>
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* each op on cpo's extends to an op on continuous fns.

"\( X, \to, +, \) etc are functors"

**recursion and fixpoints**

* a recursive prog \( S : A \to B \) can be given a semantics as a fixpoint of a non-recursive \( \mathcal{F} : (A \to B) \to (A \to B) \).

* **Theorem**: any continuous \( S : A \to A \) on a cpo \( A \) has a least fixpoint: \( \text{Fix}(S) = \bigcup_{n \in \mathbb{N}} S^n(1) \).
Finite elements in cpo's

* \( x \in A \) is finite or compact iff

\[ x \in \bigcup X \Rightarrow \exists y \in X. x \leq y \quad \text{(for all directed } X \subseteq A) \]

* finite = "approx the lub \( \Rightarrow \) approx an element".

Algebraic cpo's

* a cpo \( D \) is \( w \)-algebraic iff

1. \( K(D) = \{ x \in D \mid x \leq \text{finite} \} \) is countable
2. \( \downarrow (x) = \{ a \in K(D) \mid a \leq x \} \) is directed \( \forall x \in D \)
3. \( x = \bigcup \downarrow (x) \)

* algebraic = "elements are the lub of their finite approx".

* theorem: algebraic cpo's are completely determined (up to iso) by their compact elements:

\[ D = K(D), \text{ the "ideal completion" of } D. \]

* problem: algebraic cpo's are not closed under \( \rightarrow \)
Scott domains

* a cpo $D$ is a Scott domain iff
  1. $D$ is $w$-algebraic
  2. $UX$ exists for all consistent $X \subseteq A$

* consistent = "has an upper bound in $A$"
* memory aid: "Scott domain = a c^3po"
* Scott domains are closed under
  $X, \rightarrow, +, \otimes, o\rightarrow, \Theta, (-)_2$.

theses

* semantic domains are Scott domains
* computable functions are continuous functions

but... remember that cpo's and continuous functions are sufficient for most purposes in semantics.